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ON A CLASS OF q-SERIES RELATED TO QUADRATIC FORMS

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Abstract. We consider a class of functions, related to the cyclotomic polynomials, which are invariant under the Fourier transform; and using the Poisson summation formula, we bring out its connection with the Dedekind zeta functions associated with the classical quadratic field. Applying Mellin transform to the Dedekind zeta functions, we evaluate the Lambert series for the Kronecker symbol in terms of sums of products of classical theta functions.

1. Introduction. In this note we will consider the integrals of the form

$$\int_0^\infty t^{s-1} f(t)dt,$$

where f is a suitable rational function.

To motivate the main theme of this paper, we first begin with the identity

$$\int_0^\infty t^{s-1}\frac{t}{1+t^2}dt = \frac{\pi}{2}\sec\frac{\pi s}{2},\ -1 < Re\ s < 1. \tag{1.1}$$

This identity follows easily from the well-known identity [6, p.118]

$$\int_0^\infty \frac{t^{s-1}}{1+t}dt = \frac{\pi}{\sin \pi s},\ 0 < Re\ s < 1$$

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by the replacements of t with t^2 and s with $\frac{s+1}{2}$. One of the most interesting consequences of (1.1) is an identity known as Ramanujan's formula [5, p.11]:

$$\int_{-\infty}^{\infty} e^{-2\pi izt}\text{sech }\pi t dt = \text{sech }\pi z, \ -\frac{1}{2} < Im\ z < \frac{1}{2}. \tag{1.2}$$

To deduce (1.2) from (1.1), we choose $t = e^{-\alpha y}, \alpha > 0$. Then the left hand side of (1.1) becomes

$$\int_0^{\infty} t^{s-1}\frac{t}{1+t^2}dt = \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha sy}\text{sech }\alpha y\ dy.$$

Now let $\alpha s = 2\pi iz$, we obtain

$$\int_{-\infty}^{\infty} e^{-2\pi izt}\text{sech }\alpha t dt = \frac{\pi}{\alpha}\text{sech }\frac{\pi^2 z}{\alpha}, \tag{1.3}$$

and Ramanujan's identity follows from (1.3) by choosing $\alpha = \pi$.

We recall that the Fourier transform of the function f is defined by

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-2\pi ixt} f(t)dt.$$

The most well-known example of a function which is invariant under the Fourier transform is, perhaps, the normal distribution:

$$\int_{-\infty}^{\infty} e^{-2\pi ixt}e^{-\pi t^2}dt = e^{-\pi x^2}.$$

Making obvious change of variable, the above integral becomes

$$\int_{-\infty}^{\infty} e^{-2\pi ixt}e^{-\alpha t^2}dt = \sqrt{\frac{\pi}{\alpha}}e^{-\pi^2 x^2/\alpha}, \ \alpha > 0. \tag{1.4}$$

So, the identity (1.2) provides yet another interesting example of a function which has its Fourier transform identical to itself. Before we proceed further, let us first see the significance as well as the connection between the identities (1.3) and (1.4). This is achieved by applying the Poisson summation formula [4, p. 111]

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

to (1.3) and (1.4). We obtain

$$\sum_{n=-\infty}^{\infty} e^{-\alpha n^2} = \sqrt{\frac{\pi}{\alpha}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/\alpha} \tag{1.5}$$

and

$$\sum_{n=-\infty}^{\infty} \text{sech } \alpha n = \frac{\pi}{\alpha} \sum_{n=-\infty}^{\infty} \text{sech } \frac{\pi^2 n}{\alpha}. \tag{1.6}$$

Since sech $t = \frac{2}{e^{-t}+e^{t}}$, we can rewrite (1.6) as

$$\sum_{n=-\infty}^{\infty} \frac{e^{-n\alpha}}{1+e^{-2n\alpha}} = \frac{\pi}{\alpha} \sum_{n=-\infty}^{\infty} \frac{e^{-n\pi^2/\alpha}}{1+e^{-2n\pi^2/\alpha}}.$$

