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GENERALIZED HENSTOCK STIELTJES INTEGRAL

BY

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Abstract. The work concerns with the introduction of a generalized Henstock Stieltjes integral, the $\mathcal{H}S_k$-integral, that generalizes the concept of the $\mathcal{R}S_k^*$-integral of Ray and Das [9]. The new integral includes the generalized Lebesgue Stieltjes, the LS_k-integral of Bhattacharyya and Das [1].

1. Introduction. In [9] Ray and Das introduce a new definition of the RS_k^*-integral of Russell [12] which they call the $\mathcal{R}S_k^*$-integral. It is shown that if f is bounded and g is k-convex on $[a,b]$ with $g_+^{(k-1)}(a)$ and $g_-^{(k-1)}(b)$ existing, then

$$(1) \qquad (RS_k^*) \subset (\mathcal{R}S_k^*) \subset (LS_k)$$

where (I) stands for the class of I-integrable functions on $[a,b]$.

We introduce here a definition of a generalized Henstock Stieltjes integral, which we call the $\mathcal{H}S_k$-integral. The proposed integral includes the LS_k-integral of Bhattacharyya and Das [1] and generalizes the integrals of Pfeffer [8] in some sense or other.

For notations and definitions not produced here we refer to Russell [11, 12], Bhattacharyya and Das [1,2], and Ray and Das [9]. However, we recall certain characteristics of BV_k and k-convex functions and also the definition of the $\mathcal{R}S_k^*$-integral of Ray and Das [9].

Let a, b be fixed real numbers such that $a < b$. Let k be a positive

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integer greater 1. Let g be of bounded k-th variation on $[a,b]$. Following Theorems 15 and 19 of Russell [11], $g = g_1 - g_2$, where g_1 and g_2 are k-convex on $[a,b]$ and $D_+^{k-1}g_i(a)$, $D_-^{k-1}g_i(b)$ exist for each $i = 1,2$. In view of Corollary to Theorem 17 of Russell [11] and Theorem 2 of Russell [12] we can, without loss of generality, assume the function g to be k-convex on $[a,b]$ with $D_+^{k-1}g(a)$ and $D_-^{(k-1)}g(b)$ existing. Utilising Corollary to Theorem 17 of Russell [11] and Lemma 3.2 of Das and Das [3], it follows that $D^r g(x)$ are continuous in $[a,b]$, $1 \le r \le k-2$, for $k \ge 3$ and $D^{k-1}g(x)$ exists in $[a,b]$ except for a countable set of points. Also it is shown in the proof of Lemma 3.2 of Das and Das [3] that if $a < x < y < b$, then

$$\begin{aligned} D_+^{k-1}g(a) &\le D_-^{k-1}g(x) \le D_+^{k-1}g(x) \\ &\le D_-^{k-1}g(y) \le D_+^{k-1}g(y) \le D_-^{k-1}g(b). \end{aligned} \tag{2}$$

This shows that $D_-^{k-1}g(x)$, $D_+^{k-1}g(x)$ are monotonic non-decreasing respectively in $(a,b]$, $[a,b)$ and so are continuous in $[a,b]$ except possibly for a countable set of points. The existence of $D^{k-1}g(x)$ follows at each point of continuity of either sided derivatives. In view of Theorem 12 of Russell [11] and Theorem 1(ii) of Verblunsky [15] the $(k-1)$th Riemann* derivatives (unilateral and bilateral) can be replaced by the corresponding $(k-1)$th ordinary derivatives. We shall denote by C the subset of $[a,b]$ where $g^{(k-1)}(x)$ exists and by D the set $[a,b] \setminus C$.

Definition 1.1. (cf. Definition 2.2 of [9]). Let f and g be defined on $[a,b]$ and let g be k-convex on $[a,b]$ and $g_+^{(k-1)}(a)$, $g_\pm^{(k-1)}(b)$ exist. For any partition $P = \{a = x_0 < x_1 < \cdots < x_q = b\}$, we write gk-mesh $(P) = \max_{1\le j\le q}[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]$ and

$$\begin{aligned} S_1(P,f,g) &= \sum_{j=1}^{q} f(x_j)[g_+^{(k-1)}(x_j) - g_-^{(k-1)}(x_j)]/(k-1)! \\ &+ \sum_{j=1}^{q} f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)! \end{aligned}$$

where $\xi_j \in (x_{j-1}, x_j)$, $j = 1,2,\ldots,q$.

