BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 25, Number 2, June 1997

ON STRONGLY EXTREME AND DENTING CONTRACTIONS IN $\mathcal{L}(C(X), C(Y))$

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Abstract. A characterization of strongly extreme and denting points of the unit ball of the space of bounded operators between spaces of continuous functions is obtained. Similar results are derived for the spaces of operators on L^∞-spaces and L^1-spaces.

The geometry of Banach space can be characterized by local geometric properties of its unit ball. In the space of operators $\mathcal{L}(E, F)$ there are known only partial results even for classical Banach spaces E, F (cf. [2,3,4,5,6,7].) In this paper we continue this investigations concerning the structure of the space of operators acting on the space of continuous functions.

1. Definitions, notations and basic facts. For a Banach space E let $B(E)$ and $B^0(E)$ denote its closed and open unit balls. Let E^* denote the dual space of E. For Banach spaces E and F let $\mathcal{L}(E, F)$ denote the space of linear bounded operators from E to F with the supremum norm.

Let Q be a closed convex set in a linear space E. Point $x \in Q$ is called:

a) strongly extreme point of the set Q if the condition $\|(x_n + y_n)/2 - x\| \to 0$ for $x_n, y_n \in Q$ implies $\|x_n - x\| \to 0$

b) denting point if

$$\forall \varepsilon > 0 \quad x \notin \overline{\text{conv}}(Q \setminus B^0(x, \varepsilon)),$$

or equivalently if for every $\varepsilon > 0$ there exists an open slice S of the set Q

Received by the editors May 5, 1995 and in revised form April 20, 1996.
AMS 1980 Subject Classification: Primary 46B20, Secondary 47B38, 46E15.
Key words and phrases: Extreme point, strongly extreme point, denting point.

such that $x \in S$ and the diameter of S is less that ε, where an open slice S of the set Q is such a subset of Q that there exists a nontrivial functional $x^* \in E^*$ such that $S = \{x \in Q : x^*(x) > \sup x^*(Q) - \varepsilon\}$. We denote by ext$Q$, s-ext$Q$, dent$Q$ the sets of all extreme points, of all strongly extreme points and of all denting points of Q, respectively. It is easy to see that dent $Q \subset$ s-ext$Q \subset$ ext$Q$.

Let X, Y be compact (Hausdorff) spaces. By $C(X)$ we denote the space of all real continuous functions on X. Morris and Phelps [8] called an operator $T \in \mathcal{L}(C(x), C(Y))$, such that its conjugate T^* transforms Dirac measures (extreme points of $B(C(Y)^*)$) into extreme points of $B(C(X)^*)$ a nice operator. It is of the form

$$(Tf)(y) = r(y) \cdot f(\phi(y)), \quad f \in C(X), \quad y \in Y,$$

where $r \in C(Y)$ with $|r| \equiv 1$ and $\phi : Y \to X$ is a continuous function. The set of nice operators coincides with ext$B(\mathcal{L}(C(X), C(Y)))$ if for example X is metrizable ([1]) or Y is extremely disconnected ([10]).

By δ_x we denote the Dirac measure at x. The dual space $C(X)^*$ is identified with the space of all regular Borel measures equipped with the norm of total variation $\|\mu\| = Var_X \mu$. Recall that

$$Var_A \mu = \sup \sum_{i=1}^{n} |\mu(A_i)|$$

where supremum is taken over all finite partition of A onto Borel sets $\{A_1, A_2, \ldots, A_n\}$. Therefore $T^*\delta_y \in C(X)^*$ is a measure for $T \in \mathcal{L}(C(X), C(Y))$ and hence $(T^*\delta_y)(\{\phi(u)\})$ means the value of a measure $T^*\delta_y$ on a set $\{\phi(u)\}$ consisting of one point $\phi(u)$.

We will use the following facts.

Remark 1. For $a, b, c \in R$, if $|a| \leq 1$, $|b| \leq 1$ and $|c| = 1$ then $|\frac{a+b}{c} - c| \leq \varepsilon$ implies $|a - c| \leq 2\varepsilon$.

Remark 2. If μ is a measure on X and $u \in X$, then

$$Var_{\{u\}^c} \mu \leq \|\mu\| - |\mu(\{u\})|.$$

2. Main results.

Theorem 1. *Every nice operator in $B(\mathcal{L}(C(X), C(Y)))$ is a strongly extreme point of $B(\mathcal{L}(C(X), C(Y)))$.*

Proof: Let T_0 be a nice operator, i.e. is of the form

$$(T_0 f)(y) = r(y) \cdot f(\phi(y)).$$

Let R_n, $S_n \in B(\mathcal{L}(C(X), C(Y)))$ be such that $\|R_n + S_n)/2 - T_0\| \overset{n\to\infty}{\longrightarrow} 0$. We have

$$\sup_{y\in Y} \left\| \frac{R_n^*\delta_y + S_n^*\delta_y}{2} - r(y) \cdot \delta_{\phi(y)} \right\| \to 0.$$

We have $\|R_n^*\delta_y\| \le 1$, $\|S_n^*\delta_y\| \le 1$, so $|(R_n^*\delta_y)(\{\phi(y)\})| \le 1$, $|(S_n^*\delta_y)(\{\phi(y)\})| \le 1$.

