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SECOND ORDER QUASILINEAR ELLIPTIC PROBLEMS IN A BALL*

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Abstract. A global $W^{2,p}$ estimate for the solution of the following quasilinear elliptic problem:

$$\begin{cases} \sum_{i,j=1}^{N} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\frac{\partial u}{\partial x_i} + c(x,u)u = f(x) & \text{in} \quad B, \\ u = 0 \quad \text{on} \quad \partial B, \end{cases}$$

where B is a ball in R^N, $N \geq 3, a_{ij} = a_{ij}(x,r) \in C^{0,1}(\overline{B} \times R), a_{ij}, \frac{\partial a_{ij}}{\partial x_i}, \frac{\partial a_{ij}}{\partial r}, b_i, c \in L^\infty(B \times R)$, with $i,j = 1,2,\ldots,N$ and $c \leq 0$, and $f \in L^p(B)$, is established. As a consequence, for each p, $p \geq N$, there exists a strong solution $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ provided the oscillations of a_{ij} with respect to r are sufficiently small. Moreover, the set of solutions $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ is $W^{2,p}$ bounded.

1. Introduction. Let Ω be an open set in R^N, $N \geq 3$. $W^{m,p}(\Omega) = \{u \in L^p(\Omega) | \text{weak derivatives } D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq m\}$, $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ and $W_{loc}^{m,p}(\Omega)$ is the space consisting of functions belonging to $W^{m,p}(\Omega')$ for all $\Omega' \subset\subset \Omega$. $H^m(\Omega) = W^{m,2}(\Omega), H_0^m(\Omega) = W_0^{m,2}(\Omega)$. ∇u denotes the gradient of u. $B_R(y)$ is the open ball in R^N of radius R centered at y. $B_R^+(y) = B_R(y) \cap R_+^N = \{x = (x_1,\ldots,x_N) \in B_R(y) | x_N > 0\}$.

We investigate the existence of strong solutions to the following quasi-

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linear elliptic problem in a $C^{1,1}$ domain $\Omega \subset R^N$, $N \geq 3$:

$$(1.1) \quad \begin{cases} Lu=\sum_{i,j=1}^N a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^N b_i(x,u)\frac{\partial u}{\partial x_i}+c(x,u)u=f(x) & \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{cases}$$

where $f \in L^p(\Omega)$.

Define the mapping F in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ by letting $u = F(v)$ be the unique solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to the linear elliptic problem

$$(1.2) \quad \begin{cases} L_v u=\sum_{i,j=1}^N a_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^N b_i(x,v)\frac{\partial u}{\partial x_i}+c(x,v)u=f(x) & \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega. \end{cases}$$

The unique solvability of problem (1.2) is guaranteed by the linear existence result [4, Theorem 9.15] under appropriate coefficients conditions. We notice here that F is well-defined for $p > \frac{N}{2}$. One then intends to find a fixed point of F. Observe that the well-known regularity theorem of Agmon-Douglis-Nirenberg [1] asserts that

$$(1.3) \qquad \|u\|_{W^{2,p}} \leq C(\|u\|_{L^p} + \|L_v u\|_{L^p})$$

where C is a constant depending on the moduli of continuity of the coefficients $a_{ij}(x, v(x))$ on $\overline{\Omega}$, etc.. If $a_{ij}(x, v) = a_{ij}(x)$, then the constant C in (1.3) is independent of v and furthermore there exists a constant C independent of v such that

$$\|u\|_{W^{2,p}} \leq C\|L_v u\|_{L^p}.$$

Applying the Schauder fixed point theorem, one can readily obtain a solution. However, for the case that a_{ij} depends both on x and v, the constant C in (1.3) varies with v.

Our main idea is to make the constant in (1.3) be independent of v. To achieve this purpose, we refrain ourselves to concentrate on problem (1.1) in a ball B of R^N. Under stronger coefficients conditions on $a_{ij} = a_{ij}(x, r) \in C^{0,1}(\overline{B} \times R)$ and sufficiently small oscillations with respect to r, which together with the maximum principle [2]

$$\sup_{\Omega} |u| \leq C\|f\|_{L^N(\Omega)}$$

leads directly to the existence of problem (1.1) (Corollary 2.5) provided $p \geq N$. Moreover, solutions in some other specific domains are also concerned in this paper.

