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A NOTE ON ABSOLUTE SUMMABILITY FACTORS

BY

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Abstract. This paper deals with the generalization of a number of results due to Bor [5,6,7]. It also shows how certain results due to Mishra and Srivastava [14] and Bor [3] can be deduced from known results.

1. Let $\sum a_n$ be a given infinite series with $\{s_n\}$ as the sequence of its nth partial sums. Let $\{p_n\}$ be a sequence of positive numbers such that $P_n = p_0 + p_1 + \cdots + p_n \longrightarrow \infty$ as $n \longrightarrow \infty$. We write

$$U_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k$$

and

$$t_n = \frac{1}{n} \sum_{k=1}^{n} k a_k.$$

A series $\sum a_n$ is said to be summable $|C, 1|_k, k \geq 1$, if

$$\sum_{1}^{\infty} \frac{|t_n|^k}{n} < \infty. \tag{1.1}$$

It is said to be summable $|\overline{N}, p_n|_k, k \geq 1$ if

$$\sum_{1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |U_n - U_{n-1}|^k < \infty [1]. \tag{1.2}$$

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Mishra [12] and Mazhar [10] obtained the following theorem for summability $|C,1|_k$.

Theorem A. *If* $\{\lambda_n\}$ *is a convex sequence such that*

$$\sum_1^\infty \frac{\lambda_n}{n} < \infty \tag{1.3}$$

and if

$$\sum_1^n \frac{|s_v|^k}{v} = O(\log n), k \geq 1, \tag{1.4}$$

then $\sum a_n \lambda_n$ *is summable* $|C,1|_k$.

It is known [8,11,15] that if $\Delta^2\lambda_n \geq 0$ and (1.3) holds then

$$\sum_1^\infty n \log(n+1)\Delta^2\lambda_n < \infty, \tag{1.5}$$

$\{\lambda_n\}$ is a non-negative decreasing sequence and $\lambda_n \log n = o(1)$. With a view to obtain a more general result Mishra and Srivastava [13] proved the following:

Theorem B. *Let* $\{X_n\}$ *be a positive non-decreasing sequence and let there be sequences* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *such that*

$$|\Delta\lambda_n| \leq \beta_n \tag{1.6}$$

$$\beta_n \longrightarrow 0, n \longrightarrow \infty \tag{1.7}$$

$$\sum_1^\infty n|\Delta\beta_n|X_n < \infty \tag{1.8}$$

$$|\lambda_n|X_n = O(1). \tag{1.9}$$

If

$$\sum_1^n \frac{|s_v|^k}{v} = O(X_n), \ k \geq 1, \tag{1.10}$$

then $\sum a_n \lambda_n$ *is summable* $|C,1|_k$.

For $X_n = \log n$, $\beta_n = |\Delta\lambda_n|$. Theorem B gives Theorem A. They showed by means of an example that condition (1.8) does not imply

(1.11) $$\sum_1^\infty n|\Delta^2\lambda_n|X_n < \infty.$$

Taking

$$\lambda_n = \frac{1}{ne^n},\ \beta_n = \frac{1}{e^n},\ X_n = \frac{e^n}{n^2},\ n \geq 2,$$

$$|\Delta\lambda_n| = \frac{1}{ne^n}\left(1 - \frac{n}{(1+n)e}\right) < \beta_n,\ \beta_n \longrightarrow 0$$

and $|\Delta^2\lambda_n| \leq \frac{C}{ne^n}$, C is a positive contant, we have

$$\sum_1^\infty n|\Delta^2\lambda_n|X_n \leq C\sum_1^\infty \frac{1}{n^2} < \infty$$

while

$$\sum_1^\infty n|\Delta\beta_n|X_n = \sum_1^\infty \frac{n}{e^n}\left(1 - \frac{1}{e}\right)\frac{e^n}{n^2}$$

$$= \left(1 - \frac{1}{e}\right)\sum_1^\infty \frac{1}{n} = \infty.$$

Thus in general, conditions (1.8) and (1.11) are independent of each other and claims made earlier in [4,13] that (1.8) is a weaker condition is not true.

2. Generalizing Theorem B, Bor has proved a number of theorems for summability $|\overline{N}, p_n|_k$, $k \geq 1$. In all these theorems he imposes certain restrictions on the sequence $\{p_n\}$. Some of his theorems are given below.