The $\mathcal{RS}_k^*$-integral $\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}$, written as $(\mathcal{RS}_k^*)\ \int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}$ is the real number I, if it exists uniquely, and if for each $\epsilon > 0$, there corresponds a real number $\delta(\epsilon)$ such that for any partition P of $[a,b]$ with gk-mesh $(P) < \delta$, the inequality

$$|S_1(P,f,g) - I| < \epsilon$$

is satisfied. If the integral exists, then f is said to be $\mathcal{RS}_k^*$-integrable with respect to g, written as $(f,g) \in \mathcal{RS}_k^*[a,b]$.

Bhattacharyya and Das [1] obtain the following result.

Theorem 1.2. *Let f be bounded on $[a,b]$ and g be k-convex on $[a,b]$ with $g_+^{(k-1)}(a)$, $g_-^{(k-1)}(b)$ existing.*

(1) *If $(f,g) \in RS_k^*[a,b]$, then $(f,g) \in LS_k[a,b]$ and the two integrals agree.*

(2) *$(f,g) \in RS_k^*[a,b]$ if and only if f is continuous in $[a,b]$ except a set of gk-measure zero.*

It is further shown by an example in [1] that $(f,g) \notin RS_k^*[a,b]$ if the interval $[a,b]$ contains any point x_0 in its interior such that f is discontinuous at x_0 and $g_-^{(k-1)}(x_0)$ does not exist.

Ray and Das ([9], Theorem 2.3) obtain the following:

Theorem 1.3. *Let f be bounded, g be k-convex on $[a,b]$ and $g_+^{(k-1)}(a)$, $g_-^{(k-1)}(b)$ exist. If $(f,g) \in \mathcal{RS}_k^*[a,b]$, then $(f,g) \in LS_k[a,b]$ and*

$$(\mathcal{RS}_k^*)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = (LS_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

Further $(f,g) \in \mathcal{RS}_k^*[a,b]$ if and only if f is continuous on C except on a set of gk-measure zero.

Immediately they remark that if $f \in BV[a,b]$ and $g \in BV_k[a,b]$, then $(f,g) \in \mathcal{RS}_k^*[a,b]$.

Finally the authors ascertain by various examples that (RS_k^*) is a proper subclass of $(\mathcal{RS}_k^*)$ and that $(\mathcal{RS}_k^*)$ is a proper subclass of (LS_k) so as to obtain the chain (1).

Henceforth we shall assume g to be k-convex on $[a,b]$ and $g_+^{(k-1)}(a)$, $g_\pm^{(k-1)}(b)$ exist.

2. The HS_k integral.

Definition 2.1. Let f be defined on $[a,b]$. A partition of $[a,b]$ is a set $P=\{x_0,x_1,\ldots,x_q;\xi_1,\xi_2,\ldots,\xi_q\}$ such that $a=x_0<x_1<\cdots<x_q=b$ and $x_{j-1}\le\xi_j\le x_j$, $j=1,2,\ldots,q$. For a given positive function δ on $[a,b]$, we say P is $\delta(gk)$-fine whenever $g_-^{(k-1)}(x_j)-g_+^{(k-1)}(x_{j-1})<\delta(\xi_j)$ for all $j=1,2,\ldots,q$. Write

$$\begin{aligned}S(P,f,g)&=\sum_{j=1}^{q}f(x_j)[g_+^{(k-1)}(x_j)-g_-^{(k-1)}(x_j)]/(k-1)!\\&\quad+\sum_{j=1}^{q}f(\xi_j)[g_-^{(k-1)}(x_j)-g_+^{(k-1)}(x_{j-1})]/(k-1)!\\&=\sum_{j=1}^{q}T_j'(P,x_j)+\sum_{j=1}^{q}T_j''(P,x_{j-1},x_j)\\&=\sum_{j=1}^{q}T_j(P,x_{j-1},x_j;\xi_j)\end{aligned}$$

