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THE EQUITABLE Δ-COLOURING CONJECTURE HOLDS FOR OUTERPLANAR GRAPHS

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Abstract. In this note we prove that the Equitable Δ-Colouring Conjecture holds for outerplanar graphs.

1. Introduction. The graphs we consider are finite, simple and undirected. Let G be a graph. We use $V(G), E(G)$ and $\Delta(G)$ to denote respectively the vertex set, edge set and maximum (vertex) degree of G. If $\{v_1, v_2, \ldots, v_r\} \subseteq V(G)$, then the subgraph of G induced by $\{v_1, v_2, \ldots, v_r\}$ is denoted by $G[v_1, v_2, \ldots, v_r]$. We write $xy \in E(G)$ if the two vertices x and y of G are adjacent in G and we use $d_G(v)$ or simply $d(v)$ to denote the degree of v in G. For $U \subseteq V(G)$ and $v \in V(G)\backslash U$, we use $d_G(v, U)$ to denote the number of edges joining v to U in G. The star of size n is denoted by S_n. The complete bipartite graph with bipartition (X, Y), where $|X| = m$ and $|Y| = n$, is denoted by $K_{m,n}$.

Two graphs G and H are said to be disjoint if they have no vertex in common. The union $G \cup H$ of two disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. A planar graph G is outerplanar if it can be drawn on a plane in such a way that G has no crossing and that all its vertices lie on the boundary of the same face.

We call a (proper) vertex-colouring ϕ of a graph G an equitable colouring of G if the number of vertices in any two colour classes differ by at most

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one. If ϕ is an equitable colouring of G using k colours, then we say that ϕ is an equitable k-colouring of G. The least integer k for which G has an equitable k-colouring is defined to be the equitable chromatic number of G and is denoted by $\chi_e(G)$. The least integer k for which G has an equitable k'-colouring of G for every $k' \geq k$ is denoted by $\chi^*(G)$. A partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ is called an independence-partition of $V(G)$ if each V_i is independent in G (i.e. no two vertices of V_i are adjacent in G). Clearly if $(V_1, V_2, \ldots, V_k)$ is an equitable independence-partition of $V(G)$, then $\chi_e(G) \leq k$.

Hajnal and Szemerédi [1] proved the following theorem.

Theorem 1. (*Hajnal and Szemerédi* [1]) *Any graph G has an equitable k-colouring for any $k \geq \Delta(G) + 1$.*

Meyer [3] proved that any tree T has an equitable $(\lceil \Delta(T)/2 \rceil + 1)$-colouring and he made the following conjecture:

The Equitable Colouring Conjecture (ECC): For any connected graph G, except the complete graph and the odd cycle, $\chi_e(G) \leq \Delta(G)$.

(B. Toft ([4]) has compiled a long list of open problems on colourings of graphs. The ECC is one of the most interesting open problems listed in [4].)

In [2], Chen, Lih and Wu proved that if G is a connected graph having $\Delta(G) \geq \frac{|G|}{2}$ and G is not the complete graph, or the odd cycle, or the complete bipartite graph $K_{2m+1,2m+1}$, then G has an equitable Δ-colouring. (H. P. Yap (1994, unpublished) had independently proved a slightly stronger result that if G is a connected graph of order n having $\Delta(G) \geq \frac{n}{2}$ and G is neither the complete graph nor the complete bipartite graph, then $\chi^*(G) \leq n - \alpha'(\bar{G}) \leq \Delta(G)$, where $\alpha'(\bar{G})$ is the edge independence number of $\bar{G}$. However, a close examination of the proof of Theorem 2 given in [2] reveals that Chen, Lih and Wu had actually obtained the same stronger result.) Based on this result, Chen, Lih and Wu put forth the following conjecture:

The Equitable Δ-Colouring Conjecture: Let G be a connected graph having maximum degree Δ. Suppose G is not the complete graph, or the odd cycle, or the complete bipartite graph $K_{2m+1,2m+1}$. Then G has an equitable Δ-colouring.

Note that if a graph satisfies the Equitable Δ-Colouring Conjecture, then it also satisfies the ECC. Hence the Equitable Δ-Colouring Conjecture is stronger than the ECC.

In [5], Yap and Zhang proved that if G is a graph of order n having $\frac{n}{3}+1 \leq \Delta(G) < \frac{n}{2}$, then $\chi^*(G) \leq r+s+t \leq \Delta(G)$ for some parameters r, s and t of G. Thus the Equitable Δ-Colouring conjecture is true for graphs G of order n having $\Delta(G) \geq \frac{n}{3}+1$.

In this note we prove that outerplanar graphs satisfy the Equitable Δ-Colouring Conjecture.

