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DIFFERENTIATED SHIFT-INVARIANT INTEGRAL OPERATORS, MULTIVARIATE CASE

BY

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Abstract. In a recent paper [1] the author, together with H. Gonska, introduced some wavelet type very general multivariate integral operators over $\mathbb{R}^d$, $d \geq 1$, and studied their various properties such as shift-invariance, global smoothness preservation and optimality, convergence to the unit, their differentiation, and preservation of continuous probability multivariate distribution functions. These operators are introduced through a convolution-like iteration of another very general multivariate operator with a general multivariate scaling type function. In this article the author gives sufficient conditions, so that the partial derivatives of the above described multivariate operators possess most of the above listed nice properties of their originals. Especially a sufficient condition is given so that the "global smoothness preservation" corresponding multivariate inequality is attained, that is sharp. Finally several applications are presented, there the partial derivatives of very general specialized multivariate operators are shown to fulfill most of the above mentioned properties. In particular the partials of these operators are shown to preserve continuous multivariate probability density functions.

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1. Introduction. This article continues [1] and it is also motivated from there. In [1] we have the following setting: let $\{l_k\}$ be a sequence of positive linear multivariate operators that maps $C_U(\mathbb{R}^d)$, $d \geq 1$ (the space of uniformly continuous functions on $\mathbb{R}^d$), into $C(\mathbb{R}^d)$ such that

$$l_k(f,x) = l_0(f(2^{-k}\cdot),x), \quad \text{all } x \in \mathbb{R}^d, \ f \in C_U(\mathbb{R}^d).$$

Also, let φ be a Lebesgue measurable real valued function of compact support $\subseteq \times_{i=1}^d [-a_i, a_i]$, $a_i > 0$. We assume that $\varphi \geq 0$ and

$$\int_{\mathbb{R}^d} \varphi(x-u)du = 1, \quad \text{all } x \in \mathbb{R}^d.$$

we introduce the following sequence $\{\mathcal{L}_k\}_{k\in z}$ of positive linear multivariate operators acting on $C_U(\mathbb{R}^d)$:

$$\mathcal{L}_k(f,x) := \int_{\mathbb{R}^d} l_k(f,u)\varphi(2^k x - u)du, \text{ any } x \in \mathbb{R}^d.$$

Under suitable assumptions in [1], we established that $\mathcal{L}_k$ is a shift invariant operator, it has the "global smoothness preservation property" i.e., in moduli of continuity with respect to an arbitrary norm that $\omega_1(\mathcal{L}_k f, \delta) \leq \omega_1(f,\delta)$, any $\delta > 0$, and this inequality is sharp, furthermore $\mathcal{L}_k$ converges with rates to the unit I as $k \to +\infty$. In the last convergence the associated modulus of continuity is with respect to maximum norm. also in [1], when φ is bounded, we examined how to differentiate $\mathcal{L}_k$. Furthermore we presented several examples. However the result of [1] that mostly motivated this paper is the following theorem.

Theorem ($*$). *Let f be a continuous probabilistic multivariate distribution function from $\mathbb{R}^d$ into $[0,1]$. Assume that $(\ell_0 f)$ is also a continuous multivariate distribution function, and φ is a continuous function. Then $(\mathcal{L}_k f)_{k\in z}$ is a continuous multivariate distribution function from $\mathbb{R}^d$ into $[0,1]$.*

Illustration (on Theorem ($*$)). Assume that $\frac{\partial^m(\ell_0 f)}{\partial_{x_1}\cdots\partial_{x_m}}$, $1 \leq m \leq d$ do exist and are continuous on $\mathbb{R}^d$, then so do $\frac{\partial^m(\ell_k f)}{\partial_{x_1}\cdots\partial_{x_m}}$, $1 \leq m \leq d$. Thus (by

H. Bauer [2], pp. 103-104) we get that $\frac{\partial^d(\mathcal{L}_k f)}{\partial_{x_1}\ldots\partial_{x_d}}$ exists, and it is continuous by Lemma 1 of [1]. Assume that $(\ell_k f)$ has a continuous probability density function (p.d.f) g_k, then $g_k = \frac{\partial^d(\ell_k f)}{\partial_{x_1}\ldots\partial_{x_d}}$. Also assume that $(\mathcal{L}_k f)$ has a continuous probability density function G_k, then $G_k = \frac{\partial^d(\mathcal{L}_k f)}{\partial_{x_1}\ldots\partial_{x_d}}$. In this situation it holds:

$$G_k(x) = 2^{kd} \cdot \int_{\mathbb{R}^d} g_k(u)\varphi(2^k x - u)du, \text{ all } x \in \mathbb{R}^d, \ k \in \mathbb{Z}.$$

I.e., the above integral transforms continuous p.d.f.'s to continuous p.d.f.'s. See also that related Theorem 7 at the end of this article. So, it is worth studying the partial derivatives of $\mathcal{L}_k$ operators.

An excellent source in the study of multivariate probability distribution functions and p.d.f.'s is the book of A.N. Shiryayev [4].

Thus, in this article we study thoroughly the linear operators

$$\left(\frac{\partial^{\sum_{r=1}^m j_r}\mathcal{L}_k}{\partial x_{i_1}^{j_1}\ldots\partial x_{i_m}^{j_m}}\right)_{k\in Z}$$

acting on $C(\mathbb{R}^d)$. We prove analogous properties to the above described for $(\mathcal{L}_k)_{k\in Z}$. Under appropriate assumptions all basic results of [1] carry over here and we present several applications.

Call

$$\bullet^\partial := \frac{\partial^\rho \bullet}{\partial x_{i_1}^{j_1}\ldots\partial x_{i_m}^{j_r}}, \sum_{r=1}^m j_r := \rho \in \mathbb{N}.$$

In general it is not true that

$$(\mathcal{L}_k f)^\partial = \mathcal{L}_k(f^\partial), \quad f \in C^{(\rho)}(\mathbb{R}^d).$$

Therefore the results in [1] cannot cover the study of $\mathcal{L}_k^\partial$ operators as an application, i.e., the writing of this article is necessary.

Further Motivation. Example. Consider here $f \in C^{(\rho)}(\mathbb{R}^d)$ such that $f^\partial \in C_U(\mathbb{R}^d), d \geq 1$, and study the following specific ℓ_0-operator. Define the linear operator

$$(\theta_0 f)(x) := f(x) + f(0), \quad \text{all } x \in \mathbb{R}^d.$$

Notice that

$$(\theta_0(f^\partial))(x) = f^\partial(x) + f^\partial(0) \neq f^\partial(x) = (\theta_0 f)^\partial(x).$$

I.e., $(\theta_0 f^\partial) \neq (\theta_0 f)^\partial$. Also observe that θ_0 maps $C^{(\rho)}(\mathbb{R}^d)$ into itself. Clearly (20) is fulfilled:

$$\begin{aligned}&|(\theta_0 f)^\partial(x-u) - (\theta_0 f)^\partial(y-u)|\\ &\quad=|f^\partial(x-u) - f^\partial(y-u)| \leq \omega_1(f^\partial, \|x-y\|), \text{ all } x, y \in \mathbb{R}^d,\end{aligned}$$

where $\|\cdot\|$ is an arbitrary norm. Also (14) is satisfied:

$$|(\theta_0 f)^\partial(u) - f^\partial(y)| = |f^\partial(u) - f^\partial(y)| \leq \omega_{1,\infty}(f^\partial, \|u-y\|_\infty \leq \omega_{1,\infty}(f^\partial, a),$$

where $u, y : \|u-y\|_\infty \leq a, a > 0$ and $\omega_{1,\infty}$ is the modulus of continuity with respect to $\|\cdot\|_\infty$-norm. Furthermore the "shift-invariance condition" (19) is true:

$$(\theta_0^\partial(f(2^{-k}\cdot+\alpha)))(2^k u) = 2^{-k\rho}\cdot f^\partial(u+\alpha) = (\theta_0^\partial(f(2^{-k}\cdot)))(2^k\cdot(u+\alpha)),$$

all $k \in \mathbb{Z}, \alpha \in \mathbb{R}^d$ fixed, all $u \in \mathbb{R}^d$; $\rho \in \mathbb{N}$ be given. Here observe that θ_0 fulfills only (20):

$$|\theta_0(f, x-u) - \theta_0(f, y-u)| \leq \omega_1(f, \|x-y\|), \text{ all } x, y \in \mathbb{R}^d.$$

Hence define

$$\theta_k(f, x) := \theta_0(f(2^{-k}\cdot), x), \text{ all } x \in \mathbb{R}^d, \ k \in \mathbb{Z},$$

and

$$\Omega_k(f, x) := \int_{\mathbb{R}^d} \theta_k(f, u)\varphi(2^k x - u)du, \text{ all } x \in \mathbb{R}^d, \ k \in \mathbb{Z}.$$

Then from Theorem 1 following, Ω_k only enjoys the "global smoothness preservation" property, however Ω_k^∂ has all the nice properties: of "shift invariance", "global smoothness preservation", and convergence to $\frac{\partial^\rho}{\partial x_{i_1}^{j_1}\ldots\partial x_{i_m}^{j_m}}$-operator; see Proposition 1, Theorem 1 and Theorem 3. That is, sometimes differentiated multivariate integral operators behave nicer than their original ones.

Furthermore we have the following:

$$\theta_k(f,x) = f\left(\frac{x}{2^k}\right) + f(0),\ \ \theta_k^\partial(f,x) = 2^{-k\rho} f^\partial(2^{-k}x),$$

and

$$\theta_k(f^\partial,x) = f^\partial\left(\frac{x}{2^k}\right) + f^\partial(0).$$

Also

$$\Omega_k(f^\partial,x) = \int_{\mathbb{R}^d} \theta_k(f^\partial,u)\cdot\varphi(2^kx-u)du,$$

and

$$\Omega_k^\partial(f,x) = 2^{k\rho}\cdot\int_{\mathbb{R}^d}(\theta_k f)^\partial(u)\cdot\varphi(2^kx-u)du.$$

That is

$$\Omega_k^\partial(f,x) = \int_{\mathbb{R}^d} f^\partial\left(\frac{u}{2^k}\right)\cdot\varphi(2^kx-u)du,$$

and

$$\Omega_k(f^\partial,x) = \Omega_k^\partial(f,x) + f^\partial(0).$$

I.e., proving in general that

$$\Omega_k^\partial(f) \neq \Omega_k(f^\partial),\quad f\in C^\rho(\mathbb{R}^d),\ k\in\mathbb{Z}.$$

Finally, we notice in general that, if $\ell_0(f^\partial) = (\ell_0 f)^\partial$, $f\in C^{(\rho)}(\mathbb{R}^d)$, $d\geq 1$, then $\mathcal{L}_k(f^\partial) = (\mathcal{L}_k f)^\partial$, any $k\in\mathbb{Z}$. All the above strongly recommend to study ${\mathcal{L}_k}^\partial$, in general and separately, and that is the topic of what follows.