2. $W^{2,p}$ estimates and existence results. Recall that an operator L in (1.1) is said to be elliptic in Ω if there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,r)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for} \quad (r,\xi) \in R \times R^N \quad \text{and a.e.} \quad x \in \Omega. \tag{2.1}$$

For a fixed point $x \in R^N$, we denote osc $a_{ij}(x,r)$ the oscillation of a_{ij} with respect to r in R, that is, osc $a_{ij}(x,r)$=sup$\{a_{ij}(x,r_1)-a_{ij}(x,r_2)|r_1,r_2 \in R\}$ and let

$$\text{osc } a(x,r) = \max_{1\leq i,j\leq N} \text{osc } a_{ij}(x,r).$$

We start this section by observing an interior $W^{2,p}$ estimate in an open set $\Omega \subset R^N$ for the strong solution $u \in W^{2,p}_{\text{loc}}(\Omega) \cap L^p(\Omega)$ of the following quasilinear elliptic equation:

$$Lu = f(x) \quad \text{in} \quad \Omega \tag{2.2}$$

which will then be applied to derive a global $W^{2,p}$ estimate for the strong solutions $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ of equation (2.2) in a ball $B \subset R^N$ in Theorem 2.2.

Notice that the interior $W^{2,p}$ estimate for the linear case formulated in Theorem 9.11 [4,p.235] is derived by a uniform perturbation of the coefficients $a_{ij}(x)$ in the nneighbourhoods of finite points in Ω. In the present case that $a_{ij} = a_{ij}(x,u)$, an interior $W^{2,p}$ estimate can be established in the same line provided the oscillations of a_{ij} with respect to r are sufficiently small. Therefore, we have the following lemma in which K is a constant depending only on N,p, and satisfying

$$\|D^2w\|_{L^p(\Omega)} \leq K\|\Delta w\|_{L^p(\Omega)}, \tag{2.3}$$

where $w \in W_0^{2,p}(\Omega)$.

Lemma 2.1. *Let Ω be an open set in R^N and the coefficients of L satisfy, for a positive constant Λ,*

$$(2.4) \qquad a_{ij} \in C^{0,1}(\Omega \times R),\ b_i,\ c \in L^\infty(\Omega \times R),\ f \in L^p(\Omega);$$
$$|a_{ij}|,\ |b_i|,\ |c| \le \Lambda,$$

where $i, j = 1, \ldots, N$. Suppose that

$$(2.5) \qquad \operatorname{osc} a(x,r) \le \frac{\lambda}{4K} \quad \forall x \in \Omega,$$

where K is given by (2.3). Then if $u \in W^{2,p}_{\text{loc}}(\Omega) \cap L^p(\Omega)$, $1 < p < \infty$, is a strong solution of equation (2.1) we have for any domain $\Omega' \subset\subset \Omega$ the estimate

$$(2.6) \qquad \|u\|_{W^{2,p}(\Omega')} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}),$$

where C is a constant depending on $N, p, \lambda, \Lambda, \Omega', \Omega$ and the moduli of continuity of the coefficients $a_{ij}(x,r)$ with respect to x on Ω'.

To simply the boundary estimate, we refrain Ω to be a ball in R^N. Thus, we can further derive a local boundary estimate which together with Lemma 2.1 enables us to establish the following global estimate.

Theorem 2.2. *Let B be a ball in R^N and the operator L satisfy (2.4) with $a_{ij}(x,r) \in C^{0,1}(\overline{B} \times R)$. Suppose that*

$$(2.7) \qquad \operatorname{osc} a(x,r) \le \frac{\lambda}{4K} \quad \forall x \in B,$$

$$(2.8) \qquad \operatorname{osc} a(x,r) < \frac{\lambda}{8N^2K} \quad \forall x \in \partial B,$$

where K is given by (2.3). Then if $u \in W^{2,p}(B) \cap W^{1,p}_0(B), 1 < p < \infty$, is a strong solution of equation (2.1) in B we have the estimate

$$(2.9) \qquad \|u\|_{W^{2,p}(B)} \le C(\|u\|_{L^p(B)} + \|f\|_{L^p(B)}),$$

where C is a constant depending on $N, p, \lambda, \Lambda, \partial B, B$ and the moduli of continuity of the coefficients $a_{ij}(x,r)$ with respect to x on $\overline{B}$.