Although for simplicity, we choose $\alpha > 0$ in our derivation of (1.3), it is easy to see that by analytic continuation, all the above identities are valid for $Re\ \alpha > 0$. To understand the link between (1.5) and (1.6), we recall the definition of the theta function [6, chapter 11]

$$\theta_3(q) = \vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, q = e^{\pi i \tau} \textit{ and } Im\ \tau > 0,$$

and a well-known identity due to Jacobi

$$\vartheta_3^2(\tau) = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} = 2\sum_{n=-\infty}^{\infty} \frac{q^n}{1+q^{2n}} \tag{1.7}$$

Now by choosing $\alpha = -\pi i \tau$ with $Im\tau > 0$, we see that (1.5) is precisely the Jacobi's imaginary transformation [6, p. 474]

$$\vartheta_3(-\frac{1}{\tau}) = \sqrt{-i\tau}\vartheta_3(\tau); \tag{1.8}$$

whereas (1.6) is an analog of (1.8) for $\vartheta_3^2(\tau)$.

In view of the above examples, it is natural to inquire whether there are other functions which are invariant under Fourier transform. There are, indeed, many and the particularly important ones are these related to the Hermite functions (see [4, p. 98]). The aim of this paper is to present another class of such functions which is a natural generalization of (1.2) and more significantly brings out the connection between the quadratic fields and the theta functions.

We now outline the organization of this paper as follows. In section 2, we evaluate, using the residue theorem, the integral

$$\int_0^\infty t^{s-1}\phi(t)dt \tag{1.9}$$

where $\phi(t) = \phi_N(t) = \sum_{n=1}^{N-1} \chi(n)t^n/1 - t^N$, and χ is a nonunit primitive character modulo N. In particular, the identity (1.1) is derived from Theorem 3 by choosing $N = 4$. And for $N = 3$,

$$\int_{-\infty}^{\infty} e^{-2\pi izt}\frac{1}{1+2\cosh\frac{2\pi t}{\sqrt{3}}}dt = \frac{1}{1+2\cosh\frac{2\pi z}{\sqrt{3}}}.$$

In section 3, we further restrict the character χ to be quadratic, it thus becomes the character of an imaginary quadratic field, and the crucial quantities such as the class number h and the number w of the roots of 1 of a given quadratic field enter into the identity naturally. In section 4, the machinery of the algebraic number theory will be applied to the quadratic field to establish the role of the theta functions in various identities. In particular, the identity (1.7) is rederived as a special case for $N = 4$.

We assume the reader is reasonably familiar with the general knowledge of the analytic and algebraic number theory.

2. Evaluation of the integral (1.9). Let N be a positive integer. A function χ is called a character modulo N if, for all integers n and m,

(1) $\chi(1) = 1$.

(2) $\chi(n) = \chi(n+N)$.

(3) $\chi(mn) = \chi(m)\chi(n)$.

(4) $\chi(n) = 0$ if (n, N), the gcd of n and N, is > 1.

A character χ_0 is called the unit character modulo N if $\chi_0(n) = 1$ if $(n, N) = 1$.

Let $\zeta = \zeta_N = e^{2\pi i/N}$. The Gaussian sum $\tau_a(\chi)$ corresponding to the character χ and the integer a is defined as

$$\tau_a(\chi) = \sum_{n=1}^{N-1} \chi(n)\zeta^{na}$$

And we denote $\tau(\chi) = \tau_1(\chi)$.

Let N' be a positive integer which is divisible by N. For any character χ modulo N, we can form a character χ' modulo N' as follows

$$\chi'(a) = \begin{cases} \chi(a) & \text{if } (a, N') = 1 \\ 0 & \text{if } (a, N') > 1. \end{cases}$$

We say that χ' is induced by the character χ. Let χ be a character modulo N. If there is a proper divisor d of N and a character χ_1 modulo d which induces χ, then the character χ is called nonprimitive, otherwise it is called primitive.

We need a lemma [2, p. 334].