where

$$\begin{aligned}T_j'(P,x_j)&=f(x_j)[g_+^{(k-1)}(x_j)-g_-^{(k-1)}(x_j)]/(k-1)!\\&=f(x_j)|\{x_j\}|_{gk},\end{aligned}$$

$$\begin{aligned}T_j''(P,x_{j-1},x_j)&=f(\xi_j)[g_-^{(k-1)}(x_j)-g_+^{(k-1)}(x_{j-1})]/(k-1)!\\&=f(\xi_j)|(x_{j-1},x_j)|_{gk},\end{aligned}$$

and

$$T_j(P,x_{j-1},x_j;\xi_j)=T_j'(P,x_j)+T_j''(P,x_{j-1},x_j).$$

The HS_k-integral of f with respect to g, written as $(HS_k)\int_a^b f(x)\frac{d^kg(x)}{dx^{k-1}}$ is the real number I if for every $\epsilon>0$ there is a positive function δ on $[a,b]$ such that for every $\delta(gk)$-fine partition P of $[a,b]$, the inequality

$$|S(P,f,g)-I|<\epsilon$$

is satisfied. If the $\mathcal{HS}_k$-integral exists we write $(f,g)\in \mathcal{HS}_k[a,b]$ and

$$I=(\mathcal{HS}_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

The above definition of the $\mathcal{HS}_k$-integral differs from that of the $\mathcal{RS}_k^*$-integral in the same way as Henstock integral does from Riemann integral. The gk-mesh of a partition is nolonger constant but varies from point to point and for getting partitions we choose first the points $\xi_1,\xi_2,\ldots,\xi_q$ then $x_0,x_1,\ldots,x_q$ whereas in case of the $\mathcal{RS}_k^*$-integral we are to choose first $x_0,x_1,\ldots,x_q$ then $\xi_1,\xi_2,\ldots,\xi_q$. The point ξ_j is called the associated point of $[x_{j-1},x_j]$, and x_j, $j=1,2,\ldots,q$, the partition points. For brevity we write $P=\{[u,v];\xi\}$ where $[u,v]$ denotes a typical interval in P and ξ is the associated point of $[u,v]$. Further

$$\begin{aligned}S(P,f,g)&=\sum T'(P,v)+\sum T''(P,u,v)\\&=\sum T(P,u,v;\xi).\end{aligned}$$

Lemma 2.2. *To each positive function δ on $[a,b]$ there exists a partition $P=\{x_0,x_1,\ldots,x_q;\xi_1,\xi_2,\ldots,\xi_q\}$ of $[a,b]$ such that for all $j=1,2,\ldots,q$*

(i) $x_{j-1}\le\xi_j\le x_j$,

(ii) $g_-^{(k-1)}(x_j)-g_+^{(k-1)}(x_{j-1})<\delta(\xi_j)$,

(iii) $g_+^{(k-1)}(x)-g_-^{(k-1)}(x)<\delta(\xi_j)$ *for* $x\in(x_{j-1},x_j)$.

The proof is omitted (cf. Theorem 2.14 below).

Theorem 2.3. *If $(f,g)\in\mathcal{RS}_k^*[a,b]$, then $(f,g)\in\mathcal{HS}_k[a,b]$ and*

$$(\mathcal{HS}_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}=(\mathcal{RS}_k^*)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

Proof. To each $\epsilon>0$ there corresponds a real number $\delta(\epsilon)>0$ such that for any partition P of $[a,b]$ with gk-mesh $(P)<\delta$, the inequality

$$\left|S_1(P,f,g)-(\mathcal{RS}_k^*)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}\right|<\epsilon$$

is satisfied, where $S_1(P,f,g)$ is as in Definition 1.1.

Define a positive function $\delta(\xi)$ on $[a,b]$ such that $\delta(\xi)=\delta$ for all $\xi\in[a,b]$. Clearly every $\delta(gk)$-fine partition P of $[a,b]$ is also a partition P of gk-mesh $(P)<\delta$. Consequently every $\mathcal{HS}_k$ sum, $S(P,f,g)$, is a $\mathcal{RS}_k^*$ sum, $S_1(P,f,g)$, and the theorem is proved.

