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ON AN INTEGRO-FUNCTIONAL EQUATION ARISING IN ORDER STATISTICS

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Abstract. In this paper, an integro-functional equation is introduced to characterize the gamma density function.

1. Introduction and Main Theorem. The problem of characterizing population distribution through the independence of two statistics is of importance in mathematical statistics. The analysis of this problem requires the solving of some nonlinear integro-functional equations. To characterize the normality through the independence of a tube statistic with finite basis and the sample mean, Anosov(1964) used the corresponding integro-functional equation to establish the problem. And it is remarked in Kagan et al. (1973, p.3) that, in this circle of problems, only the simplest one has been solved so far.

Recently Hwang and Hu (1994) introduced a useful set of nonlinear transformations, and obtained the distributions of studentized order statistics by applying these transformations. In this paper, an integro-functional equation, which can be used to characterize gamma population, is established by our nonlinear transformation (1994).

For obtainning our main result, the following integral-functional equation is defined. Let $t = (t_1, \cdots, t_{n-1}), n \geq 3$, and let A_n be a closed subset of the surface of the unit sphere $\{t : t_1^2 + \cdots + t_{n-1}^2 = 1\}$ in $\mathbf{R}^{n-1}$ and let

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$\sigma_n(t)$ be a distribution function over the set A_n. And let $\lambda_i(t), 1 \leq i \leq n$, be defined and continuous on A_n and such that

$$\sum_{i=1}^{n} \lambda_i(t) = 0, \quad \sum_{i=1}^{n} \lambda_i^2(t) = 1. \tag{1.1}$$

Let $f(x)$ be a defined and continuous density on $(0, +\infty)$. Then the integro-functional equation is defined by

$$\int_{A_n} \pi_{i=1}^{n} f(x(v\lambda_i(t)+1))d\sigma_n(t) = C_n \cdot [f(x)]^n \int_{A_n} \pi_{i=1}^{n} f(v\lambda_i(t)+1)d\sigma_n(t) \tag{1.2}$$

for all $x > 0$ and for sufficiently small $v > 0$, where $C_n > 0$ is a constant.

Note that the boundedness of the continuous functions $\lambda_i(t)$, $1 \leq i \leq n$, is very important here, and holds by the relation $\sum_{i=1}^{n} \lambda_i^2(t) = 1$ as given in (1.1). The boundedness of $\lambda_i(t)$ implies that $v \cdot \lambda_i(t) + 1 > 0$ for all sufficiently small $v > 0$, and thus the integro-functional equation (1.2) is well-defined.

In this paper, we obtain what function f will be the only solution of the integro-functional equation (1.2) as the following:

Main Theorem: The gamma density function

$$f(x) = \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha}} \cdot x^{\alpha-1} \cdot e^{-x/\beta}, \quad x > 0 \tag{1.3}$$

is the only solution of the integro-functional equation (1.2), where $\alpha, \beta > 0$ are parameters.

Section 2 gives an illustration of the integro-functional equation (1.2), and the proof of the main theorem will be presented in Section 3.

2. An Illustration of Integro-Functional Equation. As a matter of fact, Hwang and Hu(1994) have already obtained an illustration of integro-functional equation in exponential density case as follows:

Let $t = (t_1, \ldots, t_{n-1})$ and A_n be given by

$$A_n = \left\{ t : \begin{array}{l} t_1^2 + \ldots + t_{n-1}^2 = 1 \\ \left(\frac{n-k+2}{n-k}\right)^{1/2} \cdot t_{k-1} \leq t_k \leq 0, \; 2 \leq k \leq n-1 \end{array} \right\} \tag{2.1}$$

and define the distribution function $\sigma_n(t)$ over the set A_n by

$$d\sigma_n(t) = a_n \cdot (-t_n)^{-(n-1)} \cdot dF_n(t), \quad t \in A_n \tag{2.2}$$

where a_n is the normalizing constant, and $F_n(t)$ is the uniform distribution over A_n. Note that this distribution function $\sigma_n(t)$ is defined in Hwang and Hu(1994).

