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THE SETS CONTAINING POINTS OF MULTIPLICITY $\underset{\sim}{c}$ OF PLANAR BROWNIAN PATHS

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Abstract. We propose a sufficient condition for a planar set to contain points of multiplicity $\underset{\sim}{c}$ on a planar Brownian path; here $\underset{\sim}{c}$ is the cardinality of the continuum. Any self-similar fractal with positive Hausdorff dimension will satisfy this condition and hence contains $\underset{\sim}{c}$ multiple points of Brownian paths. As a consequence, two independent one dimensional Brownian motions will collide at the same position uncountably often.

1. Introduction. Given a function $w : (0, \infty) \to R^d$, a point $x \in R^d$ is a k-multiple point or point of multiplicity k of w, k being a cardinal number, finite or infinite, if the cardinal number of $w^{-1}(\{x\})$ is greater than or equal to k. For example, if w is a one dimensional Brownian path, then every point in R^1 is a $\underset{\sim}{c}$-multiple point of w, here $\underset{\sim}{c}$ is the cardinal number of the continuum. In fact, we have the following interesting results due to Dvoretzky, Erdös, Kakutani, [1], [2], [3] and Taylor [2]: with probability one, a 2-dim Brownian path has points of multiplicity $\underset{\sim}{c}$; a 3-dim Brownian path has double points but no triple points; a d-dim Brownian path, $d \geq 4$, has no double points. Recently Evans [4] and Tongring [9] investigate conditions for a set to contain k-multiple points of Brownian paths for finite k. Motivated by their works, we shall propose conditions on a planar set

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which are sufficient to ensure the E contains c-multiple points of a planar Brownian path. As a consequence, we will show that two independent 1-dim Brownian motions collide uncountably often at the same point with probability one.

2. Basic Definitions. We present here some elementary definitions and concepts from potential theory. The material is mainly borrowed from Kametani [7]. First, let $|\cdot|$ denote the Euclidean norm in R^d. Let $h(t)$ be a strictly increasing continuous function defined for small positive t satisfying $\lim_{t\downarrow 0} h(t) = 0$. For $E \subseteq R^d$, let

$$m_h(E) = \liminf_{\epsilon\downarrow 0} \left\{ \sum_j h(\delta(S_j)); \cup_j S_j \supseteq E, \delta(S_j) \leq \epsilon \right\},$$

where each S_j is an open ball with diameter $\delta(S_j)$, and the infimum is taken over all possible covering of E by a sequence of open balls with diameter less than ε. $m_h(\cdot)$ is called the *Hausdorff measure* with respect to the function $h(t)$. Obviousely, if $h(t) \geq g(t)$, then $m_h \geq m_g$. When $h(t) = t^\alpha$, $0 < \alpha < \infty$, $0 \leq t < 1$, write $m_\alpha(\cdot)$ instead of $m_{t^\alpha}(\cdot)$ and call $m_\alpha(\cdot)$ the *α-dimensional measure.* Of course, $m_\alpha \geq m_{\alpha'}$, for $\alpha < \alpha'$. Let

$$\begin{aligned} s(E) &= \sup\{\alpha : m_\alpha(E) = \infty\} \\ &= \inf\{\alpha : m_\alpha(E) < \infty\}. \end{aligned}$$

$(\sup\emptyset = 0, \inf\emptyset = \infty)$. Call $s(E)$ the *Hausdorff dimension* of the set E. Let $\phi(t)$ be a strictly decreasing, continuous function defined on positive t satisfying $\lim_{t\downarrow 0}\phi(t) = \infty$. Let E be a bounded Borel set in R^d, μ a probability measure on E, and let

$$U_\mu^\phi(x) = \int_E \phi(|x-y|)\mu(dy).$$

$U_\mu^\phi(\cdot)$ is called the *potential* of ϕ with respect to the distribution μ. Let $V^\phi(E) = \inf_u \sup_x U_\mu^\phi(x)$, where the infimum is taken over all probability measures on E, and supremum is taken over all points in E. The *capacity*, $C^\phi(E)$, of E with respect to the function ϕ is defined to be $\phi^{-1}(V^\phi(E))$ with the understanding that $\phi^{-1}(\infty) = 0$. The following result is proved in

Kametani : if $m_h(E) > 0$, E is bounded Borel, and if $\int_{0+} \phi(t)dh(t) < \infty$, then $C^{\phi}(E) > 0$.

Now we will consider the case $d = 2$. We will take $\phi_k(t) = (\log \frac{1}{t})^k$, $k \geq 1$, and write $C_k(E)$ for $C^{\phi_k}(E)$. $C_k(E)$ is called the k-th *logarithmic capacity* of the set E (E is a bounded Borel set in R^2). From what we just mentioned above, we see if $s(E) > 0$, then $C_k(E) > 0$, $\forall k \geq 1$.

