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A DENSE ORBIT ALMOST IMPLIES SENSITIVITY TO INITIAL CONDITIONS

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Abstract. Let X be an infinite metric space and $f \in C^0(X,X)$. If f has a dense orbit in X, we show that f either has sensitive dependence on initial conditions (Theorem 5) or is uniformly recurrent (to be defined below) and if X is bounded then there is a nontrivial metric on X with which f is an isometry (Theorem 4). As consequences, we obtain that if f has a dense orbit in X and if (i) f is not one-to-one or (ii) f is a homeomorphism on X and f has a periodic point which is a saddle point, then f has sensitive dependence on initial conditions.

The study of chaotic dynamical systems has become increasingly popular nowadays ([2,4]). Although there has been no universally accepted mathematical definition of chaos, it is generally believed that sensitive dependence on initial conditions is the central element of chaos (see also [6]). Therefore, it would be interesting to know under what conditions sensitive dependence on initial conditions can be guaranteed. Let (X,d) be a metric space with metric d and let $f \in C^0(X,X)$. Let δ be a positive number. We say that f has δ-sensitive dependence on initial conditions if, for every point $x \in X$ and every positive number ε, there exist a point $y \in X$ with $d(x,y) < \varepsilon$ and a positive integer n such that $d(f^n(x), f^n(y)) \geq \delta$. We say that f has sensitive dependence on initial conditions if it has δ-sensitive dependence on initial conditions for some positive number δ. In this case, the number δ is also called a sensitivity constant for f. We say that f is topologically transitive if, for any two nonempty open sets U and V in X, there

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exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$. In [1], it is shown that, in an infinite metric space X, if $f \in C^0(X, X)$ is topologically transitive and the set of periodic points of f is dense in X, then f has sensitive dependence on initial conditions. In [7], it is shown that if (X, d) is a separable, second category space without isolated points, then f is topologically transitive if and only if f has a dense orbit. So, we may assume that f has a dense orbit rather than that f is topologically transitive. It may be argued that if f has a dense orbit, then for any point there is another point close by whose orbit is dense in the whole space, and so it seems that f may have sensitive dependence on initial conditions. But rotation through an irrational angle on the circle S^1 (setting the total angle of S^1 to 1) is an isometry (i.e., distances are preserved under the mapping) and thus is certainly not sensitive to initial conditions. It has no periodic point, but every point has a dense orbit. Therefore, having a dense orbit is not enough to ensure sensitive dependence on initial conditions. However, if f is not one-to-one, then we can show that having a dense orbit is enough to guarantee sensitive dependence on initial conditions. To be more specific, we show in this note, among other things, that , in an infinite metric space (X, d), a map $f \in C^0(X, X)$ with a dense orbit either has sensitive dependence on initial conditions (Theorem 5) or is uniformly recurrent (to be defined below) and if X is bounded then there is a non-trivial metric d^* on X with $d^*(x, y) \geq d(x, y)$ for all x and y in X such that f is also uniformly recurrent with respect to d^* and f is an isometry on (X, d^*), that is, $d^*(f(x)), f(y)) = d^*(x, y)$ for all x and y in X (Theorem 4).

For the rest of this paper, we always let (X, d) denote an infinite metric space and let f denote a map in $C^0(X, X)$. Let δ be a positive number and let E be a subset of X which contains at least two distinct points. We say that E is an asymptotically δ-expansive set of f if, for any two distinct points x and y of E, we have $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta$. On the other hand, we say that f has asymptotically δ-sensitive dependence on initial conditions if, for every $x \in X$ and every open neighborhood $N(x)$ of x, there is a point $y \in N(x)$ such that $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta$. We say that f has asymptotically sensitive dependence on initial conditions if it has asymptotically δ-sensitive dependence on initial conditions for some positive number δ. If f has sensitive dependence on initial conditions, it is natural to ask if, for any integer $k \geq 2$, f^k also has sensitive dependence on initial conditions (with possibly different sensitivity constants). We don't know the answer yet. However, it is easy to see that the uniform continuity of f is a sufficient condition. Some other sufficient conditions are also given below (Theorems 8 & 9).