Let $X_1, \ldots, X_n$ be iid random variables from the exponential distribution with parameter θ, that is, the density function $f(x)$ is given by $f(x) = \theta e^{-\theta x}, x > 0$ and zero otherwise. Define

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad V_n = S_n/\overline{X}_n$$

where S_n is the sample standard deviation. Now, by applying a non-linear transformation given in Hwang and Hu (1994, Theorem 2.2), we obtain that if $\overline{X}_n$ and V_n are independent then the integro-functional equation (1.2) holds for all $x > 0$ and for all $v > 0$ in the neighborhood of the origin, say $0 < v < \sqrt{n}/(n-1)$; and A_n and $\sigma_n(t)$ are given as in (2.1) and (2.2) respectively, and the functions $\lambda_i(t)$ are

$$\lambda_i(t) = \left[\frac{n-i}{n-i+1}\right]^{\frac{1}{2}} \cdot t_i - \sum_{k=1}^{i-1}\left[\frac{1}{(n-k)(n-k+1)}\right]^{\frac{1}{2}} \cdot t_k, \ 1 \le i \le n-1$$

$$\lambda_n(t) = -t_{n-1}/\sqrt{2} - \sum_{k=1}^{n-2}\left[\frac{1}{(n-k)(n-k+1)}\right]^{\frac{1}{2}} \cdot t_k,$$

the summation will be taken as zero for $i = 1$. Note that by a straightforward computation, these functions $\lambda_i(t)$ actually satisfy the relations (1.1).

3. Proof of Main Theorem. In view of the relation (1.1), it is easily to prove that if $f(x)$ is a gamma density as given in (1.3), then the integro-functional equation (1.2) holds. And it is natural to ask: under what conditions on $f(x)$ will the gamma densities be the only solution of (1.2)?

Assume that $f(x)$ is continuously twice-differentiable, it can be easily shown that the gamma density actually is the only solution of the equation (1.2). To this end, we differentiate twice with respect to the variable v on the two sides of (1.2), and then letting $v \longrightarrow 0^+$ and taking the relation (1.1) into account; the operations involved being valid in view of the conditions imposed on $f(x)$ and the boundedness of the continuous functions $\lambda_i(t)$. As a result, we obtain a second order differential equation,

$$x^2 \cdot \left[y'' \cdot y^{n-1} - (y')^2 - y^{n-2}\right] = C'_n \cdot y^n, \quad x > 0$$

where $y = f(x)$ and C'_n is a constant. And it follows from the probability character of $f(x)$ that the gamma density is the only solution of equation (1.2). Thus, this completes the proof of our main result in this case.

However, it would not be easy to solve this problem under the condition that $f(x)$ be continuous which is weaker than continuously twice-differentiable. In the following, we apply both the results of Anosov (1964) and Hwang-Hu (1994) to prove the main theorem under the condition that $f(x)$ be continuous.

Six lemmas which will be used to prove the main theorem are presented as follows:

Lemma 3.1. *Let $f(x) > 0$ be defined and continuous on (x_0, x_1), and let $u(x) = \ell n(f(x))$, and define*

$$L_{v,t}u(x) = \sum_{i=1}^{n}[u(x(v\lambda_i(t)+1)) - u(x)] \tag{3.3}$$

for sufficiently small $v > 0$ and $t \in A_n$ where $\lambda_i(t); i = 1, 2, \cdots, n$, are as defined in (1.1). *Then, the integro-functional equation* (1.2) *can be rewritten as*

$$\int_{A_n} \exp(L_{v,t}u(x))d\sigma_n(t) = h(v) \tag{3.4}$$

where $\sigma_n(t)$ is a distribution function over A_n.

$$h(v) = C_n \cdot \int_{A_n} \pi_{i=1}^{n} f(v\lambda_i(t)+1)d\sigma_n(t) \tag{3.5}$$

Proof. This lemma follows immediately from the definition of $L_{v,t}$.

Lemma 3.2. *Let $L_{v,t}$ be defined as in* (3.3). *Then,*

(1) *$L_{v,t}$ is a linear operator.*

(2) *For any linear function $\ell(x), L_{v,t}\ell(x) = 0$.*

(3) *If $w(x)$ is a convex function, then $L_{v,t}w(x) \geq 0$.*

(4) *If $w(x)$ is a concave function, then $L_{v,t}w(x) \leq 0$.*

(5) *If $w(x)$ is continuously twice-differentiable on (x_0, x_1) and $[x_2, x_3] \subset (x_0, x_1)$, then*

$$\lim_{v \longrightarrow 0+} \frac{1}{v^2} \cdot L_{v,t}w(x) = \frac{1}{2} \cdot x^2 \cdot w''(x) \tag{3.6}$$

uniformly for x in $[x_2, x_3], t \in A_n$ and

$$\lim_{v \longrightarrow 0+} \frac{1}{v^2} \int_{A_n} L_{v,t}w(x)d\sigma_n(t) = \frac{1}{2}x^2 \cdot w''(x) \tag{3.7}$$

uniformly for x in $[x_2, x_3]$.