The last important source of motivation, but not the least, is [3]. There C. Cottin and H. Gonska deal with the simultaneous approximation and global smoothness preservation of very general operators, over compact domains, in the univariate and multivariate case. They measure smoothness with higher order K-functionals and moduli of smoothness and they give important applications to univariate and multivariate Bernstein operators. Their resulting inequalities are often sharp or nearly sharp.

2. Main Results. Let $X := C_u(\mathbb{R}^d), d \geq 1$ be the space of uniformly continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$. For $f \in X$, define the first order modulus of continuity of f by

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in \mathbb{R}^d \\ \|x-y\| \leq \delta}} |f(x) - f(y)|, \ \delta > 0,$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^d$. It is known [1] that $\omega_1(f,\delta) < \infty$, $\forall \delta > 0$. Here $C(\mathbb{R}^d)$ denotes the space of continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$. Let $\{\ell_k\}_{k \in Z}$ be a sequence of linear operators from $C(\mathbb{R}^d)$ into itself such that

$$\ell_k(f,x) = \ell_0(f(2^{-k}\cdot),x), \text{ all } x \in \mathbb{R}^d, \ f \in C(\mathbb{R}^d). \tag{1}$$

Assume that

$$\ell_0 : C^{(\rho)}(\mathbb{R}^d) \to C^{(\rho)}(\mathbb{R}^d), \ \rho \in \mathbb{N} \text{ be fixed,} \tag{2}$$

where $C^{(\rho)}(\mathbb{R}^d)$ is the space of ρ-times continuously differentiable functions.

In this paper we consider only $f \in C^{(\rho)}(\mathbb{R}^d)$ such that $\frac{\partial^\rho f}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_m}^{j_m}} \in X$, where fixed $i_1, \ldots, i_m \in \{1, \ldots, d\}$ be such that $i_1 < \cdots < i_m$, and fixed $j_1, \ldots, j_m \in \mathbb{N}$ be such that $\sum_{r=1}^m j_r = \rho$.

From now on we will denote by

$$f^\partial := \frac{\partial^\rho f}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_m}^{j_m}}, \quad f \in C^{(\rho)}(\mathbb{R}^d), \tag{3}$$

where the mixed partial is defined as above.

Let φ be a bounded real valued function of compact support $\subseteq \times_{i=1}^d [-a_i, a_i]$, $a_i > 0$. We assume that $\varphi \geq 0$, φ is Lebesgue measurable and

(4) $$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{d-fold}\varphi(x_1-u_1,x_2-u_2,\ldots,x_d-u_d)du_1\ldots du_d=1,$$

$$\text{all } (x_1,\ldots,x_d)\in\mathbb{R}^d$$

((4) same as

$$\int_{\mathbb{R}^d}\varphi(x-u)du=1,\quad \text{all } x\in\mathbb{R}^d;$$

where $x=(x_1,\ldots,x_d), u=(u_1,\ldots,u_d)$). One can easily prove that

(5) $$\int_{\mathbb{R}^d}\varphi(u)du=1.$$

Examples.

(i) For $i=1,\ldots,d$ take the characteristic function

$$\varphi_i(x_i):=\chi_{[-\frac{1}{2},\frac{1}{2})}(x_i)=\begin{cases}1, & x_i\in[-\frac{1}{2},\frac{1}{2})\\ 0, & \text{else.}\end{cases}$$

Define

$$\varphi^*(x):=\prod_{i=1}^{d}\varphi_i(x_i),\ \text{all } x=(x_1,\ldots x_d)\in\mathbb{R}^d.$$

Note that φ^* is bounded, supp $\varphi^*\subseteq[-\frac{1}{2},\frac{1}{2}]^d$, $\varphi^*\geq 0$ and φ^* is Lebesgue measurable. Also

$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{d-fold}\prod_{i=1}^{d}\varphi_i(x_i-u_i)du_1\ldots du_d=1,$$

i.e.,

$$\int_{\mathbb{R}^d}\varphi^*(x-u)du=1,\ \text{all } x\in\mathbb{R}^d.$$

(ii) For $i=1,\ldots,d$ consider the hat functions

$$\varphi_i(x_i):=\begin{cases}1+x_i, & -1\leq x_i\leq 0,\\ 1-x_i, & 0\leq x_i\leq 1.\end{cases}$$

Define

$$\tilde{\varphi}(x_1,\ldots,x_d) := \prod_{i=1}^{d} \varphi_i(x_i) \geq 0, \text{ all } (x_i,\ldots,x_d) \in \mathbb{R}^d,$$

which is a bounded continuous scale function with support $\subseteq [-1,1]^d$. It is easy to see that

$$\int_{\mathbb{R}^d} \tilde{\varphi}(x-u)du = \left(\int_{-\infty}^{\infty} \varphi_i(x_i-u_i)du_i \right)^d = 1, \text{ all } x \in \mathbb{R}^d.$$

Let $\{\mathcal{L}_k\}_{k\in Z}$ be the sequence of linear operators acting on $C^{(p)}(\mathbb{R}^d)$ and defined as follows

$$\mathcal{L}_0(f,x) := \int_{\mathbb{R}^d} \ell_0(f,u)\varphi(x-u)du \tag{6}$$

and

$$\mathcal{L}_k(f,x) := \mathcal{L}_0(f(2^{-k}\cdot);2^k x), \tag{7}$$

all $x \in \mathbb{R}^d, k \in \mathbb{Z}$. I.e.,

$$\mathcal{L}_k(f,x) = \int_{\mathbb{R}^d} \ell_k(f,u)\varphi(2^k x-u)du. \tag{8}$$

Note that

$$\mathcal{L}_0(f,x) = \int_{\mathbb{R}^d} \ell_0(f,x-u)\varphi(u)du = \int_{x_{i=1}^d[-a_i,a_i]} \ell_0(f,x-u)\varphi(u)du. \tag{9}$$

In [1] we proved its Theorem 3 with the use of a result due to H. Bauer ([2], pp. 103-104; differentiation of a parametrized general integral with respect to the parameter). However we missed mentioning the so much needed assumption (2), that is, ℓ_0 maps $C^{(\rho)}(\mathbb{R}^d)$ into itself. With this correction, Theorem 3 of [1] says that : (denote by $(\ell_0 f)^\partial(x) := \frac{\partial^\rho(\ell_0 f)(x)}{\partial x_{i_1}^{j_1}\ldots\partial x_{i_m}^{j_m}}$ and by

$$\left.(\mathcal{L}_0 f)^\partial(x) := \frac{\partial^\rho(\mathcal{L}_0 f)(x)}{\partial x_{i_1}^{j_1}\ldots\partial x_{i_m}^{j_m}}\right)$$

$$\begin{aligned} \mathcal{L}_0^\partial(f,x) :=& (\mathcal{L}_0 f)^\partial(x) = \int_{X_{i=1}^d[-a_i,a_i]} (\ell_0 f)^\partial(x-u)\varphi(u)du \\ &= \int_{X_{i=1}^d[-a_i,a_i]} \ell_0^\partial(f,x-u)\varphi(u)du, \end{aligned} \tag{10}$$

the last equality exists only due to a different notation. Thus $\mathcal{L}_k^\partial(f,x)$ exists! $k \in \mathbb{Z}$. Then, obviously

$$\mathcal{L}_0^\partial(f,x) = \int_{\mathbb{R}^d} \ell_0^\partial(f,u)\varphi(x-u)du, \text{ all } x \in \mathbb{R}^d. \tag{11}$$

For our convenience we shall see the notation

$$\int_{-\vec{a}}^{\vec{a}} \bullet = \int_{X_{i=1}^d[-a_i,a_i]} \bullet \tag{12}$$

Furthermore, we obtain that

$$\begin{aligned}\mathcal{L}_k^\theta(f,x) &= 2^{k\rho} \cdot \int_{-\vec{a}}^{\vec{a}} \ell_k^\theta(f, 2^k x - u)\varphi(u)du \\ &= 2^{k\rho} \cdot \int_{\mathbb{R}^d} \ell_k^\partial(f, 2^k x - u)\varphi(u)du \\ &= 2^{k\rho} \cdot \int_{\mathbb{R}^d} \ell_k^\partial(f,u)\varphi(2^k x - u)du, \text{ all } x \in \mathbb{R}^d, k \in \mathbb{Z}.\end{aligned} \tag{13}$$

Assumption. Take $f \in C^{(\rho)}(\mathbb{R}^d), \rho \in \mathbb{N}$ fixed, such that $f^\partial \in X$. For fixed $\alpha > 0$ we assume that

$$\sup_{\substack{u,y\in\mathbb{R}^d \\ \|u-y\|_\infty \le \alpha}} |\ell_0^\partial(f,u) - f^\partial(y)| \le \omega_{1,\infty}\left(f^\partial, \frac{m\alpha+n}{2^r}\right), \tag{14}$$

is true for f as above, $m \in \mathbb{N}, n \in \mathbb{Z}_+, r \in \mathbb{Z}$, where $\omega_{1,\infty}$ is the modulus of continuity ω_1 defined with respect to $\|\cdot\|_\infty$.