Proof. For simplicity, let B be the unit ball $B_1(0)$ with its boundary $\mathcal{S}$

$$\mathcal{S} = \partial B = \left\{ x = (x_1, \dots, x_N) \in R^N | \sum_{i=1}^{N} x_i^2 = 1 \right\}.$$

For any $x^0 = (x_1^0, \dots, x_N^0) \in \mathcal{S}$ there exists an integer k, $1 \le k \le N$, such that $x_0 \in \mathcal{S}_k^+$ or $x_0 \in \mathcal{S}_k^-$, where

$$\mathcal{S}_k^+ = \{ x \in \mathcal{S} | \sum_{i \ne k} x_i^2 \le \frac{N-1}{N}, x_k > 0 \},$$

$$\mathcal{S}_k^- = \{ x \in \mathcal{S} | \sum_{i \ne k} x_i^2 \le \frac{N-1}{N}, x_k > 0 \},$$

Without loss of generality, we can assume $x_0 \in \mathcal{S}_N^+$. Write

$$\begin{aligned} x_0 = &(\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1}, \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1}, \\ &\cos\theta_2 \sin\theta_3 \cdots \sin\theta_{N-1}, \cos\theta_3 \sin\theta_4 \cdots \sin\theta_{N-1}, \\ &\cos\theta_4 \sin\theta_5 \cdots \sin\theta_{N-1}, \cdots, \cos\theta_{N-2} \sin\theta_{N-1}, \cos\theta_{N-1}) \end{aligned}$$

for some θ_i, $0 \le \theta_{N-1} \le \tan^{-1}\sqrt{N-1}, 0 \le \theta_i < 2\pi, i = 1, \cdots, N-2$, where θ_{N-1} is the angle from the positive x_N-axis to x_0. Rotate the coordinate axes, the rotated axes are denoted as the $x_1'-, \cdots, x_N'$-axis, by the mapping R_{x_0} defined by $x' = x\mathbf{O}_N$, where

$$\mathbf{O}_3 = \begin{bmatrix} \cos\theta_1 \cos\theta_2 & -\sin\theta_1 & \cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \cos\theta_2 & \cos\theta_1 & \sin\theta_1 \sin\theta_2 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix},$$

$$\mathbf{O}_k = \begin{bmatrix} & & 0 \\ \mathbf{O}_{k-1} & & \vdots \\ & & 0 \\ 0 \cdots 0 & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_{k-2} & 0 & 0 \\ 0 \cdots 0 & \cos\theta_{k-1} & \sin\theta_{k-1} \\ 0 \cdots 0 & -\sin\theta_{k-1} & \cos\theta_{k-1} \end{bmatrix}, \quad k = 4, \cdots, N,$$

here $\mathbf{I}_{k-2}$ is the $(k-2) \times (k-2)$ identity matrix, such that x_0 is converted into the point $(0, \cdots, 0, 1)$. Define a mapping $\psi = \psi_{x^0} = \psi_{(0,\dots,0,1)} \circ R_{x_0}$ in a neighbourhood $\mathcal{N} = \mathcal{N}_{x^0} = R_{x_0}^{-1}(\mathcal{N}_{(0,\dots,0,1)}) \subset R^N$, where

$$\psi_{(0,\dots,0,1)} = \frac{1}{r_0}(x_1', \dots, x_{N-1}', \sqrt{1 - \sum_{i \ne N} x_i'^2} - x_N'), \ 0 < r_0 \le \sqrt{\frac{N-1}{N}},$$

and

$$\mathcal{N}_{(0,\dots,0,1)} = \Big\{ x' \in R^N \,|\, \sum_{i\neq N} x_i'^2 < r_0^2, \sqrt{1-\sum_{i\neq N} x_i'^2} - \sqrt{r_0^2 - \sum_{i\neq N} x_i'^2} < x_N < \sqrt{1-\sum_{i\neq N} x_i'^2} + \sqrt{r_0^2 - \sum_{i\neq N} x_i'^2} \Big\}.$$