Lemma 1. *Assume that f has a dense orbit $O_f(u)$. Then, for any point $x \in X$ and any positive integer m, $\limsup_{n\to\infty} d(f^n(u), f^{n+m}(u)) \geq d(x, f^m(x))$.*

Proof. Let x and m be given and let $\langle \varepsilon_n \rangle$ be any sequence of positive numbers with $\lim_{n\to\infty}\varepsilon_n = 0$. Then, since f is continuous and since the orbit $O_f(u)$ is dense in X, there exists, for every ε_n, a positive integer k_n such that the point $f^{k_n}(u)$ is so close to x that $d(f^{k_n}(u), x) < \frac{\varepsilon_n}{2}$ and $d(f^m(f^{k_n}(u)), f^m(x)) < \frac{\varepsilon_n}{2}$. By triangle inequality, $d(f^{k_n}(u), f^{k_n+m}(u)) \geq d(x, f^m(x)) - d(f^{k_n}(u), x) - d(f^{m+k_n}(u), f^m(x)) > d(x, f^m(x)) - \varepsilon_n$. Consequently, $\limsup_{n\to\infty} d(f^n(u), f^{n+m}(u)) \geq d(x, f^m(x))$.

It is easy to see that if f has asymptotically sensitive dependence on initial conditions then it also has sensitive dependence on initial conditions. If f has a dense orbit, then the converse is also true as is shown below.

Lemma 2. *Assume that f has a dense orbit. Then f has sensitive dependence on initial conditions if and only if f has asymptotically sensitive dependence on initial conditions.*

Proof. If f has asymptotically sensitive dependence on initial conditions, then it is trivial that f also has sensitive dependence on initial conditions. So, assume that f has a dense orbit $O_f(u)$ and f has sensitive dependence on initial conditions. Then, for some positive number δ, f has δ-sensitive dependence on initial conditions. Let $x \in X$ be any point and let $N(x)$ be any open neighborhood of x. Then there exist a point $y \in N(x)$ and a positive integer k such that $d(f^k(x), f^k(y)) > \delta$. Since f is continuous and since $O_f(u)$ is dense in X, there exists a point $v \in O_f(u) \cap N(x)$

which is so close to the point x that $d(f^k(v), f^k(x)) < \frac{1}{2}[d(f^k(x), f^k(y)) - \delta]$. But since the orbit $O_f(v)$ is also dense in X, there exists a positive integer m such that $f^m(v)$ is so close to the point y that $f^m(v) \in N(x)$ and $d(f^{k+m}(v), f^k(y)) < \frac{1}{2}[d(f^k(x), f^k(y)) - \delta]$. Consequently, by triangle inequality, $d(f^k(v), f^{k+m}(v)) > d(f^k(x), f^k(y)) - d(f^k(v), f^k(x)) - d(f^{k+m}(v), f^k(y)) > \delta$. By Lemma 1, $\limsup_{n\to\infty} d(f^n(v), f^{n+m}(v)) \geq d(f^k(v), f^{k+m}(v)) > \delta$. Thus, we have either $\limsup_{n\to\infty} d(f^n(x), f^n(v)) > \frac{\delta}{2}$ or $\limsup_{n\to\infty} d(f^n(x), f^n(f^m(v)) = \limsup_{n\to\infty} d(f^n(x), f^{n+m}(v)) > \frac{\delta}{2}$. Since $v \in N(x)$ and $f^m(v) \in N(x)$, we obtain that f has asymptotically $\frac{\delta}{2}$-sensitive dependence on initial conditions.

Lemma 3. *If, for some positive number δ, f has a dense asymptotically δ-expansive set in X, then f has asymptotically $\frac{\delta}{2}$-sensitive dependence on initial conditions.*

Proof. Assume that E is a dense asymptotically δ-expansive set of f. Let x be any point of X and let $N(x)$ be any open neighborhood of x. Since E is dense in X, the set $E \cap N(x)$ contains infinitely many points. Let v and w be any two distinct points in $E \cap N(x)$. Then, $\limsup_{n\to\infty} d(f^n(v), f^n(w)) \geq \delta$. Thus, we have either $\limsup_{n\to\infty} d(f^n(x), f^n(v)) \geq \frac{\delta}{2}$ or $\limsup_{n\to\infty} d(f^n(x), f^n(w)) \geq \frac{\delta}{2}$. This shows that f has asymptotically $\frac{\delta}{2}$-sensitive dependence on initial conditions.

