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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A HIGHER ORDER LINEAR DIFFERENTIAL EQUATION

BY

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1. Introduction. Asymptotic behavior of oscillatory solutions of second order differential equations

$$y'' + p(t)y = 0 \tag{1}$$

have been widely studied. For example, if $p(t)$ tends monotonically to infinity with t, then (1) has at least one nontrivial solution that tends to zero. However, that need not be the case for all solutions [10].

However, for

$$y^{(iv)} - p(t)y = 0, \tag{2}$$

Hastings and Lazer [5] proved that all oscillatory solutions tend to zero if $p(t)$ tends monotonically to infinity with t. The author [6] proved that all oscillatory solutions of (2) are unbounded if $p(t)$ decreases monotonically to zero as t tends to infinity.

With even the lesser hypothesis $0 < m \le p(t)$, where m is constant, Švec [11] proved that there is a pair of linearly independent zero tending oscillatory solutions for

$$y^{(iv)} + p(t)y = 0. \tag{3}$$

This result has been generalized by Kiguradze [8] for equations of order $n > 2$. The author proved [7] that if (3) has an oscillatory solution then

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it has a pair of unbounded oscillatory solutions such that every nontrivial linear combination of them is unbounded provided $0 \leq p(t) \leq M$ where M is constant.

The purpose of this paper is to obtain similar results for certain equations

$$y^{(n)} + p(t)y = 0, \tag{4}$$

where $p(t)$ is assumed to be a real-valued continuous function. In particular, we shall obtain theorems analogous to those in [7]. Our results apply to nonoscillatory as well as oscillatory solutions. However, the results for nonoscillatory solutions are already known.

1. Preliminary results. In this section we will give definitions and results that will be used to prove our main theorems.

Following Elias [2], we let $\sigma(c_0, \ldots, c_n)$ denote the number of sign changes in the sequence $c_0, \ldots, c_n$ of non-zero numbers. For a solution $y \not\equiv 0$ of (4) and $x \in (0, +\infty)$, we define

$$S(y, x^+) = \lim_{t \longrightarrow x^+} \sigma(y(t), -y'(t), \ldots, (-1)^n y^{(n)}(t))$$

and

$$S(y, x^-) = \lim_{t \longrightarrow x^-} \sigma(y(t), y'(t), \ldots, y^{(n)}(t)).$$

Let $a \leq x_1 \leq \ldots \leq x_r \leq b$ be the zeros of $y(t), y'(t), \ldots, y^{(n-1)}(t)$ for a solution y to (4), where the same $x_i = c$ is used to denote zeros of two different derivatives $y^{(j)}$ and $y^{(k)}$ if and only if $y^{(j)}(c) = y^{(k)}(c) = 0$ implies either $y^{(h)}(c) = 0$ for all $j \leq h \leq k$ or $y^{(h)}(c) = 0$ for $k \leq h \leq n-1$ and $0 \leq h \leq j$. If $n(x_i)$ denotes the number of consecutive (with y following $y^{(n-1)}$) derivatives which vanish at x_i and $< q >$ denotes the greatest even integer that is not greater than q, we have the following theorem.

Theorem 1. [2] *Every solution y of* (4) *satisfies the condition*

$$n(y) \equiv S(y, a^+) + \sum_{a<x_t<b} \langle n(x_t) \rangle + S(y, b^-) \le n.$$

Moreover $S(y, b^-)$ and $n - S(y, a^+)$ are both even if $p(x) < 0$ and both odd if $p(x) > 0$.

As a consequence of Theorem 1, we have

Theorem 2. [2] *If y is a solution of (4) then $S(y, x^+)$ is a nondecreasing integer valued function of x. Also $S(y, x^-)$ is a non-increasing integer valued function of x.*

Letting

$$W(y_r, y_{r+1}, \ldots, y_s)(x) = \begin{vmatrix} y_r(x) & y_{r+1}(x) & \cdots & y_s(x) \\ y_r'(x) & y_{r+1}'(x) & \cdots & y_s'(x) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_r^{(s-r)}(x) & y_{r+1}^{(s-r)}(x) & \cdots & y_s^{(s-r)}(x) \end{vmatrix}$$

and using Theorem 1 and 2, we can obtain

Theorem 3. [3] *Let γ be an integer so that $(-1)^{n-\gamma} p(x) > 0$. Then there are linearly independent solutions $y_i(x, b)$ of (4) for $i = 0, 1, \ldots, n-1$ with the following properties:*