Hence, by Remark 1,

$$\left| \frac{R_n^*\delta_y + S_n^*\delta_y}{2}(\{\phi(y)\}) - r(y) \right| \le \left\| \frac{R_n^*\delta_y + S_n^*\delta_y}{2} - r(y)\delta_{\phi(y)} \right\|$$

implies

$$|(R_n^*\delta_y)(\{\phi(y)\}) - r(y)| \le 2\left\| \frac{R_n^*\delta_y + S_n^*\delta_y}{2} - r(y)\delta_{\phi(y)} \right\|$$

and

$$|(R_n^*\delta_y)(\{\phi(y)\})| \ge 1 - 2\left\| \frac{R_n^*\delta_y + S_n^*\delta_y}{2} - r(y)\delta_{\phi(y)} \right\|.$$

Moreover, by Remark 2, we have

$$Var_{\{\phi(y)\}^c} R_n^*\delta_y \le \|R_n^*\delta_y\| - |(R_n^*\delta_y)(\{\phi(y)\})| \le$$
$$1 - |(R_n^*\delta_y)(\{\phi(y)\})| \le 2\left\| \frac{R_n^*\delta_y + S_n^*\delta_y}{2} - r(y)\delta_{\phi(y)} \right\|.$$

Hence

$$\|R_n^*\delta_y - r(y)\delta_{\phi(y)}\| \le |(R_n^*\delta_y)(\{\phi(y)\}) - r(y)| + Var_{\{\phi(y)\}^c} R_n^*\delta_y$$
$$\le 4\left\| \frac{R_n^*\delta_y + S_n^*\delta_y}{2} - r(y)\delta_{\phi(y)} \right\|.$$

Therefore

$$\sup_{y\in Y} \|R_n^*\delta_y - T_0^*\delta_y\| \le 4 \sup_{y\in Y} \left\| \frac{R_n^*\delta_y + S_n^*\delta_y}{2} - T_0^*\delta_y \right\|.$$

Hence $\|R_n - T_0\| \to 0$, i.e. T_0 is a strongly extreme operator.

Corollary 1. *Let X be a metric space or Y be an extremely disconnected space. Then*

$$\text{s-ext}B(\mathcal{L}(C(X), C(Y))) = \text{ext}B(\mathcal{L}(C(X), C(Y))).$$

Theorem 2. *Let $T_0 \in B(\mathcal{L}(C(X), C(Y)))$ be a nice operator. If Y is an infinite space, then $T_0 \notin \text{dent}B(\mathcal{L}(C(X), C(Y)))$.*

Proof: Because Y is an infinite compact Hausdorff space, there exists a sequence of functions $h_i \in C(Y)$ with disjoint supports such that supp $h_i \subset U_i$ and $\|h_i\| = 1$ and $h_i \geq 0$. Let $T_i = (1 - h_i)T_0$. Obviously $\|T_i - T_0\| = 1$. Hence $T_i \in B(\mathcal{L}(C(X), C(Y))) \setminus B(T_0, \frac{1}{2})$. Because $0 \leq \sum_{i=1}^{n} h_i \leq 1$, we obtain

$$\left\| T_0 - \frac{1}{n}\sum_{i=1}^{n} T_i \right\| = \left\| \left(\frac{1}{n}\sum_{i=1}^{n} h_i\right) T_0 \right\| \leq \frac{1}{n}\|T_0\| \cdot \left\| \sum_{i=1}^{n} h_i \right\| = \frac{1}{n} \to 0.$$

Therefore $T_0 \in \overline{\text{conv}}(B(\mathcal{L}(C(X), C(Y))) \setminus B(T_0, \frac{1}{2}))$.

Theorem 3. Let X be a metric space or Y be an extremely disconnected space. Then

$$\text{dent}B(\mathcal{L}(C(X), C(Y))) = \begin{cases} \text{ext}B(\mathcal{L}(C(X), C(Y))), & \text{if card}(Y) < \aleph_0 \\ 0, & \text{if card}(Y) \geq \aleph_0. \end{cases}$$

Proof: For Y is infinite it follows from Theorem 2. If Y is finite, the situation is the same as in the case of one-point space for which $\mathcal{L}(C(X), \mathbb{R}) = C(X)^*$ and the desired result follows from properties of $B(C(X)^*)$.

