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PROBABILITY DISTRIBUTION AND A GENERATING FUNCTION OF LAGUERRE POLYNOMIALS

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Abstract. The intrinsic probability model leading to the non-central negative binomial distribution (NNBD) is used to give a probabilistic proof of a well-known generating function result of the generalised Laguerre polynomials.

1. Introduction. The following generating function involving the generalised Laguerre polynomials for $\nu > 0$ and x, y arbitrary:

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} L_k^{(n+\nu-1)}(x) = e^y L_k^{(\nu-1)}(x-y) \tag{1}$$

where the generalised Laguerre polynomials are defined as

$$L_n^{(\alpha)}(z) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; z), \qquad \alpha > -1$$

is rather well-known, see for example, Buchholz [1] and Hansen [2]. More interestingly, the result has also been derived recently by Chatterjee [3] using the novel approach of group theoretic method.

In this note, we shall show that the generating function in (1) is intimately connected with a class of probability distribution known as the non-central negative binomial distribution (NNBD). This distribution and many of its bivariate extensions have been discussed extensively by Lee and Ong in [4]-[9]. From the intrinsic probability model formulation of the

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NNBD, we are able to derive the result in (1) using a purely probabilistic argument.

2. The NNBD distribution. Suppose X is a negative binomial random variable (r.v.) with parameters p and t having the probability distribution (p.d.):

$$b(k) = P[X = k] = \binom{k+t-1}{k} p^k q^t, \quad k = 0, 1, 2, \ldots \tag{2}$$

where $0 < q = 1 - p < 1$ and $t > 0$. Now let t be a r.v. T with

$$T = N + \nu$$

where $\nu > 0$ is a constant, and N is a Poisson r.v. with paramter λ, i.e.,

$$\pi(n) = P[N = n] = \frac{e^{-\lambda}\lambda^n}{n!}, \quad 0 < \lambda < \infty, \quad n = 0, 1, 2, \ldots$$

From (2) we may write the conditional p.d. of X as

$$b(k|n) = P[X = k|N = n] = \binom{k+n+\nu-1}{k} p^k q^{n+\nu}.$$

By the total probability theorem, the p.d. of $X, P(k) = P[X = k]$ is then given by

$$\begin{aligned} P(k) &= \sum_{n=0}^{\infty} b(k|n)\pi(n) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \binom{k+n+\nu-1}{k} p^k q^{\nu+n}. \end{aligned} \tag{3}$$

The expression in (3) may be written as

$$P(k) = e^{-\lambda} p^k q^{\nu} \frac{(\nu)_k}{k!} {}_1F_1(k+\nu; \nu; \lambda q)$$

which, on using Kummer's transformation [10]:

$${}_1F_1(\alpha; \beta; y) = e^y {}_1F_1(\beta - \alpha; \beta; -y)$$

then becomes

$$P(k) = e^{-\lambda p} q^{\nu} p^k L_k^{(\nu-1)}(-\lambda q). \tag{4}$$

The p.d. in (4) is termed NNBD which will be used as a basis in the sequel to derive the generating function relation given in (1).

3. Probabilistic derivation. Now let N_1 and N_2 be two independent Poisson r.v.'s with parameters λ_1 and λ_2 respectively, i.e., for $i = 1, 2$

$$\pi_i(r) = P[N_i = r] = e^{-\lambda_i}\frac{\lambda_i^r}{r!}, \quad r = 0, 1, 2, \ldots$$

As before, let t be a r.v. with

$$T = N_1 + N_2 + \nu$$

The p.d. of X, conditional on N_1 and N_2, is now

$$\begin{aligned} b(k|m,n) &= P[X = k|N_1 = m, N_2 = n] \\ &= \binom{k+m+n+\nu-1}{k} p^k q^{m+n+\nu} \end{aligned} \tag{5}$$

From the independence of the r.v.'s N_1 and N_2, we see that $Q(k) = P[X = k]$ in this case is given by

$$\begin{aligned} Q(k) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b(k|m,n)\pi_1(m)\pi_2(n) \\ &= e^{-\lambda 1p - \lambda_2} p^k q^\nu \sum_{n=0}^{\infty} \frac{(\lambda_2 q)^n}{n!} L_k^{(n+\nu-1)}(-\lambda_1 q) \end{aligned} \tag{6}$$

where the inner sum is evaluated using the result in (4). On the other hand, (6) may be written in the following alternative form:

$$\begin{aligned} Q(k) &= \sum_{n=0}^{\infty} P[X = k|Z = n]P[Z = n] \\ &= \sum_{n=0}^{\infty} b(k|Z = n)P[Z = n] \end{aligned} \tag{7}$$

where $Z = N_1 + N_2$ is the sum of two independent Poisson r.v.'s. It is elementary to show that Z is again a Poisson *r.v.* having the parameter $\lambda_1 + \lambda_2$, i.e.,

$$P[Z = n] = \frac{(\lambda_1 + \lambda_2)^n}{n!} e^{-(\lambda_1+\lambda_2)}, \quad n = 0, 1, 2. \ldots$$

Using the result from (4) in (7), with λ being substituted by $\lambda_1 + \lambda_2$, we then have

(8) $$Q(k) = e^{-(\lambda_1+\lambda_2)p} p^k q^\nu L_k^{(\nu-1)}(-(\lambda_1+\lambda_2)q).$$

On equating the results in (6) and (8), and the fact that $q = 1 - p$, we obtain

(9) $$\sum_{n=0}^{\infty} \frac{(\lambda_2 q)^n}{n!} L_k^{(n+\nu-1)}(-\lambda_1 q) = e^{\lambda_2 q} L_k^{(\nu-1)}(-(\lambda_1+\lambda_2)q).$$

Since the Laguerre polynomials are defined for arbitrary values in the arguments, we may substitute $\lambda_2 q = y$ and $-\lambda_1 q = x$ in (9) to obtain the final result as stated in (1). We note in passing that these substitutions, while mathematically admissible, may give rise to situations with negative probability which, of course, would not have any probabilistic meaning in those cases.

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