Not that $\ell_k^\partial, \mathcal{L}_k^\partial$ are linear operators acting on $C(\mathbb{R}^d)$. Also notice that

$$\begin{aligned}\mathcal{L}_k^\partial(f,x) &:= (\mathcal{L}_k^\partial(f))(x) := (\mathcal{L}_k f)^\partial(x) = ((\mathcal{L}_k f)(x))^\partial \\ &=((\mathcal{L}_0(f(2^{-k}\cdot)))(2^k x))^\partial = 2^{k\rho}\cdot((\mathcal{L}_0(f(2^{-k}\cdot)))^\partial)(2^k x) \\ &= 2^{k\rho}\cdot(\mathcal{L}_0^\partial(f(2^{-k}\cdot)))(2^k x) = 2^{k\rho}\cdot\mathcal{L}_0^\partial(f(2^{-k}\cdot), 2^k x), \text{ all } x \in \mathbb{R}^d.\end{aligned} \tag{15}$$

I.e.,

$$\mathcal{L}_k^\partial(f,x) = 2^{k\rho}\cdot\mathcal{L}_0^\partial(f(2^{-k}\cdot), 2^k x), \tag{16}$$

$$\mathcal{L}_k^\partial(f,x) = 2^{k\rho}\cdot\int_{\mathbb{R}^d}(\ell_0(f(2^{-k}\cdot)))^\partial(u)\varphi(2^k x-u)du \tag{17}$$

and

(18) $$\mathcal{L}_k^{\partial}(f,x) = 2^{k\rho}\cdot \int_{\mathbb{R}^d} (\ell_k f)^{\partial}(u)\varphi(2^k x-u)du, \text{ all } x\in\mathbb{R}^d.$$

Definition 1. *Let $f_\alpha(\cdot) := f(\cdot+\alpha)$, $\alpha\in\mathbb{R}^d$, and ϕ be an operator. If $\phi(f_\alpha)=(\phi f)_\alpha$, then ϕ is called a shift invariant operator.*

Proposition 1. *Assume that*

(19) $$\ell_0^{\partial}(f(2^{-k}\cdot+\alpha);2^k u) = \ell_0^{\partial}(f(2^{-k}\cdot);2^k(u+\alpha)),$$

all $k\in\mathbb{Z}$, $\alpha\in\mathbb{R}^d$ fixed, all $u\in\mathbb{R}^d$; any $f\in C^{(\rho)}(\mathbb{R}^d)$; $\rho\in\mathbb{N}$ be given. Then $\mathcal{L}_k^{\partial}$, any $k\in\mathbb{Z}$, is a shift invariant operator.

Proof. Notice that

$$(\mathcal{L}_0 f)^{\partial}(x) = \int_{\mathbb{R}^d}(\ell_0 f)^{\partial}(x-u)\varphi(u)du, \text{ all } x\in\mathbb{R}^d.$$

We observe that

$$\begin{aligned}
(\mathcal{L}_k(f\cdot+\alpha);x))^{\partial} =& (\mathcal{L}_k(f_\alpha;x))^{\partial} = (\mathcal{L}_0(f_\alpha(2^{-k}\cdot);2^k x))^{\partial}\\
=&2^{k\rho}\cdot(\mathcal{L}_0(f_\alpha(2^{-k}\cdot)))^{\partial}(2^k x)\\
=&2^{k\rho}\cdot\int_{\mathbb{R}^d}(\ell_0(f(2^{-k}\cdot+\alpha)))^{\partial}(2^k x-u)\varphi(u)du\\
=&2^{k\rho}\cdot\int_{\mathbb{R}^d}(\ell_0(f(2^{-k}\cdot+\alpha)))^{\partial}(2^k\cdot(x-2^{-k}u))\cdot\varphi(u)du\\
=&2^{k\rho}\cdot\int_{\mathbb{R}^d}(\ell_0(f(2^{-k}\cdot)))^{\partial}(2^k\cdot(x-2^{-k}u+\alpha))\cdot\varphi(u)du\\
=&2^{k\rho}\cdot\int_{\mathbb{R}^d}(\ell_0(f(2^{-k}\cdot)))^{\partial}(2^k\cdot(x+\alpha)-u)\cdot\varphi(u)du\\
=&2^{k\rho}\cdot(\mathcal{L}_0(f(2^{-k}\cdot)))^{\partial}(2^k\cdot(x+\alpha))\\
=&((\mathcal{L}_k f)(x+\alpha))^{\partial} = (\mathcal{L}_k f)^{\partial}(x+\alpha).
\end{aligned}$$

I.e.,

$$\mathcal{L}_k^{\partial}(f_\alpha) = (\mathcal{L}_k^{\partial} f)_\alpha.$$

Next we establish the property of "global smoothness preservation" of operators $\mathcal{L}_k^{\partial}, k\in\mathbb{Z}$.

Theorem 1. *For any $f \in C^{(\rho)}(\mathbb{R}^d)$ such that $f^\partial \in X, \rho \in \mathbb{N}$ fixed, and any $u \in \mathbb{R}^d$ assume that*

$$|\ell_0^\partial(f, x-u) - \ell_0^\partial(f, y-u)| \le \omega_1(f^\partial, \|x-y\|), \text{ all } x, y \in \mathbb{R}^d, \tag{20}$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^d$. Then

$$\omega_1((\mathcal{L}_k f)^\partial, \delta)\omega_1(f^\partial, \delta), \tag{21}$$

any $\delta > 0$.

Proof. We have that

$$\begin{aligned}
&|\mathcal{L}_0^\partial(f,x) - \mathcal{L}_0^\partial(f,y)| \\
&=\left|\int_{\mathbb{R}^d} \ell_0^\partial(f,u)\varphi(x-u)du - \int_{\mathbb{R}^d} \ell_0^\partial(f,u)\varphi(y-u)du\right| \\
&=\left|\int_{\mathbb{R}^d} \ell_0^\partial(f,x-u)\varphi(u)du - \int_{\mathbb{R}^d} \ell_0^\partial(f,y-u)\varphi(u)du\right| \\
&=\left|\int_{\mathbb{R}^d} (\ell_0^\partial(f,x-u) - \ell_0^\partial(f,y-u))\varphi(u)du\right| \\
&\le \int_{\mathbb{R}^d} |\ell_0^\partial(f,x-u) - \ell_0^\partial(f,y-u)|\varphi(u)du \\
&\le \left(\int_{\mathbb{R}^d} \varphi(u)du\right) \cdot \sup_{u\in\mathbb{R}^d} |\ell_0^\partial(f,x-u) - \ell_0^\partial(f,y-u)| \\
&\le 1 \cdot \omega_1(f^\partial, \|x-y\|, \text{ any } x, y \in \mathbb{R}^d.
\end{aligned}$$

I.e., it holds

$$|\mathcal{L}_0^\partial(f,x) - \mathcal{L}_0^\partial(f,y)| \le \omega_1(f^\partial, \|x-y\|), \text{ any } x, y \in \mathbb{R}^d.$$

However

$$\begin{aligned}
&|\mathcal{L}_k^\partial(f,x) - \mathcal{L}_k^\partial(f,y)| \\
&=|2^{k\rho} \cdot (\mathcal{L}_0(f(2^{-k}\cdot)))^\partial(2^k x) - 2^{k\rho} \cdot (\mathcal{L}_0(f(2^{-k}\cdot)))^\partial(2^k y)| \\
&=|(\mathcal{L}_0(2^{k\rho} \cdot (f(2^{-k}\cdot)))^\partial(2^k x) - (\mathcal{L}_0(2^{k\rho} \cdot (f(2^{-k}\cdot))))^\partial(2^k y)| \\
&\le \omega_1((2^{k\rho} \cdot (f(2^{-k}\cdot)))^\partial; 2^k \cdot \|x-y\|) \\
&= \omega_1(f^\partial(2^{-k}\cdot); 2^k \cdot \|x-y\| = \omega_1(f^\partial, \|x-y\|).
\end{aligned}$$

I.e.,

$$|\mathcal{L}_k^{\partial}(f,x) - \mathcal{L}_k^{\partial}(f,y)| \le \omega_1(f^{\partial}, \|x-y\|),$$

all $x, y \in \mathbb{R}^d$.

Optimality of (21) follows.

Theorem 2. *Let $i_1, \dots, i_m \in \{1, \dots, d\}$ be such that $i_1 < \cdots < i_m$, and $j_1, \dots, j_m \in \mathbb{N}$ be such that $\sum_{r=1}^m j_r = \rho$. Consider $g_{j_1}(x) := \frac{x^{j_1+1}}{(j_1+1)!}$, and $g_{j_r}(x) := \frac{x^{j_r}}{j_r!}$, $r = 2, \dots, m$; all $x \in \mathbb{R}$. Denote by $f^{\partial} := \frac{\partial^{\rho} f}{\partial x_{i_1}^{j_1} \dots \partial x_{i_m}^{j_m}}$, for any $f \in C^{(\rho)}(\mathbb{R}^d)$. Denote also by $pr_{i_r} : \mathbb{R}^d \ni (x_1, \dots, x_d) \to x_{i_r}$, $r = 1, \dots, m$ the projection onto the i_r coordinate. For all $x, y, u \in \mathbb{R}^d$ assume that*

$$\begin{aligned}(22)\quad &\left(\ell_0\left(\left(\sum_{r=1}^m g_{j_r} \circ pr_{i_r}\right)(2^{-k}\cdot)\right)\right)^{\partial}(2^k x - u)\\ &-\left(\ell_0\left(\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)(2^{-k}\cdot)\right)\right)^{\partial}(2^k y - u) = 2^{-k\rho}\cdot(x_{i_1} - y_{i_1}).\end{aligned}$$

Then

$$(23)\qquad \omega_1\left(\left(\mathcal{L}_k\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)\right)^{\partial}; \delta\right) = \omega_1\left(\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)^{\partial}; \delta\right),$$

for any $\delta > 0$; $k \in \mathbb{Z}$. That is establishing that (21) is sharp!

Note. The dominant role of the i_1-coordinate in Theorem 2 was chosen without loss of generality and for simplicity. I.e., by defining $g_{j_r}, r = 1, \dots, m$ accordingly, the role of i_1-coordinate in (22) can be taken over by any other i_r-coordinate, $r = 2, \dots, m$.

Proof. Notice that pr_{i_1} is a uniformly continuous function on $\mathbb{R}^d$. Also notice that

$$\mathcal{L}_k^{\partial}(f,x) = 2^{k\rho}\cdot\int_{\mathbb{R}^d}(\ell_0(f(2^{-k}\cdot)))^{\partial}(2^k x - u)\cdot\varphi(u)du,$$

any $x \in \mathbb{R}^d$, true for all $f \in C^{(\rho)}(\mathbb{R}^d)$ (such that $f^{\partial} \in X$).