Recall that each AL-space (abstract Lebesgue space), by the Kakutani representation theorem, is norm and order isomorphic to $L^1(\mu)$ for some positive Radon (not necessarily bounded) measure μ on a locally compact space. Clearly, the space $L^\infty(\mu)$ is the dual of AL-space $L^1(\mu)$, and by another representation theorem can be represented as $C(X)$, where X is the Stonean (extremally disconnected) space (see [9], II.9).

Corollary 2. *Let $(Q_i, \mathbb{B}_i, \mu_i), i = 1, 2$ be σ-finite measurable spaces. Then*

$$\text{s-ext}B(\mathcal{L}(L^\infty(\mu_1), L^\infty(\mu_2))) = \text{ext}B(\mathcal{L}(L^\infty(\mu_1), L^\infty(\mu_2))).$$

Corollary 3. *Let $(Q_i, \mathbb{B}_i, \mu_i), i = 1, 2$ be σ-finite measurable spaces. Then*

$$\text{dent}B(\mathcal{L}(L^\infty(\mu_1), L^\infty(\mu_2))) = \begin{cases} \text{ext}B(\mathcal{L}(L^\infty(\mu_1), L^\infty(\mu_2))) & \text{if } \dim L^\infty(\mu_2) < \infty \\ 0, & \text{if } \dim L^\infty(\mu_2) = \infty. \end{cases}$$

Note that

$$\text{ext}B(\mathcal{L}(L_1(\mu_1), L^1(\mu_2)) = \{T : T^* \in \text{ext}B(\mathcal{L}(L^\infty(\mu_2), L^\infty(\mu_1)))\},$$

([6], p.151).

Corollary 4. *Let $(Q_i, \mathbb{B}_i, \mu_i), i = 1, 2$ be σ-finite measurable spaces. Then*

$$\text{s-ext}B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2)) = \text{ext}B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2)).$$

Proof: Let $T_n, R_n \in B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2))$ be such that $\frac{T_n+R_n}{2} \to T_0 \in \text{ext}B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2))$. Then $\frac{T_n^*+R_n^*}{2} \to T_0^* \in \text{ext}B(\mathcal{L}(L^\infty(\mu_2), L^\infty(\mu_1)) = \text{s-ext}B(\mathcal{L}(L^\infty(\mu_2), L^\infty(\mu_1))$, which implies $\|T_n^* - T_0^*\| \to 0$. Therefore $\|T_n - T_0\| \to 0$, i.e. $T_0 \in \text{s-ext}B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2))$.

Corollary 5. *Let $(Q_i, \mathbb{B}_i, \mu_i), i = 1, 2$ be σ-finite measurable spaces. Then*

$$\text{dent}B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2)) = \begin{cases} \text{ext}B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2)), & \text{if } \dim L^1(\mu_1) < \infty \\ 0, & \text{if } \dim L^1(\mu_1) = \infty \end{cases}$$

Proof: Let $T \in \text{ext}B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2))$. Let us assume that $\dim L^1(\mu_1) < \infty$. Then, by Corollary 3,

$$T^* \in \text{ext}B(\mathcal{L}(L^\infty(\mu_2), L^\infty(\mu_1)) = \text{dent}B(\mathcal{L}(L^\infty(\mu_2), L^\infty(\mu_1)).$$

Hence for every $\varepsilon > 0$ there exists a functional η on $B(\mathcal{L}(L^\infty(\mu_2), L^\infty(\mu_1)))$ and a slice S^* containing T^* with the diameter less than ε. Obviously, we may also consider η as a functional on $\mathcal{L}(L^1(\mu_1), L^1(\mu_2))$. This functional cuts a slice containing T with the diameter less that ε, i.e. $T \in \text{dent}B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2))$.

Let now $\dim L^1(\mu_1) = \infty$. Hence $L^\infty(\mu_1)$ contains a sequence of non-negative measurable functions (h_i) such that $\|h_i\|_\infty = 1$ and $h_i h_j = 0$ μ_1-a.e. if $i \neq j$. Similarly as the proof of Theorem 2 the operators $S_i = (1 - h_i)T^*$ fulfill $\|S_i - T^*\| = 1$ and $\|T^* - \frac{1}{n}\sum_{i=1}^n S_i\| \leq \frac{1}{n}$. It is easy to verify, that there exist $T_i \in (B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2)) \setminus B(T_0, \varepsilon))$ such that $T_i^* = S_i$. In fact, T_i is defined by $T_i f = T((1 - h_i)f)$. Therefore $\|T - \frac{1}{n}\sum_{i=1}^n T_i\| \to 0$, i.e. $T \notin \text{dent}(B(\mathcal{L}(L^1(\mu_1), L^1(\mu_2)) \setminus B(T, \varepsilon))$.

Acknowledgment. The authors wish to thank the referee for his valuable remarks.

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