Lemma 1. *Let χ be a primitive character modulo N. Then for every integer a*

$$\tau_a(\chi) = \overline{\chi(a)}\tau(\chi).$$

We now estabish a generalization of (1.1)

Theorem 1. *Let χ be a non-unit primitive character modulo N and let*

$$\phi(t) = \phi_N(t) = \sum_{n=1}^{N-1} \chi(n)t^n \Big/ 1 - t^N.$$

Then

$$\int_0^\infty t^{s-1}\phi(t)dt = \frac{-2\pi i\tau(\chi)}{N} \sum_{n=1}^{N-1} \overline{\chi(n)}(\zeta^s)^n \Big/ 1 - (\zeta^s)^N \tag{2.1}$$

where $\zeta^s = e^{2\pi is/N}$, $o <$ Re $s < 1$.

Proof. We remark that since $\sum_{n=1}^{N-1} \chi(n) = 0$ for any nonunit character whether it is primitive or not, the rational function $\phi(t)$ is analytic at $t = 1$.

The poles of $\phi(t)$ are all simple and located at the points ζ^k, $k = 1, 2, \ldots, N-1$, where $\zeta = e^{2\pi i/N}$. The residue of $\phi(t)t^{s-1}$ at ζ^k is

$$\begin{aligned}\lim_{t\to\zeta^k}(t-\zeta^k)\phi(t)t^{s-1} &= -\frac{1}{N}\zeta^{ks}\sum_{n=1}^{n-1}\chi(n)\zeta^{nk}\\ &= -\frac{1}{N}\zeta^{ks}\tau_k(\chi)\\ &= -\frac{1}{N}\zeta^{ks}\tau(\chi)\overline{\chi(k)} \quad \text{(from Lemma 1).}\end{aligned}$$

We now consider the integral

$$\int_C z^{x-1}\phi(z)dz,$$

where C is the familiar key hole contour which consists of the circles C_1 : $|z| = \rho$ and C_2 : $|z| = R$, $0 < \rho < R$, and the line segment L : $[\rho, R]$. The path of integration is described as follows: Starting at the point $z = \rho$, one first traverses along C_1 once in the clockwise direction, then follows along L to $z = R$ and traverses C_2 counterclockwise once and return to the initial point $z = \rho$ along L.

The theorem follows easily from the residue theorem by letting $R \to \infty$ and $\rho \to 0$.

Lemma 2. *Let ϕ_N be as in Theorem 1. Then*
(a) $\phi_N(1) = \frac{1}{N}\sum_{n=1}^{N-1} n\chi(n)$ and $\phi_N(1) = 0$ if $\chi(-1) = 1$,
(b) $\sum_{k=1}^{\infty}\phi_N(e^{-\alpha k}) = \sum_{n=1}^{\infty}\frac{\chi(n)e^{-\alpha n}}{1-e^{-\alpha n}}$,
(c) $\sum_{k=1}^{\infty}\phi_N(e^{\alpha k}) = \begin{cases} -\sum_{n=1}^{\infty}\frac{\chi(n)^{-\alpha n}}{1-e^{-\alpha n}} & \text{if } \chi(-1) = 1\\ \sum_{n=1}^{\infty}\frac{\chi(n)e^{-\alpha n}}{1-e^{-\alpha n}} & \text{if } \chi(-1) = -1,\end{cases}$
where Re $\alpha > 0$. Here the character χ needs not be primitive.

Proof. (a) The first part follows immediately from L'Hôptial's rule. To establish the second assertion, we note that if $\chi(-1) = 1$, then

$$\begin{aligned}\phi_N(1) &= -\frac{1}{N}\sum_{n=1}^{N-1} n\chi(n) = \frac{1}{N}\sum_{n=1}^{N-1}(-n)\chi(-n) = \frac{1}{N}\sum_{n=1}^{N-1} n\chi(n)\\ &= -\phi_N(1).\end{aligned}$$

Clearly this implies that $\phi_N(1) = 0$.