We say that f is uniformly recurrent (with respect to the metric d) if there exist a strictly increasing sequence $\langle m_i \rangle$ of positive integers and a strictly decreasing sequence $\langle \varepsilon_i \rangle$ of positive numbers such that $\lim_{i\to\infty} \varepsilon_i = 0$ and $d(x, f^{m_i}(x)) < \varepsilon_i$ for all positive integers i and all $x \in X$. Note that this definition of uniform recurrence is equivalent to that of uniform rigidity used by Glasner and Weiss [3] in the context of compact metric spaces. It is clear that if f is uniformly recurrent then every point of X is a recurrent point of f. On the other hand, if f is uniformly recurrent and has a periodic point z of least period $k > 1$ such that, for some positive integer $i, \varepsilon_i < \min\{d(z, f^n(z)) | 1 \leq n \leq k-1\}$, then it is clear that k divides m_i. This fact will be used in the proof of Part (5) of Theorem 7 below.

Theorem 4. *Assume that f is uniformly recurrent with respect to the* *metric d. Then the following hold:*

(a) *For any two distinct points x and y in X,* $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq d(x,y) > 0$. *In particular, f is one-to-one.*

(b) *If, for any two points x and y in X, the number* $d^*(x,y) = \limsup_{n\to\infty} d(f^n(x), f^n(y))$ *is finite, then the following hold:*

(1) *(X, d^*) is a metric space.*

(2) *$d^*(x,y) \geq d(x,y)$ for all x and y in X.*

(3) *f is an isometry with respect to the metric d^*.*

(4) *f is also uniformly recurrent with respect to d^*.*

Proof. Assume that f is uniformly recurrent. If there exist two distinct points x and y in X such that $\limsup_{n\to\infty} d(f^n(x), f^n(y)) < d(x,y)$, then there exists a positive number $c < 1$ such that $\limsup_{n\to\infty} d(f^n(x), f^n(y)) < cd(x,y)$. So, there is a positive integer m such that $d(f^n(x), f^n(y)) < cd(x,y)$ for all integers $n \geq m$. Since f is uniformly recurrent, there exists a positive integer $n > m$ such that $d(x, f^n(x)) < \frac{1-c}{4} d(x,y)$ and $d(y, f^n(y)) < \frac{1-c}{4} d(x,y)$. For this n, we have $d(x,y) \leq d(x, f^n(x)) + d(f^n(x), f^n(y)) + d(f^n(y), y) \leq \frac{1-c}{4} d(x,y) + cd(x,y) + \frac{1-c}{4} d(x,y) = \frac{1+c}{2} d(x,y) < d(x,y)$. This is a contradiction. Therefore, for any two distinct points x and y in X, $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq d(x,y)$.

If, for any two point x and y in X, the number $d^*(x,y) = \limsup_{n\to\infty} d(f^n(x), f^n(y))$ is finite, then it is a routine job to check that d^* is a metric for X. We omit the details. On the other hand, it is clear that $d^*(f(x), f(y)) = \limsup_{n\to\infty} d(f^n(f(x)), f^n(f(y))) = \limsup_{n\to\infty} d(f^n(x), f^n(y)) = d^*(x,y)$. So f is an isometry with respect to d^*. Finally, if, for some positive integer k and some positive number ε, $d(x, f^k(x)) < \varepsilon$ for all $x \in X$, then it is clear that $d(f^n(x), f^{n+k}(x)) < \varepsilon$ for all integers $n \geq 0$. Consequently, $\limsup_{n\to\infty} d(f^n(x), f^{n+k}(x)) \leq \varepsilon$. That is, $d^*(x, f^k(x)) \leq \varepsilon$. Therefore, the assertion that f is uniformly recurrent with respect to d^* clearly follows from the assumption that f is uniformly recurrent with respect to d.