2. Proof of the theorem. We first prove the following lemma.

Lemma 1. *Let G be the union of disjoint graphs $G_1, G_2, \ldots,$ and G_t. If G_i has an equitable k-colouring for all $i = 1, \ldots, t$, then G has an equitable k-colouring.*

Proof. To prove this lemma, we only need to prove that $G_1 \cup G_2$ has an equitable k-colouring. Let $(V_1, V_2, \ldots, V_k)$ be an equitable independence-partition of $V(G_1)$ and $(V_1', V_2', \ldots, V_k')$ be an equitable independence-partition of $V(G_2)$. Without loss of generality, we assume that $|V_1| \leq |V_2| \leq \cdots \leq |V_k|$ and $|V_1'| \leq |V_2'| \leq \cdots \leq |V_k'|$. Then $(V_1 \cup V_k', V_2 \cup V_{k-1}', \ldots, V_k \cup V_1')$ forms an an equitable independence-partition of $V(G_1) \cup V(G_2)$. Hence G has an equitable k-colouring.

The proof of Lemma 1 is complete.

Before we prove our theorem, let us make a remark. We shall embed an outerplanar graph G in a plane $\mathcal{P}$ in such a way that all its vertices $v_1, v_2, \ldots, v_n$ lie on the circumference of a cycle C in the clockwise direction and all its edges lie inside C without any crossing. Thus, if G is 2-connected, then $v_1v_2v_3\dots v_{n-1}v_nv_1$ forms a Hamilton cycle of G. Also if the connectivity of G is 1, then in the proof of the theorem, we can also imagine that G has an edge joining v_i to v_{i+1} for all $i = 1, \dots, n$, so that we can use Jordan's theorem to ensure that G has no edges v_iv_j and v_kv_l, where $i < k < j < l$. In other words, once the embedding is fixed, then G cannot have two edges v_iv_j and v_kv_l where $i < k < j < l$. For instance, we will assume that the outerplanar graph H given in Figure 1 has no edge joining v_7 and v_9, although $H + v_7v_9$ is still outerplanar.

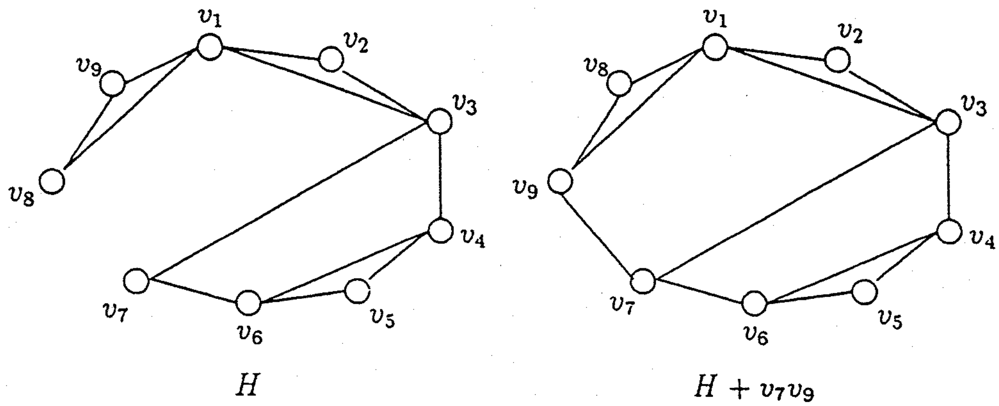

Figure 1.

Theorem 2. *Let G be a connected outerplanar graph having maximum degree $\Delta \geq 3$, then G has an equitable Δ-colouring.*

Proof. We will prove this theorem by induction on $n = |G|$. Let G be a connected outerplanar graph of order $n \geq 4$. Let $\Delta = \Delta(G)$ and $n = m\Delta + r$, where $0 \leq r \leq \Delta - 1$.

We now embed G in a plane $\mathcal{P}$ so that all its vertices $v_1, v_2, \dots, v_n$ lie on the boundary of the exterior face in the clockwise direction in such a way that G cannot have two edges v_iv_j and v_kv_l where $i < k < j < l$.

Suppose that $n \neq m\Delta + 1$. We first prove that $v_iv_{j\Delta+i} \notin E(G)$ for any $i, j \geq 1$. Suppose that $v_iv_{j\Delta+i} \in E(G)$ for some $i, j \geq 1$. Without loss of generality, we consider the case that $i = 1$.

Let $X = \{v_1, v_2, \dots, v_{j\Delta}\}$, $Y = \{v_{j\Delta+1}, v_{j\Delta+2}, \dots, v_n\}$, $G_1 = G[v_1, v_2,$

$\ldots, v_{j\Delta}]$ and $G_2 = G[v_{j\Delta+1}, v_{j\Delta+2}, \ldots, v_n]$. Since $n \neq m\Delta + 1$, we have $|V(G_2)| \geq 2$.