(b) Since $|e^{-\alpha k}| < 1$,

$$\begin{aligned}\phi_N(e^{-\alpha k}) &= \sum_{n=1}^{N-1} \chi(n)e^{-\alpha nk} \sum_{m=0}^{\infty} e^{-\alpha Nm} \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{N-1} \chi(mN+n)e^{-\alpha k(mN+n)} \quad \text{(from (2))} \\ &= \sum_{n=1}^{\infty} \chi(n)e^{-\alpha kn} = S\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=1}^{\infty} \phi(e^{-\alpha k}) &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \chi(n)e^{-\alpha kn} \\ &= \sum_{n=1}^{\infty} \frac{\chi(n)e^{-\alpha n}}{1-e^{-\alpha n}}.\end{aligned}$$

(c) The conclusion follows easily from the fact:

$$\begin{aligned}\phi(e^{\alpha k}) &= \sum_{1}^{N-1} \chi(n)e^{\alpha nk} \Big/ 1 - e^{\alpha Nk} \\ &= -\sum_{1}^{N-1} \chi(n)e^{-\alpha(N-n)k} \Big/ 1 - e^{-\alpha Nk} = \begin{cases} -S & \text{if } \chi(-1) = 1 \\ S & \text{if } \chi(-1) = -1. \end{cases}\end{aligned}$$

From the above lemma, we have

(2.2)

$$\sum_{k=-\infty}^{\infty} \phi(e^{-\alpha k}) = \begin{cases} -\frac{1}{N}\sum_{n=1}^{N-1} n\chi(n) + 2\sum_{n=1}^{\infty} \frac{\chi(n)e^{-\alpha n}}{1-e^{-\alpha n}} & \text{if } \chi(-1) = -1 \\ 0 & \text{if } \chi(-1) = 1. \end{cases}$$

We point out that the identity (2.2) holds for any character modulo N whether χ is primitive or not.

From (2.1), (2.2) and the Poisson summation formula, we obtain

Theorem 2. *Let χ be a primitive character modulo N with $\chi(-1) = -1$. Then*

$$
\begin{aligned}
&-\frac{1}{N}\sum_{n=1}^{N-1} n\chi(n) + 2\sum_{n=1}^{\infty}\frac{\chi(n)e^{n\pi i r}}{1-e^{n\pi i r}} \\
(2.3)\qquad &= \frac{2}{\tau}\frac{\tau(\chi)}{N}\left\{-\frac{1}{N}\sum_{n=1}^{N-1}\overline{n\chi(n)} + 2\sum_{n=1}^{\infty}\frac{\overline{\chi(n)}e^{n\pi i(-4/N\tau)}}{1-e^{n\pi i(-4/N\tau)}}\right\},
\end{aligned}
$$

where $Im\ \tau > 0$.

It is worthwhile to point out that except when χ is quadratic (i.e. $\chi^2 = \chi_0$), the evaluation of the Gaussian sum $\tau(\chi)$ for a general character is a very difficult problem. We refer the readers to an article by Berndt and Evans [1] for an interesting survey and the historical account of this subject. In the next section, we will examine the significance of (2.3) for the quadratic χ.

3. Quadratic characters. We first list some facts about the quadratic characters.

1. Primitive quadratic characters occur only for the moduli of the form $r, 4r$ and $8r$, where r is an odd square-free natural number.
2. Every primitive quadratic character is the character of a quadratic field. Conversely, the character of a quadratic field of discriminant D is a primitive quadratic character modulo $|D|$.
3. The characters of real quadratic fields are even, i.e. $\chi(-1) = 1$, and the characters of imaginary quadratic fields are odd, i.e. $\chi(-1) = -1$.
4. Let χ be a primitive quadratic character modulo m. Then
$$\tau(\chi) = \begin{cases} \sqrt{m} & \text{if } \chi(-1) = 1 \\ i\sqrt{m} & \text{if } \chi(-1) = -1. \end{cases}$$
5. Let χ be an odd primitive quadratic character of an imaginary quadratic field of discriminant $-D < 0$. Then the class number h of the quadratic field is
$$h = -\frac{w}{2D}\sum_{k=1}^{D-1} k\chi(k),$$
where w is the number of roots of 1 contained in the quadratic field and

$$w = \begin{cases} 6 & \text{if } D = 3 \\ 4 & \text{if } D = 4 \\ 2 & \text{for all other } D. \end{cases}$$