(a) *$y_i(x, b)$ has a zero of multiplicity exactly i at $x = a$.*

(b) *$y_i(x, b)$ has a zero of multiplicity at least $n - 1 - [i + (1 + (-1)^{i-\gamma})/2]$ at $x = b$.*

(c) *$z \in span(y_\gamma(x, b), y_{\gamma+1}(x, b), \ldots, y_{\gamma+2s+1}(x, b))$ for $0 \le s < (n - \gamma - 1)/2$ implies $\gamma + 1 \le S(z, x^+) \le \gamma + 2s + 1$ and*

$$n - (\gamma + 2s + 1) \le S(z, x^-) \le n - (\gamma + 1) \text{ for } x \in (a, b).$$

(d) *$W(y_\gamma(x, b), \ldots, y_{\gamma+2s+1}(x, b)) \ne 0$ for $a < x < b$.*

(e) *For any increasing sequence $\{b_q\}_{q=1}^{\infty}$ such that $\lim_{q \to \infty} b_q = \infty$, there is a subsequence $\{b_{q_j}\}_{j=1}^{\infty}$ such that $(y_i(x, b_{q_j}))^{(k)}$ converges to $(y_i(x))^{(k)}$ for $k, i = 0, 1, \ldots, n-1$ with convergence uniform on compact intervals and $y_0, y_1, \ldots, y_{n-1}$ is a basis for the solution space of (4) satisfying (a), (c), (d).*

An immediate consequence of Theorems 1 and 2 is the following

Theorem 4. [2] *If y is a solution of (4) then*

$$\lim_{x\longrightarrow\infty} S(y,x^+)$$

exists and is less than or equal to n.

If y is a solution of (4), we say $y\in S_k$ provided $\lim_{x\longrightarrow\infty} S(y,x^+)=k$.

We will use the following class of inequalities due to Gabushin [4]. Also see [1,9]. Here $\|f\|_p=\int_J |f(t)|^p dt$ when $1\le p<\infty$ and $\|f\|_\infty=\sup_{t\in J}|f(t)|$ with J being a given half-line.

Theorem 5. *Let n and k be integers satisfying $1\le k<n$. Let $p,r\le\infty$. There is a finite constant K such that*

$$\|y^{(k)}\|_q\le K\|y\|_p^{\alpha}\|y^{(n)}\|_r^{\beta}$$

holds if and only if

$$n/q\le(n-k)/p+k/r$$

$$\alpha=\Big(n-k-\frac{1}{r}+\frac{1}{q}\Big)/\Big(n-\frac{1}{r}+\frac{1}{p}\Big),\quad \beta=1-\alpha.$$

Here $\frac{1}{s}=0$ when $s=\infty$.

2. In this section we will consider certain even order equations (4). Thus, it will be more convenient to write the equation as

$$y^{(2n)}+py=0. \tag{5}$$

We let

$$F[y(x)]\equiv\sum_{i=0}^{n-1}(-1)^i y^{(n-i-1)}(x)y^{(n+i)}(x).$$

Differentiating F with respect to x, we obtain

Lemma 1. *If y is a solution of (5) then*

$$F'[y(x)]=[y^{(n)}(x)]^2+(-1)^n p(x)y^2(x).$$

Theorem 6. *If $(-1)^n p(x) > 0$, $|p(x)| < M$ and y is a solution of (5) such that $F[y(x)] > 0$ for $x > a$, then y is unbounded.*

Proof. Since $F'[y(x)] > 0$, if y is a solution of (5) for which $F[y(x)] > 0$, it follows that

$$\int^{\infty} F[y(x)]dx = +\infty. \tag{6}$$

But

$$\int_a^x F[y(t)]dt = G[y(x)] - G[y(a)]. \tag{7}$$

Where

$$G[y(x)] = \frac{n}{2}\left[y^{(n-1)}(x)\right]^2 - \sum_{i=0}^{n-2}(-1)^i(n-1-i)y^{(n+i)}(x)y^{(n-2-i)}(x).$$

It follows from (6) and (7) that

$$\limsup_{x\longrightarrow\infty} |y^{(j)}(x)| = \infty \quad \text{for some} \quad j = 0, \ldots, 2n-2. \tag{8}$$

If $|y(x)| < B$, then by (5) and the hypothesis

$$|y^{(2n)}(x)| = |p(x)y(x)| < MB.$$

Hence by Theorem 5, $y^{(j)}(x)$ is bounded for $j = 0, \ldots, 2n$. From this contradiction we see that y is unbounded.