Theorem C [6]. *Let $\{X_n\}$ be a positive non-decreasing sequence and let sequences $\{\beta_n\}$ and $\{X_n\}$ satisfy* (1.6), (1.7) *and* (1.9). *If*

(2.1) $$1 = O(p_n),\ n \longrightarrow \infty$$

(2.2) $$\sum_1^\infty P_n X_n |\Delta\beta_n| < \infty$$

and

(2.3) $$\sum_1^m \frac{p_n}{P_n}|s_n|^k = O(X_m),\ m \longrightarrow \infty,$$

then $\sum a_n\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

Theorem D [7]. *Let $\{X_n\}$ be a positive non-decreasing sequence and let sequences $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy* (1.6)-(1.9). *If*

(2.4) $$P_n = O(np_n),\ n \longrightarrow \infty$$

then under the condition (2.3), $\sum a_n\lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

Theorem E [5]. *Let sequences $\{p_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions* (1.6), (1.7) *and*

(2.5) $$\frac{1}{n} = O(p_n),\ n \longrightarrow \infty$$

(2.6) $$\sum_1^\infty nP_n|\Delta\beta_n| < \infty$$

(2.7) $$P_n|\lambda_n| = O(1)$$

(2.8) $$\sum_1^m p_n|s_n|^k = O(P_m),\ m \longrightarrow \infty$$

then $\sum a_n\lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

It is clear that

(i) (2.1) *and* (2.2)$\Longrightarrow$(1.8)

(ii) (2.3) *and* (2.4)$\Longrightarrow$(1.10)

(iii) (2.5) *and* (2.8)$\Longrightarrow$

(2.9) $$\sum_1^m \frac{|s_n|^k}{n} = O(P_m)$$

(iv) (2.1) *and* (2.3)$\Longrightarrow$

(2.10) $$\sum_1^m \frac{|s_n|^k}{P_n} = O(X_m),\ m \longrightarrow \infty,$$

while the converses need not be true.

In view of (ii) it follows from Theorem B that $\sum a_n\lambda_n$ is summable $|C, 1|_k$ for any sequence $\{p_n\}$ satisfying (2.3) and (2.4) and not simply for $p_n = 1$ as is the case in Theorem D.

It will also be observed that conditions (2.1), (2.4) and (2.5) are restrictive as results for $p_n = \frac{1}{n \log n}$ can not be deduced from Theorem C, D, E, and for $p_n = \frac{1}{n}$ from Theorem C, D.

In order to extend the scope of these results we prove the following more general results. To do so we need the concept of almost increasing sequence. A positive sequence $\{b_n\}$ is said to be almost increasing if there exist an increasing sequence $\{c_n\}$ and positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$. Obviously every increasing sequence is almost increasing but the converse need not be true.

Theorem 1. *Let $\{X_n\}$ be an almost increasing sequence. Then under the conditions* (1.6)-(1.10) *and* (2.3), $\sum a_n\lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

Theorem 2. *Under the conditions* (1.6)-(1.9), (2.2), (2.3) *and* (2.10) $\sum a_n\lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$, *where $\{X_n\}$ is an almost increasing sequence.*

It may be noted that since, in general, $|C, 1|_k$ and $|\overline{N}, p_n|_k$ are independent of each other, the condition (2.3) is desirable.

In all above theorems direct restrictians on the sequence $\{p_n\}$ have been removed. Hence apart from getting results for $p_n = \frac{1}{n}, \frac{1}{n \log n}$ etc our theorems are true under weaker conditions.

To prove these theorems we need the following lemmas.

Lemma 1. *Under the conditions* (1.7)-(1.8)

$$\sum_{1}^{\infty} \beta_n X_n < \infty \tag{2.11}$$

and

$$n\beta_n X_n = o(1), \quad n \longrightarrow \infty \tag{2.12}$$

where $\{X_n\}$ is an almost increasing sequence.

Writing $Ac_n \leq X_n \leq Bc_n$, where $\{c_n\}$ is a positive increasing sequence and proceeding as in [13] we can easily establish the above results.

Lemma 2. *If $\{X_n\}$ is an almost increasing sequence, then under the conditions* (1.7) *and* (2.2).

$$P_n X_n \beta_n = o(1), \ n \longrightarrow \infty \tag{2.13}$$

$$\sum_{1}^{\infty} p_n X_n \beta_n < \infty. \tag{2.14}$$

The proof is similar to that of Bor in [6] and hence omitted.

3. Proof of Theorem 1. Let T_n denote the n-th $(\overline{N}, p_n)$ mean of the series $\sum a_n \lambda_n$, then

$$\begin{aligned} T_n &= \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{s=0}^{v} a_s \lambda_s \\ &= \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v \end{aligned}$$

so that for $n \geq 1$

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v \\ &= - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \lambda_v s_v \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v s_v + \frac{p_n}{P_n} \lambda_n s_n \\ &= L_1 + L_2 + L_3, \text{ say.} \end{aligned}$$

Thus to show that $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$ it is enough to prove that

$$\sum_{1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |L_r|^k < \infty, \ r = 1, 2, 3.$$

The proofs for $r = 1, 3$ are easy and are the same as in Bor [7,481-482]. Hence we prove the case $r = 2$ only.

$$\sum_{1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|L_2|^k \leq \sum_{1}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{1}^{n-1}P_v|\Delta\lambda_v||s_v|\right)^k$$

$$\leq \sum_{1}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{1}^{n-1}P_v|\Delta\lambda_v||s_v|^k\right)\left(\sum_{1}^{n-1}P_v|\Delta\lambda_v|\right)^{k-1}$$

$$=O(1)\sum_{1}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{1}^{n-1}P_v|\Delta\lambda_v||s_v|^k,$$

since $\sum|\Delta\lambda_v| < \infty$ in view of Lemma 1 and (1.6).