6. The quadratic character of the quadratic field of discriminant D coincides with the Kronecker symbol $(\frac{D}{n})$.

All the above facts can be found in [2, p. 343 and p. 347-349]. Using the properties 4 and 5, we can restate the identity (2.1) and (2.3) as

Theorem 3. *Let χ be the character of the imaginary quadratic field of discriminant $-D$. Then*

$$\int_{-\infty}^{\infty} e^{-2\pi ixt}\phi_D(e^{-\alpha t})dt = \frac{2\pi}{\alpha\sqrt{D}}\phi_D(e^{-4\pi^2 x/\alpha D})$$

and

$$f_D(\tau) = -\frac{2}{i\tau\sqrt{D}} f_D(-\frac{4}{D\tau}), \tag{3.1}$$

where

$$f_D(\tau) = h + w\sum_{n=1}^{\infty}\left(\frac{-D}{n}\right)\frac{e^{n\pi i\tau}}{1-e^{n\pi i\tau}}.$$

Hence, by choosing $\alpha = \frac{2\pi}{\sqrt{D}}$, we see that $\phi_D(e^{-2\pi t/\sqrt{D}})$ is invariant under the Fourier transform.

4. Representation of f_D in terms of the theta functions. Let $\mathbb{Q}(\sqrt{-D})$ be the imaginary quadratic field of the discriminant $-D < 0$. The Dedekind zeta function associated with this quadratic field is defined as

$$\zeta(s,-D) = \sum_{a\neq 0}\frac{1}{N(a)^s},$$

where $N(a)$ is the norm of the ideal a of $\mathbb{Q}(\sqrt{-D})$. It is well-known that $\zeta(s,-D)$ can be written as the product of the Riemann zeta function and a Dirichlet L-series with character $\chi(n) = (\frac{-D}{n})$ (See [2, p. 343]). That is for $Re\ s > 1$

$$\zeta(s,-D) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(\sum_{n=1}^{\infty}\left(\frac{-D}{n}\right)\frac{1}{n^s}\right)$$
$$= \sum_{n=1}^{\infty}\left(\sum_{k|n}\left(\frac{-D}{k}\right)\right)\frac{1}{n^s}.$$

Thus,

$$\sum_{a\neq 0}\frac{1}{N(a)^s} = \sum_{n=1}^{\infty}\sum_{k|n}\left(\frac{-D}{k}\right)\Big/ n^s. \tag{4.1}$$

We recall that the Mellin transform of f is defined as

$$(Mf)(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} f(t)t^{s-1}dt.$$

In particular, for $f = e^{-\alpha t}, \alpha > 0$,

$$(Mf)(s) = \frac{1}{\alpha^s},$$

therefore

$$M^{-1}\left\{\frac{1}{\alpha^s}\right\} = e^{-\alpha t}.$$

Applying the inverse Mellin transform to (4.1), we obtain

$$\begin{aligned}\sum_{a\neq 0} e^{-tN(a)} &= \sum_{n=1}^{\infty}\left(\sum_{k|n}\left(\frac{-D}{k}\right)\right)e^{-nt} = \sum_{k,n=1}^{\infty}\left(\frac{-D}{k}\right)e^{-nkt}\\ &= \sum_{n=1}^{\infty}\left(\frac{-D}{n}\right)\frac{e^{-nt}}{1-e^{-nt}}.\end{aligned} \tag{4.2}$$

Now choosing $-t = \pi i\tau$, $Im\ \tau > 0$ and $q = e^{-t} = e^{\pi i \tau}$, the identity (4.2) becomes

$$\sum_{n=1}^{\infty}\left(\frac{-D}{n}\right)\frac{q^n}{1-q^n} = \sum_{a\neq 0} q^{N(a)} \tag{4.3}$$