Remark. In part (b) of the above result, f is an isometry on (X, d^*). But in general it does not have a dense orbit in (X, d^*) even if it has one in (X, d). However, if X is an infinite compact metric space, then an isometry on X which has a dense orbit in X is clearly a homeomorphism on X and so a result of Halmos and von Neumann shows that this isometry is actually topologically conjugate to a minimal rotation on a compact abelian metric group. See [8, p. 125 Theorem 5.8] for details.

Theorem 5. *Assume that f is not uniformly recurrent with respect to the metric d. If f has a dense orbit $O_f(u)$, then, for some positive number δ, $O_f(u)$ is an asymptotically δ-expansive set of f. Consequently, f has asymptotically $\frac{\delta}{2}$-sensitive dependence on initial conditions.*

Proof. Assume that the set $O_f(u)$ is not an asymptotically δ-expansive set of f. Then, for any positive number δ, there exist two positive integers $k_1 < k_2$ such that $\limsup_{n\to\infty} d(f^n(u), f^{n+(k_2-k_1)}(u)) = \limsup_{n\to\infty} d(f^n(f^{k_1}(u)), f^n(f^{k_2}(u))) < \delta$. So, let $\langle \varepsilon_i \rangle$ be any strictly decreasing sequence of positive numbers with $\lim_{i\to\infty}\varepsilon_i = 0$. Then there exists a strictly increasing sequence $\langle m_i \rangle$ of positive integers such that $\limsup_{n\to\infty} d(f^n(u), f^{n+m_i}(u)) < \varepsilon_i$ for all positive integers i. By Lemma 1, $\limsup_{n\to\infty} d(x, f^{m_i}(x)) < \varepsilon_i$. This shows that f is uniformly recurrent which contradicts the assumption. Therefore, $O_f(u)$ is a dense asymptotically δ-expansive set of f. By Lemma 3, f has asymptotically $\frac{\delta}{2}$-sensitive dependence on initial conditions.

Let $x_0 \in X$. A preorbit of x_0 is any set consisting of points $x_0, x_{-1}, x_{-2}, \ldots, x_{-n}, \ldots$, in X such that $f(x_{-m}) = x_{-(m-1)}$ for all positive integers m. The following lemma follows easily from the definition of uniform recurrence.

Lemma 6. *Let $x_0 \in X$. If x_0 has both a preorbit P and a neighborhood which is disjoint from P, then f is not uniformly recurrent.*

If f is a one-to-one continuous map from a compact metric space X onto itself, then it is easy to see that f is actually a homeomorphism on X. Now let f be a homeomorphism on the (not necessarily compact) metric space X and let z be a fixed point of f. We say that z is an attracting fixed point of f if there exists an open neighborhood $N(z)$ of z such that $\lim_{n\to\infty} f^n(x) = z$ for every $x \in N(z)$. We say that z is a repelling fixed point of f if there exists an open neighborhood $U(z)$ of z such that $\lim_{n\to\infty} f^{-n}(x) = z$ for every $x \in U(z)$. We say that z is a saddle point of f if there exist two points

$x \neq z$ and $y \neq z$ such that $\lim_{n\to\infty} f^n(x) = z$ and $\lim_{n\to\infty} f^{-n}(y) = z$. If z is a periodic point of f with least period k, we say that z is an attracting (repelling, saddle respectively) point of f if z is an attracting (repelling, saddle respectively) fixed point of f^k. It is easy to see that if f has a dense orbit then f cannot have attracting or repelling periodic points (the repelling case was pointed to me by Professor V. S. Afraimovich).

Theorem 7. *Assume that f has a dense orbit. Then the following hold:*

(1) *If f is not one-to-one, then f has asymptotically sensitive dependence on initial conditions.*

(2) *If there exist a set U in X and a point $z \in U$ such that $f(U) \supset U$ and $f(z) \notin \overline{U}$, then f has asymptotically sensitive dependence on initial conditions.*

(3) *Let $x_0 \in X$. Assume that x_0 has both a preorbit P and a neighborhood which is disjoint from P. Then f has asymptotically sensitive dependence on initial conditions.*

(4) *Assume that f is also a homeomorphism on X. If f has a periodic point which is a saddle point, then f has asymptotically sensitive dependence on initial conditions.*

(5) *If f has infinitely countably many periodic points $z_i, i \geq 1$ of different periods k_i and a positive number δ such that, for every positive integer i, $\min\{d(z_i, f^n(z_i)) | 1 \leq n \leq k_i - 1\} \geq \delta$, then f has asymptotically sensitive dependence on initial conditions.*

Proof. Part (1) follows from Theorem 4(a) and Theorem 5. Parts (2), (3), and (4) follow from Lemma 6 and Theorem 5. Part (5) follows easily from the remarks following the definition of uniform recurrence.