The converse of the above theorem is not true is shown by the following example.

Example 2.4. Let f and g be defined on $[0,1]$ by

$$\begin{aligned} f(x)&=1 && \text{if } x \text{ is rational}\\ &=0 && \text{if } x \text{ is irrational;}\\ g(x)&=x^k/2k && \text{for } 0\le x\le \frac{1}{2}\\ &=x^k/k && \text{for } \frac{1}{2}<x\le 1. \end{aligned}$$

It is shown in Example 2.2 of Ray and Das [9] that $(f,g)\notin\mathcal{RS}_k^*[\alpha,\beta]$, $0\le\alpha<\beta\le 1$. Given $\epsilon>0$ we label all the rational numbers in $[0,1]$ as $r_1,r_2,\ldots$ and define $\delta(r_j)=\epsilon/2^j$ for $j=1,2,\ldots$ and $\delta(\xi)=1$ otherwise. Then clearly $\xi=\frac{1}{2}$ is always a partition point of any $\delta(gk)$-fine partition P of $[a,b]$ and $(f,g)\in\mathcal{HS}_k[\alpha,\beta]$ for every $[\alpha,\beta]\subset[0,1)$. Infact,

$$\begin{aligned} (\mathcal{HS}_k)\int_\alpha^\beta f(x)\frac{d^k g(x)}{dx^{k-1}} &= 0 && \text{if } 0\le\alpha<\beta<\frac{1}{2}\\ &=\tfrac{1}{2} && \text{if } 0\le\alpha<\beta=\frac{1}{2}\\ &=\tfrac{1}{2} && \text{if } 0\le\alpha<\frac{1}{2}<\beta<1\\ &=0 && \text{if } \frac{1}{2}<\alpha<\beta<1. \end{aligned}$$

(Ray and Das [9], however, consider $g(x)=x^k/k$ for x in $[0,1]$. In that case the $\mathcal{HS}_k$-integral equals 0 for each $[\alpha,\beta]\subset[0,1)$). Indeed $(f,g)\in\mathcal{HS}_k[\alpha,\beta]$, $0\le\alpha<\beta\le 1$ for any k-convex functions with $g_+^{(k-1)}(0)$,

$g_{\pm}^{(k-1)}(1)$ existing. If D is the set of points of non-existence of $g^{(k-1)}(x)$ in $[0,1]$, then in view of Lemma 2.1 of Bhattacharyya and Das [2] the series

$$\sum_{x\in D\cap(0,1)} [g_{+}^{(k-1)}(x) - g_{-}^{(k-1)}(x)]/(k-1)!$$

converges to λ, $0 \le \lambda \le [g_{-}^{(k-1)}(1) - g_{+}^{(k-1)}(0)]/(k-1)!$. Clearly for the function f in Example 2.4, the HS_k-integral is 0, λ or μ $(0 < \mu < \lambda)$ according as D contains no rational point (or D is empty), D contains an infinite number of rational points or D contains only a finite number of rational points in $[0,1]$.

Following Henstock [5], Lee [7] standard properties of the HS_k-integral are immediate. We simply state them below for completeness.

Theorem 2.5. (a)*If* $(f_i, g) \in HS_k[a,b]$, $i = 1,2\ldots,n$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are real numbers, then*

$$\Big(\sum_{i=1}^{n} \lambda_i f_i, g\Big) \in HS_k[a,b]$$

and

$$(HS_k)\int_a^b \Big(\sum_{i=1}^{n} \lambda_i f_i\Big)(x)\frac{d^k g(x)}{dx^{k-1}} = \sum_{i=1}^{n} \lambda_i (HS_k)\int_a^b f_i(x)\frac{d^k g(x)}{dx^{k-1}}.$$