To see the significance of (4.3), we take $D = 3$. Since $\mathbb{Q}(\sqrt{-3})$ has the unique factorization property, all ideals are principal and each has the form $(n + m\zeta)$, where $\zeta = \frac{1+\sqrt{-3}}{2}$ and $n, m \in \mathbb{Z}$. Since there are 6 units in $Q(\sqrt{-3}) : \pm 1, \pm(\frac{-1\pm\sqrt{-3}}{2})$, there is a six to one correspondence between the sets of algebraic integers and the ideals in $\mathbb{Q}(\sqrt{-3})$. That is, the algebraic integer $n + m\zeta$ and $e(n + m\zeta)$, e a unit, all correspond to the same ideal $(n + m\zeta)$. The norm of the ideal is defined as

$$N(n + m\zeta) = (n + m\zeta)(n + m\bar{\zeta}) = n^2 + mn + m^2.$$

Therefore (4.3) yields

$$\sum_{n=1}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1 - q^n} = \frac{1}{6} \sum_{(n,m)\neq(0,0)} q^{n^2+mn+m^2}. \tag{4.4}$$

The factor $\frac{1}{6}$ is to take into account of the above mentioned six to one correspondence between the sets of ideals and the algebraic integers. Since all the ideals are principal, the class number h of $\mathbb{Q}(\sqrt{-3})$ is equal to one, and (4.4) can be written as

$$1 + 6\sum_{n=1}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1 - q^n} = \sum_{n,m=-\infty}^{\infty} q^{n^2+mn+m^2}$$

The left hand side of (4.5) is precisely the special case of f_D in (3.1) for $D = 3$.

It is well-known that there are exactly 9 imaginary quadratic fields with class number $h = 1$, the discriminants of these quadratic fields are

$$-D = -3, -4, -7, -8, -11, -19, -43, -67 \text{ and } -163.$$

Except for $-D = -4$ and -8, all the remaining discriminants satisfy

$$-D \equiv 1 (\text{mod } 4).$$

The norms for $-D = -4$ and -8 are, respectively,

$$N(m + ni) = m^2 + n^2 \text{ and } N(m + i\sqrt{2}n) = m^2 + 2n^2,$$

and for the remaining cases the norms of the quadratic integers (and the ideals) have the form

$$N(m + n\zeta_D) = m^2 + mn + \left(\frac{D+1}{4}\right)n^2$$

where $\zeta_D = \frac{1+\sqrt{-D}}{2}$.

To express the right hand side of (4.3) in terms of the theta functions, we need a

Lemma 4.

$$\sum_{n,m=-\infty}^{\infty} q^{m^2+mn+ln^2} = \vartheta_2(q)\vartheta_2(q^D) + \vartheta_3(q)\vartheta_3(q^D),$$

where $l = (D+1)/4$.

Proof. We recall first that

$$\vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \text{ and } \vartheta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}.$$

Now, we note that

$$m^2+mn+ln^2 = \begin{cases} (m+k)^2 + Dk^2 & \text{if } n = 2k \\ (m+k+\frac{1}{2})^2 + D(k+\frac{1}{2})^2 & \text{if } n = 2k+1, \end{cases}$$

hence

$$\begin{aligned} \sum_{n,m=-\infty}^{\infty} q^{m^2+mn+ln^2} &= \sum_{m=-\infty}^{\infty} \sum_{n,\text{even}} + \sum_{m=-\infty}^{\infty} \sum_{n,\text{odd}} \\ &= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{-\infty} q^{(m+k)^2} q^{Dk^2} \\ &\quad + \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} q^{(m+k+\frac{1}{2})^2} q^{D(k+\frac{1}{2})^2} \\ &= \vartheta_3(q)\vartheta_3(q^D) + \vartheta_2(q)\vartheta_2(q^D). \end{aligned}$$