Clearly G_1 and G_2 are outerplanar graphs. Suppose that G_1 (or G_2) is not connected. We consider a connected component G' of G_1. If $\Delta(G') = \Delta$, then by the induction hypothesis (the star S_Δ is a connected graph of least order and having maximum degree Δ, which can be equitably coloured with Δ colours), G' has an equitable Δ-colouring. On the other hand, if $\Delta(G') < \Delta$, then by Theorem 1, G' also has an equitable Δ-colouring. Hence, by Lemma 1, both G_1 and G_2 can be equitably coloured with Δ colours $1, 2, \ldots, \Delta$.

Let ϕ_i be an equitable Δ-colouring of G_i for $i = 1, 2$. Let $V_1, V_2, \ldots, V_\Delta$ be the colour classes of ϕ_1 (if $v \in V_i$, then $\phi_1(v) = i$) and $V_1', V_2', \ldots, V_\Delta'$ be the colour classes of ϕ_2 (if $v \in V_i'$, then $\phi_2(v) = i$).

Since $|X| = j\Delta$ and $|Y| = (m-j)\Delta + r$, we have $|V_i| = j$ and $m - j \leq |V_i'| \leq m - j + 1$, for all $i = 1, 2, \ldots, \Delta$. Since $v_1 v_{j\Delta+1} \in E(G)$, we know that the only edges joining G_1 and G_2 are the edges joining v_1 to Y and those joining $v_{j\Delta+1}$ to X. Let $v_1 \in V_1$ and $v_{j\Delta+1} \in V_\Delta'$. We now consider two cases separately.

Case 1. $d_{G_1}(v_1) \geq 1$.

From $d_{G_1}(v_1) \geq 1$, it follows that $d(v_1, Y) \leq \Delta - 1$. Thus v_1 is not adjacent to any vertex of a colour class of ϕ_2. By permutation of colours, if necessary, we assume that this colour class is V_1'. If $d_{G_2}(v_{j\Delta+1}) \geq 1$, then similarly, $v_{j\Delta+1}$ is not adjacent to any vertex of a colour class, say V_Δ, of ϕ_1. Thus we can combine ϕ_1 and ϕ_2 to obtain an equitable Δ-colouring of G. On the other hand, suppose that $d_{G_2}(v_{j\Delta+1}) = 0$ and $d(v_{j\Delta+1}) = \Delta$ (if $d(v_{j\Delta+1}) < \Delta$, then the above argument applies also). Since G is connected, $d(v_1, \{v_{j\Delta+2}, v_{j\Delta+3}, \ldots, v_n\}) \geq 1$ and as before we can combine an equitable Δ-colouring of $G[v_2, v_3, \ldots, v_{j\Delta+1}]$ and an equitable Δ-colouring of $G[v_{j\Delta+2}, v_{j\Delta+3}, \ldots, v_n, v_1]$ to obtain an equitable Δ-colouring of G.

Case 2. $d_{G_1}(v_1) = 0$.

Since G is connected, we now have $d(v_{j\Delta+1}.\{v_2, v_3, \dots, v_{j\Delta}\}) \geq 1$. In this case, as before, we can combine an equitable Δ-colouring of $G[v_2, \dots, v_{j\Delta+1}]$ and an equitable Δ-colouring of $G[v_{j\Delta+2}, v_{j\Delta+3}, \dots, v_n, v_1]$ to obtain an equiitable Δ-colouring of G. Hence $v_i v_{j\Delta+i} \notin E(G)$ for any $i, j \geq 1$.

Finally, we construct an equitable Δ-colouring of G having colour classes $V_1, V_2, \dots, V_\Delta$ as follows:

$$V_1 = \{v_1, v_{\Delta+1}, \dots, v_{m\Delta+1}\}, \dots, V_r = \{v_r, v_{\Delta+r}, \dots, v_{m\Delta+r}\},$$

$$V_{r+1} = \{v_{r+1}, v_{\Delta+r+1}, \dots, v_{(m-1)\Delta+r+1}\}, \dots, V_\Delta = \{v_\Delta, \dots, v_{m\Delta}\}.$$

Thus G has an equitable Δ-colouring.

Suppose that $n = m\Delta + 1$. If G has a 2-connected block H, then H has at least two vertices of degree 2 (this is a well-know result on 2-connected outerplanar graphs). Thus G has at least one vertex x of degree 2. Hence we can perform the induction on $G - x$ because each colour class of an equitable Δ-colouring of $G - x$ has exactly m vertices and because $\Delta \geq 3$. Otherwise G has a pendant vertex y and we can also perform the induction on $G - y$.

The proof of Theorem 2 is complete.

The following theorem follows immediately from Theorem 2 and Lemma 1.

Theorem 3. *Let G be an outerplanar graph having maximum degree Δ. Then G has an equitable Δ-colouring.*

To end this note we pose the following open problem. If the answer to this problem is affirmative, then it generalizes Meyer's result on trees.

Open Problem. Is it true that if G is an outerplanar graph having maximum degree $\Delta \geq 3$, then $\chi^*(G) \leq \lceil \frac{\Delta}{2} \rceil + 1$?

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