Here choose $f := \tilde{f} := \prod_{r=1}^m g_{j_r} \circ pr_{i_r} \in C^{(\rho)}(\mathbb{R}^d)$. Furthermore we get that

$$\tilde{f}^{\partial}(x) = \left(\prod_{r=1}^{m} g_{j_r} \circ pr_{i_r}\right)^{\partial}(x) = x_{i_1}.$$

Really we have

$$\begin{aligned}\left(\prod_{r=1}^{m} g_{j_r} \circ pr_{i_r}\right)^{\partial}(x) &= \left(\left(\prod_{r=1}^{m}(g_{j_r} \circ pr_{i_r})\right)(x)\right)^{\partial} \\ &= \prod_{r=1}^{m}((g_{j_r} \circ pr_{i_r})(x))^{(j_r)} \\ &= \left(\frac{(x_{i_1})^{j_1+1}}{(j_1+1)!}\right)^{(j_1)} \cdot \prod_{r=2}^{m}\left(\frac{(x_{i_r})^{j_r}}{j_r!}\right)^{(j_r)} = x_{i_1}.\end{aligned}$$

Obviously $\tilde{f}^{\partial} = (\prod_{r=1}^{m} g_{j_r} \circ pr_{i_r})^{\partial} \in X$, i.e., it is uniformly continuous. Thus

$$\begin{aligned}\mathcal{L}_k^{\partial}(\tilde{f}, x) - \mathcal{L}_k^{\partial}(\tilde{f}, y) =& 2^{k\rho} \cdot \int_{\mathbb{R}^d} [(\ell_0(\tilde{f}(2^{-k}\cdot)))^{\partial}(2^k x - u) \\ & - (\ell_0(\tilde{f}(2^{-k}\cdot)))^{\partial}(2^k y - u)] \cdot \varphi(u)du \\ & \text{(by (5) and (22))} \\ =& x_{i_1} - y_{i_1} = \left(\prod_{r=1}^{m} g_{j_r} \circ pr_{i_r}\right)^{\partial}(x) \\ & - \left(\prod_{r=1}^{m} g_{j_r} \circ pr_{i_r}\right)^{\partial}(y) = \tilde{f}^{\partial}(x) - \tilde{f}^{\partial}(y), \ k \in \mathbb{Z}.\end{aligned}$$

That is,

$$\mathcal{L}_k^{\partial}(\tilde{f}, x) - \mathcal{L}_k^{\partial}(\tilde{f}, y) = \tilde{f}^{\partial}(x) - \tilde{f}^{\partial}(y),$$

all $x, y \in \mathbb{R}^d, k \in \mathbb{Z}$, establishing (23).

The convergence of $\frac{\partial^{\rho}\mathcal{L}_k}{\partial x_{i_1}^{j_1} \cdots \partial x_{i_m}^{j_m}}$ $(\sum_{r=1}^{m} j_r = \rho)$ to $\frac{\partial^{\rho}}{\partial x_{i_1}^{j_1} \cdots \partial x_{i_m}^{j_m}}$ operator, as $k \to +\infty$, with rates is studied in the following.

Theorem 3. *For $f \in C^{(\rho)}(\mathbb{R}^d), d \geq 1$, such that $f^{\partial} \in C_U(\mathbb{R}^d)$, under the assumption* (14), *it holds*

$$|\mathcal{L}_k^{\partial}(f, x) - f^{\partial}(x)| \leq \omega_{1,\infty}\left(f^{\partial}, \frac{ma+n}{2^{k+r}}\right), \tag{24}$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}_+, k, r \in \mathbb{Z}$, *and* $a := \max(a_i)$, $i = 1, \ldots, d$.

Proof. We observe that

$$
\begin{aligned}
&|\mathcal{L}_k^{\partial}(f,x) - f^{\partial}(x)| \\
&= \left| 2^{k\rho} \cdot (\mathcal{L}_0(f(2^{-k}\cdot)))^{\partial}(2^k x) - f^{\partial}(2^{-k}(2^k x)) \right| \\
&= \left| 2^{k\rho} \cdot \int_{\mathbb{R}^d} (\ell_0(f(2^{-k}\cdot)))^{\partial}(u)\varphi(2^k x - u)du \right. \\
&\quad \left. - \int_{\mathbb{R}^d} f^{\partial}(2^{-k}(2^k x)) \cdot \varphi(2^k x - u)du \right| \\
&= \left| \int_{\mathbb{R}^d} (\ell_0(2^{k\rho} \cdot f(2^{-k})))^{\partial}(u) \cdot \varphi(2^k x - u)du \right. \\
&\quad \left. - \int_{\mathbb{R}^d} f^{\partial}(2^{-k}(2^k x)) \cdot \varphi(2^k x - u)du \right| \\
&= \left| \int_{\mathbb{R}^d} [(\ell_0(2^{k\rho} \cdot f(2^{-K}\cdot)))^{\partial}(u) - f^{\partial}(2^{-k}(2^k x))] \cdot \varphi(2^k x - u)du \right| \\
&\le \int_{2^k x - \vec{a}}^{2^k x + \vec{a}} |(\ell_0(2^{k\rho} \cdot f(2^{-K}\cdot)))^{\partial}(u) - f^{\partial}(2^{-k}(2^k x))| \cdot \varphi(2^k x - u)du \\
&\le \sup_{u \in X_{i=1}^d [2^k x_i - a_i 2^k x_i + a_i]} |(\ell_0(2^{k\rho} \cdot f(2^{-k}\cdot)))^{\partial}(u) - f^{\partial}(2^{-k}(2^k x))| \\
&\quad \cdot \int \cdots \int_{2^k x_i - a_i \text{ all } i=1,\ldots,d}^{2^k x_i + a_i} \cdots \int \varphi(2^k x - u)du.
\end{aligned}
$$

I.e., (4)

$$
\begin{aligned}
&|\mathcal{L}_k^{\partial}(f,x) - f^{\partial}(x)| \\
&\le \sup_{u \in X_{i=1}^d [2^k x_i - a_i 2^k x_i + a_i]} |(\ell_0(2^{k\rho} \cdot f(2^{-k}\cdot)))^{\partial}(u) - f^{\partial}(2^{-k}(2^k x))| =: \otimes.
\end{aligned}
$$

Consider $g := f(2^{-k}\cdot) \in C^{(\rho)}(\mathbb{R}^d)$. It holds that $g^{\partial} \in X$. We have

$$g^{\partial} = 2^{-k\rho} \cdot f^{\partial}(2^{-k}\cdot)$$

and

$$2^{k\rho} \cdot g^{\partial} = f^{\partial}(2^{-k}\cdot) \in X.$$

Set $y_i := 2^k x_i$. Therefore

$$\bigotimes = \sup_{\substack{\text{all } u\in\mathbb{R}^d:\\ u-2^kx\in X_{i=1}^d[-a_i,a_i]}} |(\ell_0(2^{k\rho}\cdot g))^\partial(u)-(2^{k\rho}\cdot g)^\partial(2^kx)|$$

$$= \sup_{\substack{\text{all } u\in\mathbb{R}^d:\\ |u_i-2^kx_i|\le a_i}} |(\ell_0(2^{k\rho}\cdot g))^\partial(u)-(2^{k\rho}\cdot g)^\partial(2^kx)|$$

$$\le \sup_{\substack{\text{all } u\in\mathbb{R}^d:\\ |u_i-y_i|\le \max(a_k)=:a, i=1,\dots,d}} |(\ell_0(2^{k\rho}\cdot g))^\partial(u)-(2^{k\rho}\cdot g)^\partial(y)|$$

$$= \sup_{\substack{\text{all } u\in\mathbb{R}^d:\\ \|u-y\|_\infty\le a}} |(\ell_0(2^{k\rho}\cdot g))^\partial(u)-(2^{k\rho}\cdot g)^\partial(y)|$$

$$\le \omega_{1,\infty}\left((2^{k\rho}\cdot g)^\partial, \frac{ma+n}{2^r}\right) \text{ (by assumption (14))}$$

$$= \omega_{1,\infty}\left(f^\partial(2^{-k}\cdot), \frac{ma+n}{2^r}\right)$$

$$= \omega_{1,\infty}\left(f^\partial, \frac{ma+n}{2^{k+r}}\right).$$

We have established (24) that

$$|\mathcal{L}_k^\partial(f,x)-f^\partial(x)|\le \omega_{1,\infty}\left(f^\partial, \frac{ma+n}{2^{k+r}}\right).$$

3. Applications. In the following we present four examples of multivariate operators

$$\mathcal{L}_k^\partial := \frac{\partial^\rho \mathcal{L}_k}{\partial x_{i_1}^{j_1}\dots\partial x_{i_m}^{j_m}}, \quad \sum_{r=1}^m j_r=\rho\in\mathbb{N},\ k\in\mathbb{Z},$$

where the associated multivariate operator ℓ_k is specified. These operators fulfill the assumptions of all main results in §2. The basic function φ here will be as general as in §2 and with the same properties as there. However only for the $(A_k)_{k\in\dot{Z}}$ operators, φ will be also an even continuous function, $\varphi(-x)=\varphi(x)$, all $x\in\mathbb{R}^d$. These multivariate operators appeared for the first time in [1].

Next we introduce these operators, all defined for each $k\in\mathbb{Z}$.

(i)

(25) $$(A_kf)(x) := \int_{\mathbb{R}^d} r_k^f(u)\varphi(2^kx-u)du, \text{ all } x\in\mathbb{R}^d,$$

where

$$r_k^f(u) := 2^{kd} \cdot \int_{\mathbb{R}^d} f(t)\varphi(2^k t - u)dt \tag{26}$$

is continuous in u (by Lemma 1 of [1]). I.e., here

$$\ell_k(f,u) = r_k^f(u), \text{ all } u \in \mathbb{R}^d. \tag{27}$$

(ii)

$$(B_k)f(x) := \int_{\mathbb{R}^d} f\left(\frac{u}{2^k}\right)\varphi(2^k x - u)du, \text{ all } x \in \mathbb{R}^d, \tag{28}$$

i.e., here

$$\ell_k(f,u) = f\left(\frac{u}{2^k}\right), \text{ all } u \in \mathbb{R}^d, \tag{29}$$

is continuous, in u.

(iii)

$$(L_k f)(x) := \int_{\mathbb{R}^d} c_k^f(u)\varphi(2^k x - u)du, \text{ all } x \in \mathbb{R}^d, \tag{30}$$

where

$$c_k^f(u) := 2^{kd} \cdot \int \cdots \int_{2^{-k}_{u_i}}^{2^{-k}_{u_{i+1}}} \cdots \int f(t)dt, \tag{31}$$

is (by Lemma 1 of [1]) continuous in u, i.e., here

$$\ell_k(f,u) = c_k^f(u), \text{ all } u \in \mathbb{R}^d. \tag{32}$$

(iv)

$$(\Gamma_k f)(x) := \int_{\mathbb{R}^d} \gamma_k^f(u)\varphi(2^k x - u)du, \text{ all } x \in \mathbb{R}^d, \tag{33}$$

$$\gamma_k^f(u) := \sum_{j_1=0}^{n_1} \cdots \sum_{j_d=0}^{n_d} w_{j_1,\ldots,j_d} \cdot f\left(\frac{u_1}{2^k} + \frac{j_1}{2^k \cdot n_1}, \ldots, \frac{u_d}{2^k} + \frac{j_d}{2^k \cdot n_d}\right), \tag{34}$$

$$(n_1,\ldots,n_d) \in \mathbb{N}^d, w_{j_1,\ldots,j_d} \geq 0, \sum_{j_1=0}^{n_1} \cdots \sum_{j_d=0}^{n_d} w_{j_1,\ldots,j_d} = 1.$$

I.e., here

$$\ell_k(f,u) = \gamma_k^f(u) \tag{35}$$

is continuous in $u \in \mathbb{R}^d$.