We have, in addition to (4.5),

$$1 + 4\sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) \frac{q^n}{1-q^n} = \vartheta_3^2(q) \tag{4.6}$$

$$1 + 2\sum_{n=1}^{\infty} \left(\frac{-8}{n}\right) \frac{q^n}{1-q^n} = \vartheta_3(q)\vartheta_3(q^2) \tag{4.7}$$

and

$$1+2\sum_{n=1}^{\infty}\left(\frac{-D}{n}\right)\frac{q^n}{1-q^n}=\vartheta_2(q)\vartheta_2(q^D)+\vartheta_3(q)\vartheta_3(q^D) \tag{4.8}$$

for $D=7,11,19,43,67$ and 163.

The identity (4.6) is precisely the identity (1.7) which we encountered earlier.

In general, we have

Theorem 4. *Let $-D$ be the discriminant of an imaginary quadratic field. Then*

$$\begin{aligned} f_D(q) &= h+w\sum_{n=1}^{\infty}\left(\frac{-D}{n}\right)\frac{q^n}{1-q^n} \\ &= h+w\sum_{a\neq 0,\ \text{ideal}} q^{N(a)} \\ &= \sum_{i=1}^{h}\sum_{m,n=-\infty}^{\infty} q^{Q_i(m,n)}, \end{aligned} \tag{4.9}$$

where $Q_i(m,n), i=1,2,\ldots,h$, are the inequivalent quadratic forms of the given quadratic field.

A very readable account of the correspondence between the ideals and the inequivalent quadratic forms of a given quadratic field can be found in Chapter 12 of [3]. To express the last sum of the above identity in terms of the theta functions, we need to know the explicit inequivalent forms of the quadratic forms of the quadratic field $\mathbb{Q}(\sqrt{-D})$. For example, if $-D=-20$, the class number $h=2$, so there are two non-equivalent quadratic forms with the discriminant $-20: n^2+5m^2$ and $2n^2+2mn+3m^2$. And (4.9) yields

$$\begin{aligned} 2+2\sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n} &= \vartheta_3(q)\vartheta_3(q^5)+\vartheta_2(q^2)\vartheta_2(q^{10})+\vartheta_3(q^2)\vartheta_3(q^{10}) \\ &= \sum_{n,m=-\infty}^{\infty}\left(q^{n^2+5m^2}+q^{2n^2+2mn+3m^2}\right). \end{aligned}$$

5. Remarks on the real quadratic fields. The identity (4.1) (and, hence, (4.3)) is also valid for any real quadratic fields (See [2, p. 343]). However, it does not seem possible to express the sum $\sum q^{N(a)}$ in terms of the theta functions, the complication is caused mainly from the fact that every real quadratic field has infinitely many units. To illustrate this difference between the real and imaginary quadratic fields, we consider $\mathbb{Q}(\sqrt{5})$. It has class number $h = 1$ and the quadratic integers are of the form

$$\alpha = m + n\eta$$

with norm $N(\alpha) = \alpha\alpha' = m^2 + mn - n^2$, where $\alpha' = m + n\eta'$, $\eta = \frac{1+\sqrt{5}}{2}$ and $\eta' = \frac{1-\sqrt{5}}{2}$. The set of units is the multiplicative group $\{\eta^k : k \in \mathbb{Z}\}$. Therefore every integer α has infinitely many associates $\eta^k\alpha, k \in \mathbb{Z}$, and all these associates correspond to the same ideal $a = (\alpha)$ (generated by α). To establish a one-to-one correspondence between the ideals and a representative of $\eta^k\alpha$, the algebraic integers are represented geometrically in $\mathbb{R}^2$; then there exists a fundamental domain X in $\mathbb{R}^2$ with the property that for every quadratic integer α, there is a unique element among the associates $\{\eta^k\alpha : k \in \mathbb{Z}\}$ belonging to X (for details, see [2, p. 313-316]).