Proof. (1) and (2) follow immediately from the definition of $L_{v,t}$. Let $x(v\lambda_i(t) + 1)$, $1 \leq i \leq n$, belong to the domain of the function $w(x)$, thus the relation $\lambda_1(t) + \cdots + \lambda_n(t) = 0$ as given in (1.1) gives

$$x = \frac{1}{n} \sum_{i=1}^{n} x(v\lambda_i(t) + 1)$$

and the results (3) and (4) follow from the well-known property of convex (or concave) function. And (5) follows by a direct computation and the relations (1.1), the computation involved being valid in view of the conditions imposed on $w(x)$ and the boundedness of the continuous functions $\lambda_i(t)$.

Lemma 3.3. *Under the conditions of* Lemma 3.1, *and $h_n(v)$ as defined in* (3.5), *and let $\phi(x) = e^x - 1 - x$. Then,*

(1)

$$\int_{A_n} L_{v,t}u(x)d\sigma_n(t) + \int_{A_n} \phi(L_{v,t}u(x))d\sigma_n(t) = h_n(v) - h_n(0) \tag{3.8}$$

(2)

$$\int_{A_n} L_{v,t}u(x)d\sigma_n(t) \leq h_n(v) - h_n(0) \tag{3.9}$$

(3) *For sufficiently small $v > 0$ such that $|L_{v,t}u(x)| \leq 1$, then*

$$(3.10) \qquad h_n(v) - h_n(0) \leq \int_{A_n} L_{v,t}u(x)d\sigma_n(t) + \int_{A_n} [L_{v,t}u(x)]^2 d\sigma_n(t)$$

Proof. By the definition of $L_{v,t}$, we have $L_{0,t}v(x) = 0$ and $h_n(0) = 1$, and (3.8) follows from (3.4) and the definition of ϕ. (3.9) and (3.10) follows from (3.3), (3.8) and the inequalities $e^x \leq 1+x$ for all $x \in R$, $e^x \leq 1+x+x^2$ for $|x| \leq 1$.

Lemma 3.4. *Let $u(x)$ be defined and continuous on the interval $[x_0^*, x_1^*]$. Then, either it is convex or there exists a linear function $\ell(x)$ such that the difference $g(x) = u(x) - \ell(x)$ has a local maximum in (x_0^*, x_1^*).*

Proof. The proof can be found in Kagan, Linnik and Rao (1973), p.146.

In the proof of Lemma 3.5 and 3.6 below, we need an averaging operation, and define it as follows. The averaging operation transforms $u(x)$ into

$$u_\epsilon(x) = \int_{-\infty}^{+\infty} K_\epsilon(x-y)u(y)dy$$

where the kernel K_ϵ is chosen such that it has the following properties: it is nonnegative and twice continuously differentiable on $(-\epsilon, \epsilon)$, and zero outside, and

$$\int_{-\infty}^{+\infty} K_\epsilon(x)dx = 1$$

such an averaging leaves constants invariant, and commutes with integration with respect to t and with the linear operator $L_{v,t}$. The properties of the kernel K_ϵ are widely used in mathematical analysis, see for example Wheeden and Zygmund (1977).

Lemma 3.5. *Under the conditions of* Lemma 3.1, *and let $u(x)$ satisfy* (3.4). *Then, the function $u(x)$ is either convex or concave function.*

Proof. Let $[x_0^*, x_1^*]$ be any closed interval such that $[x_0^*, x_1^*] \subset (x_0, x_1)$, it follows from Lemma 3.4 that $u(x)$ is either convex or there exists a linear function $\ell(x)$ such that $g(x) = u(x) - \ell(x)$ has a local maximum in (x_0^*, x_1^*). For the latter case, we shall prove that $u(x)$ must be concave. To this end, let x^* be a local maximum point for $g(x)$, then in view of the relations (1.1) we get $g(x^*(v\lambda_i(t)+1)) \leq g(x^*)$, $1 \leq i \leq n$, for sufficiently small $v > 0$, and consequently $L_{v,t}g(x^*) \leq 0$. Since $u(x) = g(x) + \ell(x)$, it follows from (3.4), (3.11) and (2) of Lemma 3.2 that

$$h_n(v) = \int_{A_n} \exp(L_{v,t}g(x^*))d\sigma_n(t) \leq \int_{A_n} 1 d\sigma_n(t) = h_n(0),$$