Then ψ is a diffeomorphism from $\mathcal{N}$ onto the unit ball $B_1(0)$ in R^N such that $\psi(\mathcal{N}\cap B) \subset R_+^N$, $\psi(\mathcal{N}\cap \partial B) \subset \partial R_+^N$, $\psi \in C^{1,1}(\mathcal{N})$, $\psi^{-1} \in C^{1,1}(B_1(0))$. Under the mapping $y = \psi(x) = (\psi_1(x),\dots,\psi_N(x))$, let $\tilde{u}(y) = u(x)$ and $\tilde{L}\tilde{u}(y) = Lu(x)$, where

$$\tilde{L}\tilde{u} = \sum_{i,j=1}^N \tilde{a}_{ij}(y,\tilde{u}(y)) \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} + \sum_{i=1}^N \tilde{b}_i(y,\tilde{u}(y)) \frac{\partial \tilde{u}}{\partial y_i} + \tilde{c}(y,\tilde{u}(y))\tilde{u}(y) = \tilde{f}(y) \text{ in } B_1^+(0)$$

and

$$\tilde{a}_{ij}(y,\tilde{u}(y)) = \sum_{r,s} \frac{\partial \psi_i}{\partial x_r}\frac{\partial \psi_j}{\partial x_s} a_{rs}(x,u(x)),$$
$$\tilde{b}_i(y,\tilde{u}(y)) = \sum_{r,s} \frac{\partial^2 \psi_i}{\partial x_r \partial x_s} a_{rs}(x,u(x)) + \sum_r \frac{\partial \psi_i}{\partial x_r} b_r(x,u(x)),$$
$$\tilde{c}(y,\tilde{u}(y)) = c(x,u(x)), \ \tilde{f}(y) = f(x),$$

so that $\tilde{L}$ satisfies conditions similar to (2.1) and (2.4) with constants $\tilde{\lambda}$, $\tilde{\Lambda}$ depending on λ, Λ and ψ. Furthermore, $\tilde{u} \in W^{2,p}(B_1^+(0))$, $\tilde{u} = 0$ on $B_1(0) \cap \partial R_+^N$ in the sense of $W^{1,p}(B_1^+(0))$.

Notice that $D\psi = D\psi_{(0,\dots,0,1)} DR_{x_0}$ and $\tilde{a} = (D\psi)a(D\psi)^T$, where

$$D\psi = \left[\frac{\partial \psi_i}{\partial x_j}\right], \ D\psi_{0,\dots,0,1)} = \left[\frac{\partial \psi_i}{\partial x_j'}\right], \ DR_{x_0} = \left[\frac{\partial x_i'}{\partial x_j}\right], \ \tilde{a} = [\tilde{a}_{ij}], \ i,j = 1,\dots,N.$$

We can obtain from a further computation of $\tilde{a}$ that

$$\text{osc } \tilde{a}(0,r) < \frac{N^2}{r_0^2} \cdot \text{osc } a(x_0,r). \tag{2.10}$$

Now we will choose $\tilde{\lambda} > 0$ properly. For all $\xi = (\xi_1,\dots,\xi_N) \in R^N$,

$$\sum_{i,j=1}^{N} \tilde{a}_{ij}\xi_i\xi_j = \xi\tilde{a}\xi^T = (\xi(D\psi))a(\xi(D\psi))^T \geq \lambda|\xi(D\psi)|^2$$

$$= \frac{\lambda}{r_0^2}\Big(\sum_{i\neq N}\xi_i^2 + (1+\sum_{i\neq N}X_i^2)\xi_N^2 - 2\sum_{i\neq N}\xi_i\xi_N X_i\Big)$$

$$\geq \frac{\lambda}{r_0^2}\Big((1-\epsilon)\sum_{i\neq N}\xi_i^2 + (1+(1-\frac{1}{\epsilon})\sum_{i\neq N}X_i^2)\xi_N^2\Big)$$

for any $\epsilon > 0$, where $X_i = \frac{x_i'}{\sqrt{1-\sum_{i\neq N} x_i'^2}}$, $i = 1,\ldots,N-1$. Choose $0<\epsilon<1$ such that $1+(1-\frac{1}{\epsilon})\sum_{i\neq N}X_i^2 > 1-\epsilon$, i.e., $\sum_{i\neq N}X_i^2 < \frac{\epsilon^2}{1-\epsilon}$ and so $\tilde{\lambda} = \frac{\lambda(1-\epsilon)}{r_0^2}$. Since $\sum_{i\neq N}X_i^2 < \frac{r_0^2}{1-r_0^2}$ in $\mathcal{N}_{(0,\ldots,0,1)}$, we can take $\frac{\epsilon^2}{1-\epsilon} = \frac{r_0^2}{1-r_0^2}$ to obtain

$$\tilde{\lambda} = \lambda\cdot\frac{2-r_0^2-\sqrt{4r_0^2-3r_0^4}}{2r_0^2(1-r_0^2)}. \tag{2.11}$$