$$= O(1)\sum_{1}^{m}P_v|\Delta\lambda_v||s_v|^k\sum_{n=v+1}^{\infty}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{1}^{m}|\Delta\lambda_v||s_v|^k = O(1)\sum_{1}^{m}\beta_v|s_v|^k$$

$$= O(1)\left(\sum_{1}^{m-1}\Delta\{\nu\beta_v\}\sum_{r=1}^{\nu}\frac{|s_r|^k}{r} + m\beta_m\sum_{r=1}^{m}\frac{|s_r|^k}{r}\right)$$

$$= O(1)\sum_{1}^{m-1}\nu|\Delta\beta_v|X_\nu + O\left(\sum_{1}^{m-1}\beta_{v+1}X_v\right)$$
$$+ O(m\beta_mX_m)$$

$= O(1)$ in view of (1.8), (2.11) and (2.12). This completes the proof of Theorem 1.

Using Lemma 2 and proceeding as in the proof of Theorem 1, replacing $\sum\beta_v|s_v|^k$ by $\sum_1^m\beta_\nu P_\nu\frac{|s_v|^k}{P_\nu}$ we can easily prove Theorem 2.

Taking $X_n = P_n$ in Theorem 1 we have the following result:

Corollary 1. *Under the conditions* (1.6), (1.7), (2.6), (2.7), (2.9) *and*

$$\sum_{1}^{m}\frac{p_n}{P_n}|s_n|^k = O(P_m) \tag{2.15}$$

$\sum a_n\lambda_n$ *is summable* $|\overline{N},p_n|_k,\ K \geq 1$.

Since (2.5) and (2.8)$\Longrightarrow$ (2.9) and (2.8)$\Longrightarrow$(2.15), Theorem E follows from Corollary 1.

4. In 1984, Mishra and Srivastava [14] obtained the following theorem on $|\overline{N}, p_n|$ summability.

Theorem F. *Under the conditions* (1.6)-(1.9)

$$P_n = O(np_n) \tag{4.1}$$

$$P_n \Delta p_n = O(p_n p_{n+1}) \tag{4.2}$$

and

$$\sum_1^n \frac{|s_v|}{v} = O(X_n), \tag{4.3}$$

$\sum a_n \frac{\lambda_n P_n}{np_n}$ *is summable* $|\overline{N}, p_n|$, *where* $\{X_n\}$ *is a positive non-decreasing sequence.*

This result was subsequently extended to $|\overline{N}, p_n|_k$, $k \geq 1$ by Bor [3] in 1987. His theorem is as follows:

Theorem G. Under the conditions of Theorem F with (4.3) replaced by (1.10), the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

Theorem F is the case $k = 1$ of Theorem G. It is easy to see that (4.1) and (4.2) are equivalent to (4.1) and

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{4.4}$$

Thus a generalized version of Theorem G is:

Theorem 3. *Under the conditions* (1.6)-(1.10), (4.1) and (4.4), the series $\sum \frac{a_n \lambda_n P_n}{np_n}$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$, where $\{X_n\}$ is an almost increasing sequence.

To deduce this result we need the following inclusion theorem which Bor [2] obtained in 1985 as an extension of a result of Khan and Alauddin [9].

Theorem H. *If*

(4.5) $$\frac{p_n}{P_n} = O\left(\frac{q_n}{Q_n}\right)$$

(4.6) $$\frac{q_n P_n \mu_n}{p_n Q_n} = O(1)$$

(4.7) $$P_n \Delta\mu_n = O(p_n),$$

then $\sum a_n \mu_n$ *is summable* $|\overline{N}, q_n|_k$*, whenever* $\sum a_n$ *is summable* $|\overline{N}, p_n|_k$*, where* $\{q_n\}$ *is a positive sequence such that* $Q_n = q_0 + q_1 + \cdots + q_n \longrightarrow \infty$ *as* $n \longrightarrow \infty$.

Replacing p_n by 1 and q_n by p_n in the above theorem we find that if

$$P_n = O(np_n), \ \frac{np_n\mu_n}{P_n} = O(1) \quad \text{and} \quad \Delta\mu_n = O\left(\frac{1}{n}\right),$$

then $\sum a_n \mu_n$ is summable $|\overline{N}, p_n|_k$ whenever $\sum a_n$ summable $|C, 1|_k$. Writing $\mu_n = \frac{P_n}{np_n}$ it then follows that if

$$\frac{P_n}{np_n} = O(1) \quad \text{and} \quad \Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right),$$

then $\sum a_n \frac{P_n}{np_n}$ is summable $|\overline{N}, p_n|_k$ whenever $\sum a_n$ is summable $|C, 1|_k$. Now in view of Theorem 1 (for $p_n = 1$), $\sum a_n \lambda_n$ is summable $|C, 1|_k$ and hence $\sum a_n \frac{\lambda_n P_n}{np_n}$ is summable $|\overline{N}, p_n|_k$. This proves Theorem 3.

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