(b) *If* $(f, g_i) \in HS_k[a,b]$, $i = 1,2,\ldots,n$ *and* $\mu_1, \mu_2, \ldots, \mu_n$ *are real numbers, then* $(f, \sum_{i=1}^{n} \mu_i g_i) \in HS_k[a,b]$ *and*

$$(HS_k)\int_a^b f(x)\frac{d^k(\sum_{i=1}^{n} \mu_i g_i)(x)}{dx^{k-1}} = \sum_{i=1}^{n} \mu_i (HS_k)\int_a^b f(x)\frac{d^k g_i(x)}{dx^{k-1}}.$$

Theorem 2.6. *If* $(f,g) \in HS_k[a,c]$ *and* $(f,g) \in HS_k[c,b]$ *where* $a < c < b$, *then* $(f,g) \in HS_k[a,b]$ *and*

$$(HS_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = (HS_k)\int_a^c f(x)\frac{d^k g(x)}{dx^{k-1}} + (HS_k)\int_c^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

Theorem 2.7. (Cauchy test). $(f,g) \in HS_k[a,b]$ *if and only if for every* $\epsilon > 0$, *there is a* $\delta(\xi) > 0$ *such that for any two* $\delta(gk)$*-fine partitions* P *and*

P', *the inequality*

$$|S(P, f, g) - S(P', f, g)| < \epsilon$$

holds.

Theorem 2.8. *If* $(f, g) \in HS_k[a, b]$, *then* $(f, g) \in HS_k[c, d]$ *for each* $[c, d] \subset [a, b]$.

Theorem 2.9. *If* $f(x) = 0$ gk*-almost everywhere in* $[a, b]$, *then* $(f, g) \in HS_k[a, b]$ *and* $(HS_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = 0$.

Theorem 2.10. *If* $(f, g) \in HS_k[a, b]$, $(\Psi, g) \in HS_k[a, b]$ *and* $f(x) \leq \Psi(x)$ gk*-almost everywhere in* $[a, b]$, *then*

$$(HS_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} \leq (HS_k) \int_a^b \Psi(x) \frac{d^k g(x)}{dx^{k-1}}.$$

In view of Theorem 2.8, we see that if $(f, g) \in HS_k[a, b]$, then $(f, g) \in HS_k[a, x]$ for every $x \in (a, b]$. We define the HS_k primitive F of f on $[a, b]$ by

$$\begin{aligned} F(x) &= (HS_k) \int_a^x f(t) \frac{d^k g(t)}{dt^{k-1}} \qquad && \text{if } a < x \leq b \\ &= 0 && \text{if } x = a. \end{aligned}$$

Hence if $(f, g) \in (HS_k)[a, b]$, then there exists a function F on $[a, b]$ such that for every $\epsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $\delta(gk)$-fine partition $P = \{[u, v]; \xi\}$ of $[a, b]$ the inequality

$$\left|(P) \sum [F(v) - F(u) - T(P, u, v; \xi)]\right| < \epsilon \tag{3}$$

holds.

Theorem 2.11. (Saks–Henstock Lemma). *If* $(f, g) \in HS_k[a, b]$, *then there is a function* F *on* $[a, b]$ *such that for every* $\epsilon > 0$ *there is a positive function* δ *on* $[a, b]$ *such that for every* $\delta(gk)$*-fine partition* $P = \{[u, v]; \xi\}$ *of* $[a, b]$

$$(P) \quad \sum |F(v) - F(u) - T(P, u, v; \xi)| < 4\epsilon.$$

The following Cauchy extension formula can be proved similarly as in Ray [10].

Theorem 2.12 *Let $(f, g) \in HS_k[a, d]$ for each $d \in (a, b)$. If*

$$\lim_{d \to b} (HS_k) \int_a^d f(x) \frac{d^k g(x)}{dx^{k-1}}$$

exists and equals h, then $(f, g) \in HS_k[a, b]$ and

$$(HS_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = h + f(b)[g_+^{(k-1)}(b) - g_-^{(k-1)}(b)].$$

We conclude the section providing certain observations.