Assume that f has a dense orbit. If f has a non-recurrent point, then f is not uniformly recurrent. So, by Theorem 5 above, f has sensitive dependence on initial conditions (see also [5]). If we also have $f(X) = X$, then, for every positive integer k, f^k also has sensitive dependence on initial conditions. This is shown below.

Theorem 8. *Assume that f has a dense orbit $O_f(u)$ and a non-recur-*

rent point. If $(X) = X$, then, for every positive integer k, f^k has asymptotically sensitive dependence on initial conditions.

Proof. Let v be a non-recurrent point of f. Then, since $f(X) = X$, let $\langle v_k \rangle$ be a sequence of points in X such that $v_0 = v$ and $f(v_k) = v_{k-1}$ for all positive integers k. For any integer $m \geq 0$, let $\beta_m = \inf\{d(v_m, f^n(v_m)) | n \geq 1\}$. For any positive integer k, let $\delta_k = \min\{\beta_m | 0 \leq m \leq k-1\}$. Since v is a non-recurrent point of f, so is v_m for every integer $m \geq 0$. Hence, $\beta_m > 0$ and so $\delta_k > 0$ for every positive integer k.

Now let k be a fixed positive integer. Let $0 \leq i < j$ be any two fixed integers and let ε be any positive number with $\varepsilon < \delta_k$. Since the orbit $O_f(u)$ is dense in X, there exists a strictly increasing sequence $\langle n_s \rangle$ of positive integers such that, for every such n_s, there is an integer m (depending on n_s) with $0 \leq m \leq k-1$ such that $f^{kn_s+i}(u)$ is so close to v_m that $d(f^{kn_s+i}(u), v_m) < \frac{\varepsilon}{2}$ and $d(f^{kn_s+j}(u), f^{j-i}(v_m)) = d(f^{j-i}(f^{kn_s+i}(u)), f^{j-i}(v_m)) < \frac{\varepsilon}{2}$. Thus, $d(f^{kn_s+i}(u), f^{kn_s+j}(u)) \geq d(v_m, f^{j-i}(v_m)) - d(f^{kn_s+i}(u), v_m) - d(f^{kn_s+j}(u), f^{j-i}(v_m)) > \beta_m - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \geq \delta_k - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain that $\limsup_{n\to\infty} d(f^{nk+i}(u), f^{nk+j}(u)) \geq \delta_k$. Since $0 \leq i < j$ are arbitrary, we see that the dense set $O_f(u)$ is an asymptotically δ_k-expansive set of f^k.

Now assume that f is topologically transitive and the set of all periodic points of f is dense in X. It is shown in [1,7] that f has sensitive dependence on initial conditions. In fact, more can be said. In the following, we show that, for every positive integer k, f^k has sensitive dependence on initial conditions.

Theorem 9. *Assume that f is topologically transitive and the set of all periodic points of f is dense in X. Then, for every positive integer k, f^k has asymptotically sensitive dependence on initial conditions.*

Proof. Let k be any positive integer and let $\alpha_k = \sup\{\inf\{d(f^i(y), f^i(z)) | 0 \leq i, 0 \leq j \leq k-1\}\}$, where the supreme is taken over all $y \neq z$ in X. Then it is clear that $\alpha_k > 0$. Let δ_k be any fixed positive number satisfying $\delta_k < \alpha_k$ and let V be any nonempty open set in X. Then, since $\delta_k < \alpha_k$, there exist points $y \neq z$ in X such that $\delta_k < \inf\{d(f^i(y), f^j(z)) | 0 \leq i,$