3. Main results. Our main results in this paper will be stated in Theorem 3.3. First, we state two lemmas.

Lemma 3.1. *Let E be a bounded Borel set in R^2. Then $C_{2k}(E) > 0$, if and only if k independent planar Brownian motions will have a common double point in E with probability one.*

Proof. By looking at the proof of Theorem 1 in Tongring [9], we see that k independent planar Brownian motions will have a common double point in E if and only if a planar Brownian motion will have a $2k$-kultiple point in E. By Theorem 5.1 in Fitzsimmons and Salisbury [5], the lemma now follows.

Let $C = C[0,1] = \{f : f : [0,1] \to R^2,\ f \text{ is continuous}\}$ and Let $\Delta = \{(u,v) : 0 \leq u < v < 1\}$. Let $(W_t)_{t\geq 0}$ be a planar Brownian motion. For each $(u,v) \in \Delta$, let

$$ {}_uW_v(t) = \begin{cases} W(u+t) - W(u), & \text{if } 0 \leq t < v-u, \\ W(v) - W(u), & \text{if } t > v-u. \end{cases} $$

Let $l(dudv)$ be the intersection time of (W_t), i.e., formally,

$$ l(dudv) = \delta_{\{0\}}(W(u) - W(v))dudv $$

The following lemma is the Theorem 2.1 in Le Gall [8].

Lemma 3.2. *Suppose Φ is a positive Borel function on $C^3 = C \times C \times C$ and B a Borel set in Δ. Then*

$$E\left[\int_B \Phi({}_oW_{u,u}W_{v,v}W_1)l(dudv)\right]$$
$$= (2\pi)^{-1}\int_B \frac{dudv}{(v-u)}E[\Phi(U_1^u, L^{(v-u)}, U_2^{1-v})]$$

Where $L^{(v-u)}$ is a Brownian bridge of length $v-u, U_1, U_2$ are two independent Brownian motions starting from the origin, U^u denotes the stopped process of U stopped at time u.

Theorem 3.3. *Let E be a Borel set in R^2. Suppose for each $x \in E$,*
(i) *there is a bounded Borel set $E_x \subseteq E$ such that $C_{2k}(E_x) > 0$, $\forall k \geq 1$,*
(ii) *there is a positive sequence $a_n(x)$ decreasing to 0 such that*

$$E_x \supseteq a_n(x)(E_x - x) + x = \{a_n(x)(y-x) + x : y \in E_x\}.$$

Then almost all paths of (W_t) will have $\underline{c}$-multiple points in E.

Proof. By the strong Markov property, we may assume (W_t) starts from a $x(0) \in E$. By(i), there exist a $x(1) \in E_{x(0)}$ and two times t_{11}, t_{12} such that $W(t_{11}) = W(t_{12}) = x(1)$. By (ii) and the scaling propety of (W_t), we may assume $0 < t_{11} < t_{12} < 1$. By lemma 3.2, we may consider ${}_0W_{t_{11}}$ and ${}_{t_{12}}W_1$ as two independent planar Brownian motions. Again by (i) and (ii), there exist $x(2) \in E_{x(1)}$ and $t_{21}, t_{22}, t_{23}, t_{24}$ such that $W(t_{21}) = W(t_{22}) = W(t_{23}) = W(t_{24}) = x(2)$, $0 < t_{21} < t_{22} < t_{11} < t_{12} < t_{23} < t_{24} < 1$, and $t_{11} - t_{21} < \frac{1}{2}$, $t_{24} - t_{12} < \frac{1}{2}$, and $({}_0W_{t_{21}}, {}_{t_{22}}W_{t_{11}}, {}_{t_{12}}W_{t_{23}}, {}_{t_{24}}W_1)$ can be considered as four independent planar Brownian motons starting from $x(2)$. Continue the arguments, we get a double sequence $\{t_{nj} : n \geq 1, 1 \leq j \leq 2^n\}$ in [0,1] such that for each n and each j, there are t_{n+1l}, t_{n+1l+1} such that if j is odd, then $t_{n+1l} < t_{n+1l+1} < t_{nj}$, $t_{nj} - t_{n+1l} < (\frac{1}{2})^n$ and if j is even, then $t_{nj} < t_{n+1l} < t_{n+1l+1}, t_{n+1l+1} - t_{nj} < (\frac{1}{2})^n$. Now, let

$$K = \cap_{m=1}^{\infty}(\cup_{n=m}^{\infty}\{t_{nj} : 1 \leq j \leq 2^n\})^-,$$

where the superscript$-$means taking closure. It's not hard to see K is a perfect set (each neighborhood of any point in K contains another point of K) and hence uncountable.

Any line segment, any self-similar fractal defined in Hutchinson [6] will fulfill our assumptions. In particular, by considering E to be the diagonal line, we obtain

Corollary 3.4. *Two independent 1-dim Brownian paths will collide uncountably often at the same position. That is, if $W_1(t), W_2(t)$ are two independent Brownian paths, then there is a $x \in R^1$ and a uncountable time set $T \subseteq (0, \infty)$ such that $W_1(t) = W_2(t) = x, \forall t \in T$.*

Remark. It seems reasonable to conjecture that any set E of positive Hausdorff dimension will contain points of multiplicity $\underline{c}$ for a planar Brownian motion, but we cannot prove that.

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