It is clear that (1) and (7) are fulfilled by the above examples. Notice also that the above l_0's (see (26), (29), (31) and (34)) map $C^{(\rho)}(\mathbb{R}^d)$ into itself.

Proposition 2. *The operators* A_k^∂, B_k^∂, L_k^∂, Γ_k^∂, $k \in \mathbb{Z}$ *are shift invariant operators.*

Proof. For each of the above operators we only need to prove (19), i.e., that

$$\ell_0^\partial(f(2^{-k}\cdot+\alpha); 2^k u) = \ell_0^\partial(f(2^{-k}\cdot); 2^k(u+\alpha)),$$

all $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}^d$ fixed, all $u \in \mathbb{R}^d$; any $f \in C^{(\rho)}(\mathbb{R}^d)$; $\rho \in \mathbb{N}$ be given.

i) A_k**-operators** (φ-continuous and even): Notice here that

$$\ell_0(f,u) = \int_{\mathbb{R}^d} f(u-t)\varphi(t)dt, \text{ all } u \in \mathbb{R}^d.$$

Since $f \in C^{(\rho)}(\mathbb{R}^d)$ (by the result of H. Bauer [2], pp. 103-104) we get that

$$\ell_0^\partial(f,u) = \int_{\mathbb{R}^d} f^\partial(u-t)\varphi(t)dt, \text{ all } u \in \mathbb{R}^d.$$

Thus

$$\begin{aligned}\ell_0^\partial(f(2^{-k}\cdot+\alpha); x) &= \int_{\mathbb{R}^d} (f(2^{-k}\cdot(x-t)+\alpha))^\partial \cdot \varphi(t)dt \\ &= 2^{-k\rho}\cdot \int_{\mathbb{R}^d} f^\partial(2^{-k}\cdot(x-t)+\alpha)\cdot\varphi(t)dt.\end{aligned}$$

Hence

$$\begin{aligned}\ell_0^{\partial}(f(2^{-k}\cdot+\alpha);2^ku)&=2^{-k\rho}\cdot\int_{\mathbb{R}^d}f^{\partial}(2^{-k}\cdot(2^ku-t)+\alpha)\cdot\varphi(t)dt\\&=2^{-k\rho}\cdot\int_{\mathbb{R}^d}f^{\partial}(u-2^{-k}t+\alpha)\cdot\varphi(t)dt\\&=2^{-k\rho}\cdot\int_{\mathbb{R}^d}f^{\partial}(2^{-k}\cdot 2^k\cdot(u+\alpha)-2^{-k}\cdot t)\varphi(t)dt\\&=2^{-k\rho}\cdot\int_{\mathbb{R}^d}f^{\partial}(2^{-k}\cdot(2^k\cdot(u+\alpha)-t))\cdot\varphi(t)dt\\&=\ell_0^{\partial}(f(2^{-k}\cdot);2^k\cdot(u+\alpha)),\ \text{establishing (19).}\end{aligned}$$

The last equality is true by

$$\ell_0^{\partial}(f(2^{-k}\cdot);x)=2^{-k\rho}\cdot\int_{\mathbb{R}^d}f^{\partial}(2^{-k}(x-t))\cdot\varphi(t)dt.$$

ii) B_k**-operators:** here $\ell_0(f,u)=f(u)$, that is $\ell_0^{\partial}(f,u)=f^{\partial}(u)$, all $u\in\mathbb{R}^d$. Therefore

$$\ell_0^{\partial}(f(2^{-k}\cdot+\alpha))(u)=2^{-k\rho}\cdot f^{\partial}(2^{-k}u+\alpha).$$

Hence

$$\begin{aligned}\ell_0^{\partial}(f(2^{-k}\cdot+\alpha))(2^ku)&=2^{-k\rho}\cdot f^{\partial}(2^{-k}2^ku+\alpha)\\&=2^{-k\rho}\cdot f^{\partial}(u+\alpha)=2^{-k\rho}\cdot f^{\partial}(2^{-k}(2^k(u+\alpha)))\\&=\ell_0^{\partial}(f(2^{-k}\cdot);2^k(u+\alpha)),\end{aligned}$$

Proving (19).

iii) L_k**-operators:** Here we have that

$$\ell_0(f,u)=\int\cdots\int_{u_i}^{u_i+1}f(t)dt=\int\cdots\int_0^1\cdots\int f(t+u)dt,$$

i.e.,

$$\ell_0(f,u)=\int_{\vec{0}}^{\vec{1}}f(t+u)dt,\ \text{all } u\in\mathbb{R}^d.$$

Hence

$$\ell_0^{\partial}(f,u)=\int_{\vec{0}}^{\vec{1}}f^{\partial}(t+u)dt.$$

Furthermore

$$\ell_0^\partial(f(2^{-k}\cdot+\alpha);x)=2^{-k\rho}\cdot\int_{\vec{0}}^{\vec{1}} f^\partial(2^{-k}\cdot(t+x)+\alpha)dt,\ \text{all } x\in\mathbb{R}^d.$$

Thus

$$\begin{aligned}&\ell_0^\partial(f(2^{-k}\cdot+\alpha);2^k u)\\&=2^{-k\rho}\cdot\int_{\vec{0}}^{\vec{1}} f^\partial(2^{-k}\cdot(t+2^k u)+\alpha)dt\\&=2^{-k\rho}\cdot\int_{\vec{0}}^{\vec{1}} f^\partial(2^{-k}\cdot(2^k\cdot(u+\alpha)+t))dt\\&=\ell_0^\partial(f(2^{-k}\cdot);2^k\cdot(u+\alpha)),\ \text{establishing (19).}\end{aligned}$$

The last equality comes from

$$\ell_0^\partial(f(2^{-k}\cdot);u)=2^{-k\rho}\cdot\int_{\vec{0}}^{\vec{1}} f^\partial(2^{-k}\cdot(u+t))dt.$$

iv) **Γ_k-operators:** Here

$$(\ell_0 f)(u)=\sum_{j_1=0}^{n_1}\cdots\sum_{j_d=0}^{n_d} w_{j_1,\ldots,j_d}\cdot f\left(u_1+\frac{j_1}{n_1},\cdots,u_d+\frac{j_d}{n_d}\right),$$

$$(n_1,\ldots,n_d)\in\mathbb{N}^d, w_{j_1,\ldots,j_d}\geq 0,\ \sum_{j_1=0}^{n_1}\cdots\sum_{j_d=0}^{n_d} w_{j_1,\ldots,j_d}=1.$$

Thus

$$\ell_0^\partial(f,u)=\sum_{j_1=0}^{n_1}\cdots\sum_{j_d=0}^{n_d} w_{j_1,\ldots,j_d}\cdot f^\partial\left(u_1+\frac{j_1}{n_1},\cdots,u_d+\frac{j_d}{n_d}\right),\ \text{any } u\in\mathbb{R}^d.$$

We observe that

$$\begin{aligned}&\ell_0^\partial(f(2^{-k}\cdot+\alpha))(2^k u)\\&=\sum_{j_1=0}^{n_1}\cdots\sum_{j_d=0}^{n_d} w_{j_1,\ldots,j_d}\cdot 2^{-k\rho}\cdot f^\partial\left(2^{-k}\cdot\left(2^k\cdot u+\frac{\vec{j}}{\vec{n}}\right)+\alpha\right)\\&=2^{-k\rho}\cdot\sum_{j_1=0}^{n_1}\cdots\sum_{j_d=0}^{n_d} w_{j_1,\ldots,j_d}\cdot f^\partial\left(2^{-k}\cdot\left(2^k\cdot(u+\alpha)+\frac{\vec{j}}{\vec{n}}\right)\right)\\&=\ell_0^\partial(f(2^{-k}\cdot);2^k\cdot(u+\alpha)),\end{aligned}$$

proving again (19). The last equality is true by

$$\ell_0^\partial(f(2^{-k}\cdot);u)=2^{-k\rho}\cdot\sum_{j_1=0}^{n_1}\cdots\sum_{j_d=0}^{n_d}w_{j_1,\dots,j_d}\cdot f^\partial\left(2^{-k}\cdot\left(u+\frac{\vec{j}}{\vec{n}}\right)\right).$$

In the following we show that A_k^∂, B_k^∂, L_k^∂ and Γ_k^∂ fulfill the property of global smoothness preservation.

Theorem 4. *Let $f\in C^{(\rho)}(\mathbb{R}^d)$ such that $f^\partial\in C_U(\mathbb{R}^d)$. Then*

$$\omega_1((A_kf)^\partial,\delta)\le\omega_1(f^\partial,\delta),\tag{36}$$

$$\omega_1((B_kf)^\partial,\delta)\le\omega_1(f^\partial,\delta),\tag{37}$$

$$\omega_1((L_kf)^\partial,\delta)\le\omega_1(f^\partial,\delta),\tag{38}$$

and

$$\omega_1((\Gamma_kf)^\partial,\delta)\le\omega_1(f^\partial,\delta),\tag{39}$$

any $\delta>0$; $k\in\mathbb{Z}$.