$0 \le j \le k-1\}$. Let $\beta_k = \inf\{d(f^i(y), f^j(z)) | 0 \le i,\ 0 \le j \le k-1\}$ and let ε be any positive number with $\varepsilon < \frac{1}{2}(\beta_k - \delta_k)$. Let W be any nonempty open neighborhood of z such that if $x \in W$ then $d(f^t(x), f^t(z)) < \varepsilon$ for all $0 \le t \le k-1$. Since f is topologically transitive, there exists a positive integer i such that $f^i(V) \cap W \neq \emptyset$. In particular, since the set of periodic points of f is dense in X, there exists a periodic point v_1 of f in V with least period r such that $f^i(v_1) \in W$ and so $d(f^{i+t}(v_1), f^t(z)) < \varepsilon$ for all $0 \le t \le k-1$. Similarly, there exist a positive integer j and a periodic point v_2 of f in V such that $d(f^{j+t}(v_2), f^t(y)) < \varepsilon$ for all $0 \le t \le r+k-1$.

Let $A_k = \{x \in X | d(x, f^t(y)) < \varepsilon$ for some integer $0 \le t \le k-1\}$ and let $B_k = \{x \in X | d(x, f^t(y)) < \varepsilon$ for some integer $0 \le t \le r+k-1\}$. Also let $C_k = \{x \in X | d(x, f^t(z)) < \varepsilon$ for some integer $0 \le t \le k-1\}$. Then it is clear that, for any $u \in B_k$ and any $w \in C_k$, we have $d(u, w) \ge \beta_k - 2\varepsilon$. Since v_2 is periodic, it is obvious that there exists a strictly increasing sequence $\langle m_s \rangle$ of positive integers such that $f^{km_s}(v_2) \in A_k$ for all positive integers m_s. For every such positive integer m_s, there exists, since v_1 is also periodic, an integer n_s in the interval $[m_s, m_s + \frac{r}{k}]$ such that $f^{kn_s}(v_1) \in C_k$. Since, for every such positive integer n_s, we also have $f^{kn_s}(v_2) \in B_k$, we obtain that $d(f^{kn_s}(v_1), f^{kn_s}(v_2)) \ge \beta_k - 2\varepsilon > \delta_k$. This shows that

$$d((f^k)^m(v_1), (f^k)^m(v_2)) > \delta_k \text{ for infinitely many positive integers } m.$$

Consequently, f^k has asymptotically $\frac{\delta_k}{2}$-sensitive dependence on initial conditions.

Remarks.

(1) If $k = 1$, then the above implies that, for any positive number $\delta < \frac{\alpha_1}{2}$, where α_1 is defined as in the above proof, f has asymptotically δ-sensitive dependence on initial conditions. This sensitivity constant δ is better than those found in [1] and [7].

(2) By taking y and z to be periodic points of f with disjoint orbits and letting $\alpha = \inf\{\alpha_k | 1 \le k\}$, where α_k's are defined as in the above proof, we obtain that $\alpha > 0$. Therefore, if δ is any fixed positive number which is strictly less than $\frac{\alpha}{2}$, then, for every positive integer k, f^k has asymptotically δ-sensitive dependence on initial conditions.

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References

1. J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, *On Devaney's definition of chaos*, American Mathematical Monthly **99** (1992), 332-334.

2. R. L. Devaney, *An introduction to chaotic dynamical systems, second edition*, Addison-Wesley, Menlo Park, California, 1989.

3. E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity **6** (1993), 1067-1075.

4. D. Gulick, *Encounter with chaos*, McGraw-Hill, New York, 1992.

5. Y. Hu and J. Ruan, *A note on Devaney's definition of chaos (in Chinese*), J. Fudan Univ. (Natural Science) **34** (1995), 127-131.

6. C. knudsen, *Chaos without nonperiodicity*, American Mathematical Monthly **101** (1994), 563-565.

7. S. Silverman, *On maps with dense orbits and the definition of chaos*, Rocky Mount. J. Math. **22** (1992), 353-375.

8. P. Walters, *Ergodic theory-introductory lectures*, Lecture Notes in Math. Vol.458, Springer-Verlag, Berlin, New York, 1975.

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