and from (3.9), we get

$$\int_{A_n} L_{v,t}u(x)d\sigma_n(t) \leq h_n(v) - h_n(0) \leq 0$$

for all sufficiently small $v > 0$. Now, carrying out the averaging operation, it implies

$$\int_{A_n} L_{v,t}u_\epsilon(x)d\sigma_n(t) \leq 0,$$

by dividing this relation through by v^2, and letting $v \longrightarrow 0^+$. Since $u_\epsilon(x)$ is continuously twice-differentiable, it follows from (3.7) that $u_\epsilon''(x) \leq 0$, i.e., $u_\epsilon(x)$ is a concave function. As $\epsilon \longrightarrow 0$, $u_\epsilon(x) \longrightarrow u(x)$ uniformly for all x in $[x_0^*, x_1^*]$, and the limit of concave functions is itself such a function. Thus, we have established Lemma 3.5.

Lemma 3.6. *Under the conditions of* Lemma 3.5, *the function* $u(x)$ *has the form*

$$u(x) = a + b\ell n(x) + c \cdot x \tag{3.12}$$

for any closed interval $[x_2, x_3]$ *such that* $[x_2, x_3] \subset (x_0, x_1)$, *where* a, b, c *are constants.*

Proof. It follows from Lemma 3.5 and (3), (4) of Lemma 3.2 that $L_{v,t}u(x) \geq 0$(or ≤ 0), according as the function $u(x)$ is convex (or concave). Since $|L_{v,t}u(x)| \leq 1$ for sufficiently small $v > 0$, and so

$$[L_{v,t}u(x)]^2 \leq \pm \sup_{\substack{x_2\leq u\leq x_3 \\ t\in A_n}} |L_{v,t}u(y)| \cdot L_{v,t}u(x)$$

and from (3.9) and (3.10)

$$\int_{A_n} L_{v,t}u(x)d\sigma_n(t) \leq h_n(v) - h_n(0)$$
$$\leq \int_{A_n} L_{v,t}u(x)d\sigma_n(t) \pm \sup_{\substack{x_2\leq u\leq x_3 \\ t\in A_n}} |L_{v,t}u(y)| \cdot \int_{A_n} L_{v,t}u(x)d\sigma_n(t)$$

where the sign + is taken in the convex case and the sign $-$ in the other. Next, by carrying out the averaging operation, dividing above inequalities through by v^2, and letting $v \longrightarrow 0^+$, then it follows from (3.7) and

$$\sup_{\substack{x_2\leq u\leq x_3 \\ t\in A_n}} |L_{v,t}u(y)| \longrightarrow 0 \quad \text{as} \quad v \longrightarrow 0^+$$

that

$$\frac{1}{2}x^2 \cdot u''_\epsilon(x) \leq \underline{\lim}_{v\to 0^+} \frac{h_n(v) - h_n(0)}{v^2} \leq \overline{\lim}_{v\to 0^+} \frac{h_n(v) - h_n(0)}{v^2} \leq \frac{1}{2}x^2 \cdot u''_\epsilon(x).$$

Hence, the limit exists (say equal to c_1), i.e., $\lim_{v\longrightarrow 0^+}[h(v) - h(0)]/v^2 = c_1$, and that $u''_\epsilon(x) = 2c_1/x^2$; so that $u_\epsilon(x) = a_\epsilon + b_\epsilon \ell n(x) + c_\epsilon \cdot x$. As $\epsilon \longrightarrow 0$, $u_\epsilon(x) \longrightarrow u(x)$ uniformly for x in $[x_2, x_3]$, and the limit must be the form (3.11). Thus, we have established lemma 3.6.

Proof of Main Theorem. Since $f(x)$ is not identical to zero, and is continuous on $(0, +\infty)$, we may assume that $f(x) > 0$ on some subinterval (x_0, x_1). It follows from Lemma 3.1, 3.6 and $u(x) = \ell n(f(x))$ that $f(x)$ must be representable in the form $f(x) = a_1 \cdot x^{b_1} \cdot \exp(c_1 \cdot x)$ for any closed interval $[x_2, x_3]$ such that $[x_2, x_3] \subset (x_0, x_1)$, where a_1, b_1, c_1 are constants. Now, by the continuity of $f(x)$, it must be representable in that form throughout $x > 0$. And it follows from the probability character that $f(x)$ is the gamma density as given in (1.3). Thus, this completes the proof of main theorem.

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