Proof. It will be enough to check (20) for each particular case. I.e., it is enough to prove for each operator that

$$|\ell_0^\partial(f,x-u)-\ell_0^\partial(f,y-u)|\le\omega_1(f^\partial,\|x-y\|),\ \text{all } x,y,u\in\mathbb{R}^d.$$

i) **A_k-operators:** Here we have that

$$\ell_0^\partial(f,u)=\int_{\mathbb{R}^d}f^\partial(u-t)\varphi(t)dt,\ \text{all } u\in\mathbb{R}^d.$$

Thus

$$\begin{aligned}&|\ell_0^\partial(f,x-u)-\ell_0^\partial(f,y-u)|\\&=\left|\int_{\mathbb{R}^d}(f^\partial(x-u-t)-f^\partial(y-u-t))\cdot\varphi(t)dt\right|\\&\le\int_{\mathbb{R}^d}|f^\partial(x-u-t)-f^\partial(y-u-t)|\cdot\varphi(t)dt\\&\le\omega_1(f^\partial,\|x-y\|)\cdot\int_{\mathbb{R}^d}\varphi(t)dt\\&=\omega_1(f^\partial,\|x-y\|)\cdot 1\qquad\text{(by (5))}.\end{aligned}$$

That is, (20) is established.

ii) B_k-**operators:** Here $\ell_0^{\partial}(f,u) = f^{\partial}(u)$, all $u \in \mathbb{R}^d$. Hence

$$|\ell_0^{\partial}(f, x-u) - \ell_0^{\partial}(f, y-u)| = |f^{\partial}(x-u) - f^{\partial}(y-u)| \le \omega_1(f^{\partial}, \|x-y\|).$$

Thus (20) is proved to be true.

iii) L_k-**operators:** Here we obtain

$$\ell_0^{\partial}(f,x) = \int_{\vec{0}}^{\vec{1}} f^{\partial}(u+t)dt, \text{ all } u \in \mathbb{R}^d.$$

Therefore

$$\begin{aligned}|\ell_0^{\partial}(f, x-u) - \ell_0^{\partial}(f, y-u)| &= \left| \int_{\vec{0}}^{\vec{1}} (f^{\partial}(x-u+t) - f^{\partial}(y-u+t))dt \right| \\ &\le \int_{\vec{0}}^{\vec{1}} |f^{\partial}(x-u+t) - f^{\partial}(y-u+t)|dt \\ &\le \int_{\vec{0}}^{\vec{1}} \omega_1(f^{\partial}, \|x-y\|)dt = \omega_1(f^{\partial}, \|x-y\|).\end{aligned}$$

I.e., (20) is again established.

iv) Γ_k-**operators:** Here we get that

$$\ell_0^{\partial}(f,u) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} w_{j_1,\ldots,j_d} \cdot f^{\partial}\left(u + \frac{\vec{j}}{\vec{n}}\right), \text{ all } u \in \mathbb{R}^d.$$

So that

$$\begin{aligned}&|\ell_0^{\partial}(f, x-u) - \ell_0^{\partial}(f, y-u)| \\ &= \left| \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} w_{j_1,\ldots,j_d} \cdot \left(f^{\partial}\left(x-u+\frac{\vec{j}}{\vec{n}}\right) - f^{\partial}\left(y-u+\frac{\vec{j}}{\vec{n}}\right)\right)\right| \\ &\le \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} w_{j_1,\ldots,j_d} \cdot \left| f^{\partial}\left(x-u+\frac{\vec{j}}{\vec{n}}\right) - f^{\partial}\left(y-u+\frac{\vec{j}}{\vec{n}}\right)\right| \\ &\le \left(\sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} w_{j_1,\ldots,j_d}\right) \cdot \omega_1(f^{\partial}, \|x-y\|) \\ &= 1 \cdot \omega_1(f^{\partial}, \|x-y\|), \qquad \text{by (34).}\end{aligned}$$

That is, one more time (20) is verified.

Theorem 5. *Inequalities* (36), (37), (38) *and* (39) *are sharp, in the sense that they hold as equalities when*

$$f=\prod_{r=1}^{m} g_{j_r}\circ pr_{i_r} \text{ (where } g_{j_r},\, pr_{i_r} \text{ as in Theore 2)}$$

i.e., they are attained.

Proof. Here $i_1,\dots,i_m\in\{1,\dots,d\}: i_1<\cdots<i_m$, $j_1,\dots,j_m\in\mathbb{N}$ such that $\sum_{r=1}^m j_r=\rho$. Consider $g_{j_1}(x):=\frac{x^{j_1+1}}{(j_1+1)!}$ and $g_{j_r}(x):=\frac{x^{j_r}}{j_n}$, $r=2,\dots,m$, all $x\in\mathbb{R}$. Also denote by $pr_{i_r}:\mathbb{R}^d\ni(x_1,\dots,x_d)\to x_{i_r}$, $r=1,\dots,m$, the projection onto the i_r coordinate. Here we only need to prove (22) for each operator. Then by (23) the theorem is true.

i) A_k**-operators:** Here we have

$$\begin{aligned}(\ell_0 f)^{\partial}(u)=\ell_0^{\partial}(f,u)&=\int_{\mathbb{R}^d} f^{\partial}(u-t)\varphi(t)dt\\ &=\int_{\mathbb{R}^d}(f(u-t))^{\partial}\varphi(t)dt.\end{aligned}$$

That is,

$$\begin{aligned}\underline{0}(u):&=\left(\ell_0\left(\prod_{r=1}^m g_{j_r}(pr_{i_r}(2^{-k}\cdot))\right)\right)^{\partial}(u)\\
&=\int_{\mathbb{R}^d}\left[\left(\prod_{r=1}^m g_{j_r}(pr_{i_r}(2^{-k}\cdot))\right)(u-t)\right]^{\partial}\cdot\varphi(t)dt\\
&=\int_{\mathbb{R}^d}\left[\prod_{r=1}^m g_{j_r}(2^{-k}\cdot(u_{i_r}-t_{i_r}))\right]^{\partial}\cdot\varphi(t)dt\\
&=\int_{\mathbb{R}^d}\Bigg[2^{-k(j_1+1)}\cdot\frac{(u_{i_1}-t_{i_1})^{j_1+1}}{(j_1+1)!}\cdot 2^{-kj_2}\cdot\frac{(u_{i_2}-t_{i_2})^{j_2}}{j_2!}\\
&\qquad\cdots 2^{-kj_m}\cdot\frac{(u_{i_m}-t_{i_m})^{j_m}}{j_m!}\Bigg]^{\partial}\cdot\varphi(t)dt\\
&=2^{-k}\cdot 2^{-k\rho}\cdot\int_{\mathbb{R}^d}(u_{i_1}-t_{i_1})\cdot\varphi(t)dt.\end{aligned}$$

I.e.,

$$\underline{0}(u)=2^{-k(\rho+1)}\cdot\int_{\mathbb{R}^d}(u_{i_1}-t_{i_1})\varphi(t)dt,\ \text{all } u\in\mathbb{R}^d.$$

Thus

$$\begin{aligned}\underline{0}(2^k x - u) - \underline{0}(2^k y - u) =& 2^{-k(\rho+1)} \int_{\mathbb{R}^d} [(2^k x_{i_1} - u_{i_1} - t_{i_1}) \\ & - (2^k y_{i_1} - u_{i_1} - t_{i_1})] \cdot \varphi(t) dt \\ =& 2^{-k\rho} \cdot (x_{i_1} - y_{i_1}) \cdot \int_{\mathbb{R}^d} \varphi(t) dt \\ =& 2^{-k\rho} \cdot (x_{i_1} - y_{i_1}) \cdot 1 \qquad \text{by (5)).}\end{aligned}$$

Hence

$$\underline{0}(2^k x - u) - \underline{0}(2^k y - u) = 2^{-k\rho} \cdot (x_{i_1} - y_{i_1}), \qquad \text{all } x, y \in \mathbb{R}^d,$$

proving (22).

ii) B_k**-operators:** Here $\ell_0^\partial(f, u) = f^\partial(u)$, all $u \in \mathbb{R}^d$. Thus

$$\begin{aligned}\underline{0} :=& \left(\ell_0\left(\prod_{r=1}^m g_{j_r}(pr_{i_r}(2^{-k}\cdot))\right)\right)^\partial (u) \\ =& \left[\left(\prod_{r=1}^m g_{j_r}(pr_{i_r}(2^{-k}\cdot))\right)(u)\right]^\partial \\ =& \left[\prod_{r=1}^m g_{j_r}(2^{-k} \cdot u_{i_r})\right]^\partial = 2^{-k(\rho+1)} \cdot u_{i_1}.\end{aligned}$$

I.e.,

$$\underline{0}(u) = 2^{-k(\rho+1)} \cdot u_{i_1}, \qquad \text{all } u \in \mathbb{R}^d.$$

Therefore

$$\begin{aligned}\underline{0}(2^k x - u) - \underline{0}(2^k y - u) &= 2^{-k(\rho+1)} \cdot ((2^k x_{i_1} - u_{i_1}) - (2^k y_{i_1} - u_{i_1})) \\ &= 2^{-k\rho} \cdot (x_{i_1} - y_{i_1}),\end{aligned}$$

establishing (22).

iii) L_k**-operators:** Here we have

$$\ell_0^\partial(f, u) = \int_{\vec{0}}^{\vec{1}} (f(u+t))^\partial dt, \qquad \text{all } u \in \mathbb{R}^d.$$

We observe that

$$\begin{aligned}\underline{0}(u) :&= \left[\ell_0\left(\prod_{r=1}^{m} g_{j_r}(pr_{i_r}(2^{-k}\cdot))\right)(u)\right]^{\partial}\\ &= \int_{\vec{0}}^{\vec{1}}\left[\left(\prod_{r=1}^{m} g_{j_r}(pr_{i_r}(2^{-k}\cdot))\right)(u+t)\right]^{\partial} dt\\ &= \int_{\vec{0}}^{\vec{1}}\left[\prod_{r=1}^{m} g_{j_r}(2^{-k}\cdot(u_{i_r}+t_{i_r}))\right]^{\partial} dt\\ &= 2^{-k(\rho+1)}\cdot\int_{\vec{0}}^{\vec{1}}(u_{i_1}+t_{i_1})dt.\end{aligned}$$

Thus

$$\underline{0}(u) = 2^{-k(\rho+1)}\cdot\int_{\vec{0}}^{\vec{1}}(u_{i_1}+t_{i_1})dt, \qquad \text{all } u\in\mathbb{R}^d.$$

Consequently,

$$\begin{aligned}\underline{0}(2^kx-u)-\underline{0}(2^ky-u) =&2^{-k(\rho+1)}\cdot\int_{\vec{0}}^{\vec{1}}[(2^kx_{i_1}-u_{i_1}+t_{i_1})\\ &-(2^ky_{i_1}-u_{i_1}+t_{i_1})]\cdot dt\\ =&2^{-k\rho}\cdot(x_{i_1}-y_{i_1}).\end{aligned}$$

One more time (22) was proved true.

iv) Γ_k-**operators:** Here we get that

$$(\ell_0)^{\partial}(u) = \ell_0^{\partial}(f,u) = \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\cdot\left(f\left(u+\frac{\vec{j}}{\vec{n}}\right)\right)^{\partial}, \qquad \text{all } u\in\mathbb{R}^d.$$

Consider

$$\begin{aligned}\underline{0}(u) :&= \left[\ell_0\left(\prod_{r=1}^{m} g_{j_r}(pr_{i_r}(2^{-k}\cdot))\right)(u)\right]^{\partial}\\ &= \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\cdot\left[\prod_{r=1}^{m}(g_{j_r}(pr_{i_r}(2^{-k}\cdot)))\left(u+\frac{\vec{j}}{\vec{n}}\right)\right]^{\partial}\\ &= \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\cdot\left[\prod_{r=1}^{m} g_{j_r}\left(2^{-k}\cdot\left(u_{i_r}+\frac{j_{i_r}}{n_{i_r}}\right)\right)\right]^{\partial}\\ &= 2^{-k(\rho+1)}\cdot\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\cdot\left(u_{i_1}+\frac{j_{i_1}}{n_{i_1}}\right).\end{aligned}$$

That is,

$$\underline{0}(u) = 2^{-k(\rho+1)} \cdot \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}} \cdot \left(u_{i_1} + \frac{j_{i_1}}{n_{i_1}}\right), \qquad \text{all } u \in \mathbb{R}^d.$$

Therefore

$$\begin{aligned}\underline{0}(2^k x - u) - \underline{0}(2^k y - u) &= 2^{-k(\rho+1)} \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}} \\ &\cdot \left[\left(2^k x_{i_1} - u_{i_1} + \frac{j_{i_1}}{n_{i_1}}\right) - \left(2^k y_{i_1} - u_{i_1} + \frac{j_{i_1}}{n_{i_1}}\right)\right] \\ &= 2^{-k\rho} \cdot \left(\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\right) \cdot (x_{i_1} - y_{i_1}) = 2^{-k\rho} \cdot 1 \cdot (x_{i_1} - y_{i_1}).\end{aligned}$$

Hence (22) is true again.