Theorem 7. *If $(-1)^n p(x) > 0$ and $|p(x)| < M$ then there are linearly independent solutions y_i for $i = n, n+1, \ldots, 2n-1$ of (5) so that every nontrivial linear combination of them is unbounded. Further, for each $i = n, n+1, \ldots, 2n-1, y_i$ can be chosen to have a zero at $x = a$ of order exactly i. For $p(x)$ positive y_i and y_{i+1} are in S_{i+1} for $i = n, n+2, \ldots, 2n-2$ while for negative $p(x), y_{2n-1}$ is in S_{2n}, y_i and y_{i+1} are in S_{i+1} for $i = n, n+2, \ldots, 2n-3$.*

Proof. Let y be any nontrivial linear combination of $y_n, y_{n+1}, \ldots, y_{2n-1}$ which are given by Theorem 3. Now y has a zero of multiplicity at least n at $x = a$. Thus by Lemma 1, $F[y(x)] > 0$ for $x > a$. The result now follows from Theorem 6.

Combining Theorem 7 with results of Kiguradze [8], we have the following

Corollary 1. *Suppose* $(-1)^n p(x) > 0$ *and* $0 < \varepsilon < |p(x)| < M$. *Then every solution of* (5) *is either unbounded or tends to zero.*

3. We will now consider certain odd order equations and will write (4) in the form

$$y^{(2n+1)} + py = 0. \tag{9}$$

We let

$$F_2[y(x)] = \left[\frac{y^{(n)}(x)}{2}\right]^2 + \sum_{i=1}^{n}(-1)^i y^{(n+i)}(x)y^{(n-i)}(x).$$

Differentiating F_2 with respect to x, we have

Lemma 2. *If* y *is a solution of* (9) *then*

$$F_2'[y(x)] = (-1)^{n+1}p(x)y^2(x).$$

Theorem 8. *If* $(-1)^{n+1}p(x) > 0, |p(x)| < M$ *and* y *is a solution of* (9) *such that* $F_2[y(x)] > 0$ *for* $x > a$, *then* y *is unbounded.*

Proof. Assume that $y(x)$ is bounded where $F_2[y(x)] > 0$. It then follows that $y^{(2n+1)}(x)$ is also bounded. From Theorem 5,

$$y^{(k)}(x) \quad \text{is bounded for} \quad 0 \le k \le 2n+1. \tag{10}$$

It now follows that $F_2[y(x)]$ is bounded. Consequently, from Lemma 2

$$\int_a^\infty |p(x)|y^2(x)dx < \infty.$$

Thus,

$$\int_a^\infty \left[y^{(2n+1)}(t)\right]^2 dt = \int_a^\infty p^2(t)y^2(t)dt < M\int_a^\infty |p(t)|y^2(t)dt < \infty.$$

Again applying Theorem 5,

$$\|y^{(k)}\|_4 \le K\|y\|_\infty^\alpha \|y^{(2n+1)}\|_2^\beta$$

for $n+1 \leq k < 2n+1$. Consequently,

$$\int_a^\infty \left[y^{(k)}(t)\right]^4 dt < \infty \quad \text{for} \quad n+1 \leq k < 2n+1. \tag{11}$$

Because $\left[y^{(k)}(x)\right]^2$ has a bounded derivative, it follows from (11) that

$$\lim_{x \longrightarrow \infty} y^{(k)}(x) = 0 \quad \text{for} \quad n+1 \leq k < 2n+1. \tag{12}$$

From (10) and (12)

$$\lim_{x \longrightarrow \infty} \sum_{i=1}^{n} (-1)^i y^{(n+i)}(x) y^{(n-i)}(x) = 0. \tag{13}$$

If $y(x)$ is an oscillatory solution of (9), let $\{x_j\}_{j=1}^\infty$ be a sequence of zeros of $y^{(n)}(x)$ that diverges to ∞. It then follows that

$$\lim_{j \longrightarrow \infty} F_2\left[y(x_j)\right] = 0,$$

contrary to the fact that $F_2[y(x)]$ is positive and increasing.