Observation 2.13. Given an arbitrary function $\delta > 0$ (independent of the notion of integration) in $[a, b]$, there always exists, in view of Lemma 2.2, a $\delta(gk)$-fine partition $P = \{a = x_0 < x_1 \cdots < x_q = b; \xi_1, \xi_2, \ldots, \xi_q\}$ of $[a, b]$. If $x_{j-1} < \xi_j < x_j$ for some j, we can replace $[x_{j-1}, x_j]$ by two intervals $[x_{j-1}, \xi_j]$ and $[\xi_j, x_j]$, so that ξ_j would be a partition point and this will not change the sum $S(P, f, g)$ in Definition 2.1. In fact,

$$\begin{aligned}
&T_j(P, x_{j-1}, x_j; \xi_j) \\
&= f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)! \\
&\quad + f(x_j)[g_+^{(k-1)}(x_j) - g_-^{(k-1)}(x_j)]/(k-1)! \\
&= \Big\{f(\xi_j)[g_-^{(k-1)}(\xi_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)! \\
&\qquad + f(\xi_j)[g_+^{(k-1)}(\xi_j) - g_-^{(k-1)}(\xi_j)]/(k-1)!\Big\} \\
&\quad + \Big\{f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(\xi_j)]/(k-1)! \\
&\qquad + f(x_j)[g_+^{(k-1)}(x_j) - g_-^{(k-1)}(x_j)]/(k-1)!\Big\}
\end{aligned}$$

and so

$$(4) \qquad T_j(P, x_{j-1}, x_j; \xi_j) = T_j(P, x_{j-1}, \xi_j; \xi_j) + T_j(P, \xi_j, x_j; \xi_j)$$

Ray [10] obtains a definition of $\mathcal{HS}_k^*$-integral with the approximating sum

$$S^*(P,f,g) = \sum_{j=1}^{q} T_j''(P,x_{j-1},x_j)$$
$$= \sum_{j=1}^{q} f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)!,$$

and shows that if $(f,g) \in \mathcal{RS}_k^*[a,b]$, then $(f,g) \in \mathcal{HS}_k^*[a,b]$. Further

$$(\mathcal{RS}_k^*)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = (\mathcal{HS}_k^*)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} + A,$$

where

$$A = \sum_{j=1}^{\infty} f(x_j)[g_+^{(k-1)}(x_j) - g_-^{(k-1)}(x_j)]/(k-1)!, \quad x_j \in D.$$

The accomodation of the additive term $\sum_{j=1}^{q} T_j'(P,x_j)$ in the approximating sum of this article

$$S(P,f,g) = \sum_{j=1}^{q} T_j(P,x_{j-1},x_j;\xi_j) = \sum_{j=1}^{q} T_j' + \sum_{j=1}^{q} T_j''$$

gives the Theorem 2.3 providing the equality of the two integrals. (See also Theorem 3.2).

The consideration of $\delta(gk)$-fine partition arises as an influence of such partition in the definition of the $\mathcal{RS}_k^*$-integral. The $\mathcal{RS}_k^*$-integral is not an ordinary Riemann-Stieltjes integral because of the gk-mesh replacing the mere length of each subinterval. In view of additive nature of $T_j(P,x_{j-1},x_j;\xi_j)$ (See relation (4)) we can always assume an associated point to be a partition point. So one could develop the theory with left-hand or right-hand interval-point function, basically giving the same integral. That a $\delta(gk)$-fine partition induces some δ^*-fine partition and vise-versa; and that the $\mathcal{HS}_k$-integral become exactly the gauge integral, originated independently in Henstock [4] and Kurzweil [6], are shown by the following theorem.

The authors are thankful to the referee for the suggestion of the following theorem along with the proof.

Theorem 2.14. *The HS_k-integral is the gauge integral* (Henstock-Kurzweil integral) *induced by the k-convex function g.*

Proof. In view of relation (2) and subsequent discussion thereat, it follows that for u fixed and $v \to u+$, and for v fixed and $u \to v-$, $g_-^{(k-1)}(v) - g_+^{(k-1)}(u) \to 0$. Since g is k-convex we have $g_-^{(k-1)}(v) - g_+^{(k-1)}(u) \geq 0$. So given $\delta(x) > 0$, there is a $\delta^*(x) > 0$ such that if $v - u < \delta^*(x)$, $x = u$ or $x = v$, then $g_-^{(k-1)}(v) - g_+^{(k-1)}(u) < \delta(x)$. Thus a δ^*-fine partition is a δ (gk)-fine partition.