With an observation in the proof of Theorem 9.13 [4, p.239], the oscillations of $\tilde{a}_{ij}(0,r)$ with respect to $r\in R$, corresponding to condition (2.4), must be less than $\frac{\tilde{\lambda}}{8K}$, that is,

$$\text{osc}\ \tilde{a}(0,r) \leq \frac{\tilde{\lambda}}{8K}. \tag{2.12}$$

In view of (2.10) and (2.11), then inequality (2.12) holds provided

$$\text{osc}\ a(x_0,r) \leq \frac{\lambda}{16N^2K}\cdot\frac{2-r_0^2-\sqrt{4r_0^2-3r_0^4}}{1-r_0^2}. \tag{2.13}$$

Since the right hand side of (2.13) increases to $\frac{\lambda}{8N^2K}$ as $r_0 \longrightarrow 0$, there exists r_0 small enough such that, under hypothesis (2.8), inequality (2.13) holds. Thus, in the same deduction as in the proof of Lemma 2.1, we obtain, on returning to our original coordinates, a local boundary estimate in a neighbourhood, say $\widetilde{\mathcal{N}}$. For an arbitrary ball B in R^N, by means a linear transformation from B onto the unit ball and following the arguments as above, we can also arrive at such estimate. Finally, by covering ∂B with a finite number of such neighbourhoods $\widetilde{\mathcal{N}}$ and using also the interior estimate (2.6), the desired estimate (2.9) follows immediately.

Corollary 2.3. *Under the hypotheses of Theorem 2.2 with B replaced by the ellipsoid*

$$E = \{x = (x_1, \ldots, x_N) \in R^N | \sum_{i=1}^{N} (\frac{x_i - c_i}{r_i})^2 < 1\}.$$

and with (2.8) replaced by

$$\text{osc } a(x,r) < \frac{\min r_i}{\max r_i} \cdot \frac{\lambda}{8N^2 K} \quad \forall x \in \partial E \tag{2.14}$$

then the same conclusion (2.9) remains valid.

Proof. Let $T : R^N \longrightarrow R^N$ be given by

$$T(x) = (\frac{x_1 - c_1}{r_1}, \ldots, \frac{x_N - c_N}{r_N}).$$

Then T is a diffeomorphism from E onto the unit ball $B_1(0)$ in R^N. For any $x^0 = (x_1^0, \ldots, x_N^0) \in \partial E$ there exists an integer $k, 1 \leq k \leq N$, such that $x_0 \in \Gamma_k^+$ or $x_0 \in \Gamma_k^-$, where $\Gamma_k^+ = T^{-1}(\mathcal{S}_k^+)$, $\Gamma_k^- = T^{-1}(\mathcal{S}_k^-)$. Thus, there is a neighbourhood $\mathcal{U} = \mathcal{U}_{x_0} = T^{-1}(\mathcal{N}_{T(x_0)})$ and a diffeomorphism $\phi = \phi_{x_0} = \psi_{T(x_0)} \circ T$ from $\mathcal{U}$ onto the unit ball $B_1(0)$ in R^N such that $\phi(\mathcal{U} \cap E) \subset R_+^N, \phi(\mathcal{U} \cap \partial E) \subset \partial R_+^N$, $\phi \in C^{1,1}(\mathcal{U})$, $\phi^{-1} \in C^{1,1}(B_1(0))$. The desired estimate (2.9) can be derived similarly by following the proof in Theorem 2.2.

Remark 2.4. With the same arguments stated above, estimate (2.9) can be written in a more general form

$$\|u\|_{W^{2,p}(B)} \leq C(\|u\|_{L^p(B)} + \|L_v u\|_{L^p(B)}) \tag{2.15}$$

for real-valued Carathédory functions v, where

$$L_v u = \sum_{i,j=1}^{N} a_{ij}(x,v) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,v) \frac{\partial u}{\partial x_i} + c(x,v)u$$

and the constant C is independent of v.