The identity (4.3) takes the form

$$\sum_{a \neq 0} q^{N(a)} = \sum_{n=1}^{\infty} \left(\frac{5}{n}\right) \frac{q^n}{1-q^n}, \tag{5.1}$$

where $N(a) = |N(\alpha)| = |m^2 + mn - n^2|$. To write out the sum on the left hand side of (5.1), we need to characterize all the algebraic integers belonging to the fundamental domain X: Let $\alpha = m + \eta n \in X$. Then $\alpha \in X$ if and only if $\alpha > 0$ and

$$1 \leq \left|\frac{\alpha}{\alpha'}\right| < \eta^2.$$

We need a

Lemma 5. *$\alpha \in X$ if and only if either $m > 2n \geq 0$ or $n + 2m \geq 0$ and $m < 0$.*

Proof. To simplify the computation, we note that $\eta\eta' = -1$ and $\eta + \eta' = 1$. Consider the case $\alpha' < 0$. Then $\alpha \in X$ if and only if $\alpha + \alpha' \geq 0$, $\alpha + \eta^2\alpha' < 0$ and $\alpha > 0$. Now observe that

$$0 \leq \alpha + \alpha' = m + n\eta + m + n\eta' = 2m + n(\eta' + \eta) = 2m + n$$

and

$$0 > \alpha + \eta^2 \alpha' = m + n\eta + \eta^2(m + n\eta') = m(1 + \eta^2).$$

Hence, we have $m < 0$ and $2m + n \geq 0$. Clearly, under these condition,

$$\alpha = m + n\eta \geq m + (-2m)\eta = (-m)\eta + (-m)(\eta - 1) > 0.$$

Thus, for $\alpha' < 0$, $\alpha \in X$ if and only if $m < 0$ and $2m + n \geq 0$.

The case $\alpha' > 0$ yields $m > 2n \geq 0$ and we leave the simple detail to the reader.

From the lemma, we write

$$\sum_{a \neq 0} q^{N(a)} = \sum_{m > 2n \geq 0} + \sum_{\substack{n+2m \geq 0 \\ m < 0}} q^{|m^2 + mn - n^2|}. \tag{5.2}$$

And

$$\begin{aligned} \sum_{m > 2n \geq 0} q^{|m^2+mn-n^2|} &= \sum_{\substack{n=0 \\ m>0}} + \sum_{\substack{n \geq 1 \\ m = 2n+k, k \geq 1}} \\ &= \sum_{m=1}^{\infty} q^{m^2} + \sum_{\substack{n=1 \\ k=1}}^{\infty} q^{5n^2 + 5nk + k^2} \end{aligned} \tag{5.3}$$

Similarly,

$$\begin{aligned} \sum_{\substack{n+2m \geq 0 \\ m<0}} q^{|m^2+mn-n^2|} &= \sum_{k=1}^{\infty} \sum_{\substack{n=2k+l \\ l \geq 0}} q^{|k^2 - kn - n^2|} \text{ (by letting } m = -k) \\ &= \sum_{k=1}^{\infty} q^{5k^2} + \sum_{\substack{k=1 \\ l=1}}^{\infty} q^{5K^2 + 5kl + l^2} \end{aligned} \tag{5.4}$$

Using the fact that $\sum_1^\infty q^{n^2} = \frac{1}{2}(\vartheta_3(q) - 1)$, we conclude from (5.1), (5.2), (5.3) and (5.4)

$$\sum_{n=1}^{\infty} \left(\frac{5}{n}\right) \frac{q^n}{1 - q^n} = -1 + \frac{1}{2}(\vartheta_3(q) + \vartheta_3(q^5)) + 2 \sum_{\substack{n=1 \\ m=1}}^{\infty} q^{5n^2 + 5mn + m^2}$$

It is precisely the constraint $m, n \geq 1$ on the last series (caused by the infinitude of the units in the real quadratic fields) which prevents us from expressing the above series in terms of simple combination of the theta functions. We add that the simplicity of the identity (3.1) is due to the fact that f_D is expressible as a finite sums of the theta functions; therefore any analog of (3.1) for the real quadratic fields is most likely very complicated!

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