Conversely, given $\delta^*(x) > 0$, we look for a suitable $\delta(x) > 0$. When $x = u$ there is a greatest $s \geq u$ for which $\lim_{v \to s+}\{g_-^{(k-1)}(v) - g_+^{(k-1)}(u)\} = 0$. If $s > u$ then $g_-^{(k-1)}(v) - g_+^{(k-1)}(u) = 0$ for $u \leq v \leq s$ and in $[v, s]$ we can use any partition, the contribution to the sum being 0. If $s = u$ we take $\delta(x) > 0$ arbitrarily small so that $g_-^{(k-1)}(v) - g_+^{(k-1)}(u)$ is small enough to give $v - u < \delta^*(x)$. Similarly for $x = v$ and thus a suitable $\delta(x) > 0$ follows.

This proves the theorem.

Remark 2.15. Following Theorem 2.14 we can define the HS_k-integral equivalently as follows:

Let f be defined on $[a, b]$ and g be k-convex on $[a, b]$ with $g_+^{(k-1)}(a)$, $g_\pm^{(k-1)}(b)$ existing. The HS_k-integral of f with respect to g is the real number I if for every arbitrary $\epsilon > 0$ there is a positive number δ, called a gauge, on $[a, b]$ such that for every δ-fine partition $P = \{a = x_0, x_1, \ldots, x_q = b; \xi_1, \xi_2, \ldots, \xi_q\}$, $\xi_j \in [x_{j-1}, x_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$, of $[a, b]$, the inequality

$$\Big|\Big\{\sum_{j=1}^{q} f(x_j)[g_+^{(k-1)}(x_j) - g_-^{(k-1)}(x_j)]/(k-1)! + \sum_{j=1}^{q} f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)!\Big\} - I\Big| < \epsilon$$

holds. If the integral exists we write $(f, g) \in HS_k[a, b]$ and

$$I = (HS_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}}.$$

We define S, T_j, T'_j, T''_j as in Definition 2.1 so that the above inequality can equivalently be written as

$$|S(P,f,g)-I|<\epsilon \qquad \text{or,} \qquad \Big|\sum_{j=1}^{q} T_j(P,x_{j-1},x_j;\xi_j)-I\Big|<\epsilon$$

or equivalently

$$\Big|\sum_{j=1}^{q} T'_j(P,x_j)+\sum_{j=1}^{q} T''_j(P,x_{j-1},x_j)-I\Big|<\epsilon.$$

Saks-Henstock Lemma (Theorem 2.11) and Cauchy's extension theorem (Theorem 2.12) are then immediate.

3. The HS_k-integral includes the LS_k-integral. We show that the LS_k-integral is included in the HS_k-integral and the two integrals agree. To this end, we state a lemma on LS_k-integrability whose proof follows from that of Theorem 7.2 of Saks ([14], p.73) utilising gk-measurability and linearity of the LS_k-integrals. (For gk-measurability and LS_k-integrability we refer Bhattacharyya and Das [1])

Lemma 3.1. *If $(f,g)\in LS_k[a,b]$, then there exists for each $\epsilon>0$, a lower semi-continuous function h and an upper semi-continuous function ϕ in $[a,b]$ such that*

$$h(x)\geq f(x), \quad \phi(x)\leq f(x) \quad \text{at each } x\in[a,b]$$

and

$$(LS_k)\int_a^b \{h(x)-f(x)\}\frac{d^k g(x)}{dx^{k-1}}<\epsilon/3,$$
$$(LS_k)\int_a^b \{f(x)-\phi(x)\}\frac{d^k g(x)}{dx^{k-1}}<\epsilon/3.$$

In fact, Theorem 7.2 of Saks [14] proves the result for non-negative function f. It is sufficient to consider $f=f^{+}+f^{-}$ where

$$f^{+}=\max(f,0) \qquad \text{and} \quad f^{-}=\max(-f,0)$$

in case of function of arbitrary sign.