The operators

$$\frac{\partial^\rho A_k}{\partial x_{i_1}^{j_1} \dots \partial x_{i_m}^{j_m}}, \frac{\partial^\rho B_k}{\partial x_{i_1}^{j_1} \dots \partial x_{i_m}^{j_m}}, \frac{\partial^\rho L_k}{\partial x_{i_1}^{j_1} \dots \partial x_{i_m}^{j_m}}, \frac{\partial^\rho \Gamma_k}{\partial x_{i_1}^{j_1} \dots \partial x_{i_m}^{j_m}}$$

converge as $k \to +\infty$, to $\frac{\partial^\rho}{\partial x_{i_1}^{j_1} \dots \partial x_{i_m}^{j_m}}$, with rates as follows.

Theorem 6. *Let $f \in C^{(\rho)}(\mathbb{R}^d)$, $\rho \in \mathbb{N}$ be fixed, such that $f^\partial \in X$. Also call $a := \max(a_1, \dots, a_d) > 0$. Then*

$$|A_k^\partial(f,x) - f^\partial(x)| \le \omega_{1,\infty}\left(f^\partial, \frac{a}{2^{k-1}}\right), \tag{40}$$

$$|B_k^\partial(f,x) - f^\partial(x)| \le \omega_{1,\infty}\left(f^\partial, \frac{a}{2^k}\right), \tag{41}$$

$$|L_k^\partial(f,x) - f^\partial(x)| \le \omega_{1,\infty}\left(f^\partial, \frac{a+1}{2^k}\right), \tag{42}$$

and

$$|\Gamma_k^\partial(f,x) - f^\partial(x)| \le \omega_{1,\infty}\left(f^\partial, \frac{a+1}{2^k}\right), \tag{43}$$

any $k \in \mathbb{Z}$.

Proof. Here we only need to verify assumption (14) when $\alpha = a$, for each of the operators. I.e., we need to prove for the different ℓ_0's that

$$\sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} |\ell_0^\partial(f,u)-f^\partial(y)| \le \omega_{1,\infty}\big(f^\partial, \frac{ma+n}{2^r}\big),$$

for some $m\in\mathbb{N}, n\in\mathbb{Z}_+, r\in\mathbb{Z}$, where $\omega_{1,\infty}$ is the first modulus of continuity with respect to $\|\cdot\|_\infty$.

i) A_k**-operators:** Here we have

$$\ell_0^\partial(f,u)=\int_{\mathbb{R}^d} f^\partial(u-t)\varphi(t)dt, \qquad \text{all } u\in\mathbb{R}^d.$$

Thus

$$\begin{aligned}
|\ell_0^\partial(f,u)-f^\partial(y)| &= \left|\int_{\mathbb{R}^d}(f^\partial(u-t)-f^\partial(y))\cdot\varphi(t)dt\right|\\
&= \left|\int_{\mathbb{R}^d}(f^\partial(t)-f^\partial(y))\cdot\varphi(u-t)dt\right|\\
&\le \int_{\mathbb{R}^d}|(f^\partial(t)-f^\partial(y)|\cdot\varphi(u-t)dt\\
&\le \int_{\mathbb{R}^d}\omega_{1,\infty}(f^\partial,\|t-y\|_\infty)\cdot\varphi(u-t)dt\\
&\quad \text{(here we have}\\
&\quad \|t-y\|_\infty\le\|t-u\|_\infty+\|u-y\|_\infty\le a+a=2a)\\
&\le \int_{\mathbb{R}^d}\omega_{1,\infty}(f^\partial,2a)\cdot\varphi(u-t)dt\\
&=\omega_{1,\infty}(f^\partial,2a)\cdot 1, \text{ by (4).}
\end{aligned}$$

That is,

$$\sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} |\ell_0^\partial(f,u)-f^\partial(y)| \le \omega_{1,\infty}(f^\partial,2a),$$

proving (14).

ii) B_k**-operators:** Here $\ell_0^\partial(f,u)=f^\partial(u)$. Then

$$\begin{aligned}
\sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} |\ell_0^\partial(f,u)-f^\partial(y)| &= \sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} |f^\partial(u)-f^\partial(y)|\\
&\le \sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} \omega_{1,\infty}(f^\partial,\|u-y\|_\infty)\le\omega_{1,\infty}(f^\partial,a).
\end{aligned}$$

Thus (14) is verified again.

iii) L_k**-operators:** Here we get

$$\ell_0^\partial(f,u)=\int_{\vec{0}}^{\vec{1}} f^\partial(t+u)dt,\ \text{all } u\in\mathbb{R}^d.$$

Therefore

$$\begin{aligned}
&\sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} |\ell_0^\partial(f,u)-f^\partial(y)| \\
&\qquad = \sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} \left|\int_{\vec{0}}^{\vec{1}} f^\partial(t+u)dt-f^\partial(y)\right| \\
&\qquad = \sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} \left|\int_{\vec{0}}^{\vec{1}} (f^\partial(t+u)-f^\partial(y))dt\right| \\
&\qquad \le \sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} \int_{\vec{0}}^{\vec{1}} |f^\partial(t+u)-f^\partial(y)|dt \\
&\qquad \le \sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} \int_{\vec{0}}^{\vec{1}} \omega_{1,\infty}(f^\partial;\|t+u-y\|_\infty)dt \\
&\qquad \le \int_{\vec{0}}^{\vec{1}} \omega_{1,\infty}(f^\partial;\|t\|_\infty+a)dt \\
&\qquad \le \omega_{1,\infty}(f^\partial,1+a).
\end{aligned}$$

Hence

$$\sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} |\ell_0^\partial(f,u)-f^\partial(y)|\le\omega_{1,\infty}(f^\partial,a+1),$$

proving (14).

iv) Γ_k-**operators:** Here we have that

$$\ell_0^\partial(f,u)=\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\cdot f^\partial\left(u+\frac{\vec{j}}{\vec{n}}\right),\ \text{all } u\in\mathbb{R}^d.$$

Thus

$$\begin{aligned}
&\sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} |\ell_0^\partial(f,u)-f^\partial(y)| \\
&\qquad = \sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} \left|\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\cdot\left(f^\partial\left(u+\frac{\vec{j}}{\vec{n}}\right)-f^\partial(y)\right)\right|
\end{aligned}$$

$$\le \sup_{\substack{u,y\in\mathbb{R}^d \\ \|u-y\|_\infty \le a}} \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}} \cdot \left| f^{\partial}\left(u+\frac{\vec{j}}{\vec{n}}\right) - f^{\partial}(y) \right|$$

$$\le \sup_{\substack{u,y\in\mathbb{R}^d \\ \|u-y\|_\infty \le a}} \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}} \cdot \omega_{1,\infty}\left(f^{\partial}; \left\| u+\frac{\vec{j}}{\vec{n}} - y \right\|_\infty\right)$$

$$\le \sup_{\substack{u,y\in\mathbb{R}^d \\ \|u-y\|_\infty \le a}} \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}} \cdot \omega_{1,\infty}\left(f^{\partial}; \left\| \frac{\vec{j}}{\vec{n}} \right\|_\infty + \|u-y\|_\infty\right)$$

$$\le \left(\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\right) \cdot \omega_{1,\infty}(f^{\partial}, 1+a)$$

$$= 1 \cdot \omega_{1,\infty}(f^{\partial}, 1+a), \text{ by (34).}$$

Finally we have gotten that

$$\sup_{\substack{u,y\in\mathbb{R}^d \\ \|u-y\|_\infty \le a}} |\ell_0^{\partial}(f,u) - f^{\partial}(y)| \le \omega_{1,\infty}(f^{\partial}, a+1),$$

which proves one more time (14).

Remark 1. In [1] Remark 3, we have made the following conclusion: Assume that

$$\frac{\partial^{\sum_{r=1}^m j_r} f(x)}{\partial x_{i_1}^{j_1} \cdots \partial x_{i_r}^{j_r}} \ge 0, \text{ all } x \in \mathbb{R}^d,$$

then

$$\frac{\partial^{\sum_{r=1}^m j_r}}{\partial x_{i_1}^{j_1} \cdots \partial x_{i_r}^{j_r}} \begin{Bmatrix} r_0^f(x), \\ c_0^f(x), \\ \gamma_0^f(x) \end{Bmatrix} \ge 0, \text{ all } x \in \mathbb{R}^d;$$

and finally it holds that

$$\frac{\partial^{\sum_{r=1}^m j_r}}{\partial x_{i_1}^{j_1} \cdots \partial x_{i_r}^{j_r}} \begin{Bmatrix} A_k f, \\ B_k f, \\ L_k f, \\ \Gamma_k f \end{Bmatrix} \ge 0, \text{ all } x \in \mathbb{R}^d, \text{ all } k \in \mathbb{Z}.$$

The above are true when $f \in C^{(\rho)}(\mathbb{R}^d)$, $\sum_{r=1}^m j_r = \rho$. There (in [1]) we missed to make this last assumption.