For the moment, we suppose $a_{ij} \in C^{0,1}(\overline{B} \times R)$, $a_{ij}, \frac{\partial a_{ij}}{\partial x_i}, \frac{\partial a_{ij}}{\partial r}, b_i, c$ are bounded Carathédory functions, with $c \leq 0$, and $f \in L^p(B)$, with

$p \geq N$. Consider the mapping F which assigns to $v \in W^{2,p}(B) \cap W_0^{1,p}(B)$ the solution $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ to the equation

$$L_v u = f(x) \quad \text{in } B. \tag{2.16}$$

Since $W^{2,p}(B) \cap W_0^{1,p}(B)$ is imbedded in $H_0^1(B)$. By the ellipticity of L, the mapping $F : W^{2,p}(B) \cap W_0^{1,p}(B) \longrightarrow W^{2,p}(B) \cap W_0^{1,p}(B)$ is continuous in the topology of $H^1(B)$[3]. By virtue of (2.15), together with the maximum principle for equation (2.16)

$$\sup_B |u| \leq M \|f\|_{L^N}, \tag{2.17}$$

where M is a constant depending on N, diam B, λ and Λ[2], (the maximum principle is only valid for $p \geq N$), we have the following existence result.

Corollary 2.5. *Let B be a ball in R^N and suppose $a_{ij} \in C^{0,1}(\overline{B} \times R)$, $a_{ij}, \frac{\partial a_{ij}}{\partial x_i}, \frac{\partial a_{ij}}{\partial r}, b_i, c \in L^\infty(B \times R)$, with $i, j = 1, \ldots, N$ and $c \leq 0$. Then if $f \in L^p(B)$, with $p \geq N$, under hypotheses* (2.7) *and* (2.8), *there exists a solution $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ to problem* (1.1). *Moreover, the set of solutions $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ is $W^{2,p}$ bounded.*

Proof. Consider the solution $u = F(v)$ for $v \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Since $f \in L^p(B)$, it follows from (2.15) and (2.17) that there exists a constant $k > 0$ such that

$$\|u\|_{W^{2,p}} \leq k \quad \text{for all } u = F(v),\ v \in W^{2,p}(B) \cap W_0^{1,p}(B). \tag{2.18}$$

Let

$$\mathcal{K} = \{v \in W^{2,p}(B) \cap W_0^{1,p}(B) | \|v\|_{W^{2,p}} \leq k\}.$$

Then F is a continuous mapping from $\mathcal{K}$ into $\mathcal{K}$ in the topology of $H^1(B)$. Moreover, since $W^{2,p}(B)$ is a reflexive space and $W^{1,p}(B)$ is continuously imbedded in $H^1(B)$, $\mathcal{K}$ is weakly compact in $H^1(B)$ and hence it is closed in $H^1(B)$. Also, since $W^{2,p}(B) \hookrightarrow W^{1,p}(B)$ is a compact imbedding, $\mathcal{K}$ is a compact set in $H^1(B)$. We conclude from Schauder fixed point theorem that there exists a solution of (1.1) in $\mathcal{K}$. Furthermore, we can obtain from (2.18) that the set of solutions u in $W^{2,p}(B) \cap W_0^{1,p}(B)$ is $W^{2,p}$ bounded.

Remark 2.6. Corollary 2.5 remains valid with B replaced by the ellipsoid

$$E = \{x = (x_1, \ldots, x_N) \in R^N | \sum_{i=1}^{N} (\frac{x_i - c_i}{r_i})^2 < 1\}$$

and with (2.8) replaced by (2.14).

Remark 2.7. Corollary 2.5 remains valid with B replaced by an ovaloid in R^N. (An ovaloid in R^N is a rectangle in R^N with rounded corners.)

Remark 2.8. For any bounded domain Ω with a sufficiently smooth boundary, although the diffeormorphism ψ in Theorem 2.2 is not explicitly observed, it seems that the existence of strong solutions $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to problem (1.1) in Ω remains valid provided the oscillations of a_{ij} with respect to r are sufficiently small.

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