Theorem 3.2. *If $(f,g) \in LS_k[a,b]$, then $(f,g) \in HS_k[a,b]$ and the two integrals agree.*

Proof. To each $\epsilon > 0$ arbitrary there exist, in view of Lemma 3.1, a lower semi-continuous function h an upper semi-continuous function ϕ in $[a,b]$ such that

$$-\infty \leq \phi \leq f \leq h \leq +\infty$$

and

$$(LS_k)\int_a^b \{h(x)-\phi(x)\}\frac{d^k g(x)}{dx^{k-1}} < \epsilon.$$

We can find a positive function δ on $[a,b]$ so that for each $\xi \in [a,b]$ there exists a closed interval $[u,v]$ containing ξ, $g_-^{(k-1)}(v) - g_+^{(k-1)}(u) < \delta(\xi)$ and also that for all $x \in [u,v]$,

$$h(x) > h(\xi) - \epsilon \geq f(\xi) - \epsilon,$$
$$\phi(x) < \phi(\xi) + \epsilon \leq f(\xi) + \epsilon.$$

Indeed, if $g_+^{(k-1)}(\xi) - g_-^{(k-1)}(\xi) < \delta(\xi)$ we have $u < \xi < v$ and if $g_+^{(k-1)}(\xi) - g_-^{(k-1)}(\xi) \geq \delta(\xi)$ we take $[\xi, v]$ or $[u, \xi]$.

Let $P = \{x_0, x_1, \ldots, x_q; \xi_1, \xi_2, \ldots, \xi_q\}$ be any $\delta(gk)$-fine partition of $[a,b]$. We retain $\delta(gk)$-fine partition owing to the gk-measurability concept in the definition of the LS_k-integral. Then for each $j = 1, 2, \ldots, q$

$$(5)\qquad (LS_k)\int_{(x_{j-1},x_j)} \phi \leq (LS_k)\int_{(x_{j-1},x_j)} f \leq (LS_k)\int_{(x_{j-1},x_j)} h$$

and

$$(6)\qquad (LS_k)\int_{(x_{j-1},x_j)} \phi - \epsilon|(x_{j-1},x_j)|_{gk} < f(\xi_j)|(x_{j-1},x_j)|_{gk} < (LS_k)\int_{(x_{j-1},x_j)} h + \epsilon|(x_{j-1},x_j)|_{gk}.$$

Now

$$\begin{aligned}
&\left|S(P,f,g) - (LS_k)\int_a^b f\right| \\
&\leq \sum_{j=1}^{q} \Big| f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)! \\
&\quad + f(x_j)[g_+^{(k-1)}(x_j) - g_-^{(k-1)}(x_j)]/(k-1)! - (LS_k)\int_{[x_{j-1},x_j]} f \Big| \\
&= \sum_{j=1}^{q} \Big| f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)! - (LS_k)\int_{(x_{j-1},x_j)} f \Big| \\
&< \sum_{j=1}^{q} \epsilon |(x_{j-1},x_j)|_{gk} + (LS_k)\int_{(x_{j-1},x_j)} (h-\phi), \quad \text{using (5) and (6)} \\
&< \epsilon[g_-^{(k-1)}(b) - g_+^{(k-1)}(a)]/(k-1)! + \epsilon.
\end{aligned}$$

Hence $(f,g) \in HS_k[a,b]$ and

$$(HS_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = (LS_k)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

This proves the theorem.

That (LS_k) is a proper subclass of (HS_k) follows from the following example.

Example 3.3. Let

$$\begin{aligned}
F(x) &= x^2\cos 1/x^2, && \text{if } x \neq 0 \\
&= 0, && \text{if } x = 0,
\end{aligned}$$

and

$$g(x) = x^k/k!.$$

The gk-derivative of F (see [1]) is the ordinary derivative which exists everywhere in $[0,1]$. In view of Henstock ([5], p38), $(F',\ g^{(k-1)}) \notin LS[0,1]$ but $(F'_{gk}, g) \in HS_k[0,1]$. On the otherhand, since $(F', g^{(k-1)}) \notin LS[0,1]$, it follows, utilising Theorem 2.4 of Bhattacharyya and Das [1], that $(F'_{gk}, g) \notin LS_k[0,1]$.

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