If $y(x)$ is nonoscillatory, $y^{(k)}(x)$ is monotone for $k = 0, \ldots, 2n$. Since $F_2[y(x)]$ is positive and increasing, it follows from (13) that $y^{(n)}(x)$ is bounded away from zero. In that case, $y^{(n-1)}(x)$ is unbounded, contrary to (10).

Theorem 9. *If $(-1)^{n+1}p(x) > 0$ and $|p(x)| < M$ then there are linearly independent solutions y_i for $i = n, n+1, \ldots, 2n$ of (9) so that every nontrivial linear combination of them is unbounded. Further, for each $i = n, n+1, \ldots, 2n$, y_i can be chosen to have a zero at $x = a$ of order exactly i. For $p(x)$ positive y_i and y_{i+1} are in S_{i+1} for $i = n, n+2, \ldots, 2n-1$, while for negative $p(x), y_{2n}$ is in S_{2n+1}; y_i and y_{i+1} are in S_{i+1} for $i = n, n+2, \ldots, 2n-2$.*

Proof. Let $y_i(x)$ for $i = n, \ldots, 2n$ be linearly independent solutions of (9) with zeros of order i at $x = a$. If $y(x)$ is any nontrivial linear combination of y_i for $i = n, \ldots, 2n$, then $F_2[y(a)] = 0$. Since $F_2[y(x)]$ is increasing it follows that $F_2[y(x)] > 0$ for $x > 0$. The conclusion now follows from Theorem 8.

Theorem 10. *If* $(-1)^n p(x) > 0$, $|p(x)| < M$ *and* y *is a solution of* (9) *for which* $F_2[y(a)] \leq 0$, *then* y *is unbounded.*

Proof. By Lemma 2, $F_2[y(x)]$ is decreasing for every solution y of (9). If y is so that $F_2[y(a)] \leq 0$ then $F_2[y(x)] < 0$ for $x > a$. In that case

$$\lim_{x\longrightarrow\infty} \int_a^x F_2[y(t)]dt = -\infty. \tag{14}$$

But

$$\int_a^x F_2[y(x)]dt = G[y(x)] - G[y(a)]$$

where

$$G[y(x)] \equiv (n-\frac{1}{4}) \int_a^x \left[y^{(n)}(t)\right]^2 dt + \sum_{i=0}^{n-1}(-1)^{i+1}(n-i)y^{(n+i)}(x)y^{(n-1-i)}(x)$$
$$\geq \sum_{i=0}^{n-1}(-1)^{i+1}(n-i)y^{(n+i)}(x)y^{(n-1-i)}(x).$$

By (14)

$$\lim_{x\longrightarrow\infty} G[y(x)] = -\infty.$$

Thus

$$\lim_{x\longrightarrow\infty} \sum_{i=0}^{n-1}(-1)^{i+1}(n-i)y^{(n+i)}(x)y^{(n-1-i)}(x) = -\infty. \tag{15}$$

As in Theorem 8, since $|p(x)| < M$, the assumption that y is bounded implies $y^{(2n+1)}$ is bounded. Thus, as before $y^{(k)}$ is bounded for $1 \leq k \leq 2n$, which is contrary to (15).

As before, we have the following

Theorem 11. *If* $(-1)^n p(x) > 0$ *and* $|p(x)| < M$ *then there are a linearly independent solutions* y_i *for* $i = n+1, n+2, \ldots, 2n$ *of* (9) *so that every nontrivial linear combination of them is unbounded. Further, for each* $i = n+1, n+2, \ldots, 2n$, y_i *can be chosen to have a zero at* $x = a$ *of order exactly* i. *For* $p(x)$ *positive* y_i *and* y_{i+1} *are in* S_{i+1} *for* $i = n+2, n+$

$4, \ldots, 2n-1$, *while for negative* $p(x), y_{2n}$ *is in* $S_{2n+1}; y_i$ *and* y_{i+1} *are in* S_{i+1} *for* $i = n+2, n+4, \ldots, 2n-1$.

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