As a related result we give

Theorem 7. *Here we assume that φ is continuous. Let f be a probability distribution function on $\mathbb{R}^d$, $d \geq 1$, such that all f, $\frac{\partial^m f}{\partial x_1 \ldots \partial x_m}$, $1 \leq m \leq d$, belong to $C(\mathbb{R}^d)$. Assume that f has a continuous probability density function (p.d.f) f^∂, that is, $f^\partial = \frac{\partial^d f}{\partial x_1 \ldots \partial x_d} \geq 0$. Then*

$$A_k^\partial f := \frac{\partial^d A_k(f)}{\partial x_1 \ldots \partial x_d}, \qquad B_k^\partial f := \frac{\partial^d B_k(f)}{\partial x_1 \ldots \partial x_d},$$

$$L_k^\partial f := \frac{\partial^d L_k(f)}{\partial x_1 \ldots \partial x_d} \quad \textit{and} \quad \Gamma_k^\partial f := \frac{\partial^d \Gamma_k(f)}{\partial x_1 \ldots \partial x_d}, \quad k \in \mathbb{Z},$$

are continuous p.d.f.'s. Similarly, they are continuous all the lower order partials of A_k, B_K, L_k and Γ_k that are corresponding to the above lower order partials of f.

Proof.

i) A_k-operators (φ is even): Here we have (by H. Bauer [2], pp. 103-104)

$$\ell_0^\partial(f,u) := \frac{\partial^d \ell_0(f,u)}{\partial x_1 \ldots \partial x_d} = \int_{\mathbb{R}^d} f^\partial(u-t)\varphi(t)dt, \text{ all } u \in \mathbb{R}^d.$$

Also all the other, corresponding to f, lower order partials of $(\ell_0 f)$ do exist and produced the same way! By Lemma 1 of [1], all these partials of $(\ell_0 f)$ are continuous functions. Therefore, by similar reasoning,

$$(A_0 f)^\partial(x) := \frac{\partial^d A_0 f(x)}{\partial x_1 \ldots \partial x_d} = \int_{\mathbb{R}^d} (\ell_0 f)^\partial(x-u)\varphi(u)du, \text{ all } x \in \mathbb{R}^d,$$

and all the associated lower order partials do exist and are continuous. Notice that, if f is a continuous distribution function, so is $f(2^{-k}\cdot)$, $k \in \mathbb{Z}$. Hence,

$$\frac{\partial^d (A_k f)(x)}{\partial x_1 \ldots \partial x_d} = 2^{kd} \cdot \frac{\partial^d A_0(f(2^{-k}\cdot))}{\partial x_1 \ldots \partial x_d}(2^k x)$$

and all the related lower-order partials of $(A_k f)$ do exist and are continuous.

Notice that

$$\frac{\partial^d(f(2^{-k}\cdot))}{\partial x_1\ldots\partial x_d} = 2^{-kd}\cdot\left(\frac{\partial^d f}{\partial x_1\ldots\partial x_d}\right)(2^{-k}\cdot)\geq 0,$$

and since

$$\int_{\mathbb{R}^d}\frac{\partial^d f(x)}{\partial x_1\ldots\partial x_d}dx = 1,$$

we obtain that

$$\int_{\mathbb{R}^d} 2^{-kd}\cdot\left(\frac{\partial^d f}{\partial x_1\ldots\partial x_d}\right)(2^{-k}\cdot x)\cdot dx = 1.$$

That is,

$$f_k^{\partial} := 2^{-kd}\cdot\left(\frac{\partial^d f}{\partial x_1\ldots\partial x_d}\right)(2^{-k}\cdot)$$

is a p.d.f. for $f(2^{-k}\cdot)$, i.e., $f_k^{\partial} = 2^{-kd}\cdot f^{\partial}(2^{-k}\cdot)$, $k\in\mathbb{Z}$.

From

$$\ell_0(f,u) = \int_{\mathbb{R}^d} f(u-t)\varphi(t)dt$$

and

$$(\ell_k f)(x) = \ell_0(f(2^{-k}\cdot))(x) = \int_{\mathbb{R}^d} f(2^{-k}(x-t))\varphi(t)dt,$$

we have that

$$(\ell_k f)^{\partial}(x) = 2^{-kd}\cdot\left(\int_{\mathbb{R}^d} f^{\partial}(2^{-k}(x-t))\varphi(t)dt\right)$$

and the lower order associated partials of $(\ell_k f)$ do exist and are continuous. I.e., we get that

$$\frac{\partial^d(\ell_k f)}{\partial x_1\ldots\partial x_d}(x) = \int_{\mathbb{R}^d}[2^{-kd}\cdot f^{\partial}(2^{-k}(x-t))]\cdot\varphi(t)dt\geq 0$$

and

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\frac{\partial^d(\ell_k f)}{\partial x_1\ldots\partial x_d}\right)(x)dx &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(2^{-kd}\cdot f^{\partial}(2^{-k}\cdot(x-t)))\cdot\varphi(t)dt\right)dx\\ &\quad\text{(by Fubini's theorem)}\\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f^{\partial}(2^{-k}x-2^{-k}t)d(2^{-k}x)\right)\cdot\varphi(t)dt\\ &= \int_{\mathbb{R}^d} 1\cdot\varphi(t)dt = 1,\ \text{by (5).}\end{aligned}$$

We have proved that $(\ell_k f)^\partial$ is a continuous p.d.f. Next notice that

$$A_k^\partial(f,x) = 2^{kd} \cdot \int_{\mathbb{R}^d} (\ell_k f)^\partial(2^k x - u)\varphi(u)du \geq 0.$$

And

$$\begin{aligned}\int_{\mathbb{R}^d} A_k^\partial(f,x)dx &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} 2^{kd}\cdot(\ell_k f)^\partial(2^k x-u)\cdot\varphi(u)du\right)dt \\ &\qquad \text{(by Fubini's theorem)} \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(\ell_k f)^\partial(2^k x-u)\cdot d(2^k x)\right)\varphi(u)\cdot du \\ &= \int_{\mathbb{R}^d} 1\cdot\varphi(u)du = 1, \text{ by (5).}\end{aligned}$$

That is, $A_k^\partial(f,x)$ is a continuous p.d.f. Also the associated lower order partials of A_k do exist and are continuous.

ii) B_k**-operators:** Here we have that

$$(B_k f)(x) = \int_{\mathbb{R}^d} f(2^{-k}(2^k x - u))\varphi(u)du$$

and

$$(B_k f)(x) = \int_{\mathbb{R}^d} f(x - 2^{-k}u)\varphi(u)du.$$

Hence

$$(B_k f)^\partial(x) = \int_{\mathbb{R}^d} f^\partial(x - 2^{-k}u)\varphi(u)du \geq 0$$

and it is continuous. Also the lower order associated partials of $(B_k f)$ do exist and are continuous. Furthermore,

$$\begin{aligned}\int_{\mathbb{R}^d}(B_k f)^\partial(x)dx &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f^\partial(x-2^{-k}u)\varphi(u)du\right)dx \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f^\partial(x-2^{-k}u)dx\right)\varphi(u)du \\ &= \int_{\mathbb{R}^d} 1\cdot\varphi(u)du = 1.\end{aligned}$$

That is, $(B_k f)^\partial$ is a continuous p.d.f.

iii) L_k**-operators:** Here we have that

$$(\ell_k f)(u) = 2^{kd}\cdot\int_{\vec{0}}^{\vec{2}^{-k}} f(t + 2^{-k}\cdot u)dt, \text{ all } u \in \mathbb{R}^d.$$

Hence

$$(\ell_k f)^{\partial}(u) = \int_{\vec{0}}^{\vec{2}^{-k}} f^{\partial}(t + 2^{-k} \cdot u)dt \geq 0,$$

and it is continuous.

Furthermore,

$$\int_{\mathbb{R}^d} (\ell_k f)^{\partial}(u)du = \int_{\mathbb{R}^d} \left(\int_{\vec{0}}^{\vec{2}^{-k}} f^{\partial}(t + 2^{-k} \cdot u)dt \right) du$$

(by Fubini's theorem)

$$= 2^{kd} \cdot \int_{\vec{0}}^{\vec{2}^{-k}} \left(\int_{\mathbb{R}^d} f^{\partial}(t + 2^{-k} \cdot u)d(2^{-k} \cdot u) \right) dt$$

$$= 2^{kd} \cdot \int_{\vec{0}}^{\vec{2}^{-k}} 1 \cdot dt = 2^{kd} \cdot 2^{-kd} = 1.$$

That is, $(\ell_k f)^{\partial}$ is a continuous p.d.f. Also, notice that

$$(L_k f)^{\partial}(x) = 2^{kd} \cdot \int_{\mathbb{R}^d} (\ell_k f)^{\partial}(2^k x - u) \cdot \varphi(u)du \geq 0$$

and it is continuous.

Furthermore,

$$\int_{\mathbb{R}^d} (L_k f)^{\partial}(x)dx = 2^{kd} \cdot \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (\ell_k f)^{\partial}(2^k x - u) \cdot \varphi(u)du \right) dx$$

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (\ell_k f)^{\partial}(2^k x - u)d(2^k \cdot x) \right) \varphi(u)du = 1.$$

I.e., $(L_k f)^{\partial}$ is a continuous p.d.f.

iv) Γ_k**-operators:** Here we have that

$$\ell_k^{\partial}(f, u) = 2^{-kd} \cdot \sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}} \cdot f^{\partial}\left(\frac{u}{2^k} + \frac{\vec{j}}{2^k \cdot \vec{n}} \right) \geq 0,$$

where $w_{\vec{j}} \geq 0$, $\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}} = 1$, by (34). And $\ell_k^{\partial}(f, u)$ is a continuous function in u. Furthermore

$$\begin{aligned}\int_{\mathbb{R}^d} \ell_k^\partial(f,u)du &= 2^{-kd}\cdot\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\cdot\int_{\mathbb{R}^d} f^\partial\left(\frac{u}{2^k}+\frac{\vec{j}}{2^k\cdot\vec{n}}\right)du\\ &=\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\cdot\int_{\mathbb{R}^d} f^\partial\left(\frac{u}{2^k}+\frac{\vec{j}}{2^k\cdot\vec{n}}\right)d(\frac{u}{2^k})\\ &=\left(\sum_{\vec{j}=0}^{\vec{n}} w_{\vec{j}}\right)\cdot 1=1.\end{aligned}$$

That is, $\ell_k^\partial(f,u)$ is a continuous p.d.f. Notice that

$$(\Gamma_k f)^\partial(x)=2^{kd}\cdot\int_{\mathbb{R}^d}(\ell_k f)^\partial(2^k x-u)\varphi(u)du\geq 0$$

and it is continuous.

Similarly, we find that

$$\int_{\mathbb{R}^d}(\Gamma_k f)^\partial(x)dx=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(\ell_k f)^\partial(2^k x-u)d(2^k x)\right)\varphi(u)du=1.$$

We have established that $(\Gamma_k f)^\partial$ is a continuous p.d.f.

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