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ON THE OPTIMALITY OF MULTILEVEL SCHWARZ METHODS WITH PARTIAL REFINEMENT

BY

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Abstract. We consider multilevel additive Schwarz methods with partial refinement. These algorithms are generalizations of the multilevel additive Schwarz methods developed by Dryja, Widlund and many others. We will give a different proof by using quasi-interpolants under some weaker assumptions on selected refinement subregions to show that this class of methods has an optimal condition number. Our proof uses some results on iterative refinement methods. As a by-product, the multiplicative versions which correspond to the FAC (Fast Adaptive Composite) algorithms with inexact solvers consisting of one Gauss-Seidel or damped Jacobi iteration have optimal rates of convergence.

1. Introduction. In this paper, we consider some solution methods of the large linear systems of algebraic equations which arise when working with elliptic finite element approximations on composite meshes. We consider the following linear, self-adjoint, elliptic problems discretized by finite element methods on a bounded Lipschitz polyhedral region Ω in R^n.

$$(1)\qquad \begin{cases} -\sum_i \sum_j \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = f & \text{in } \Omega, \\ u = u_0 & \text{on } \Gamma_D \subset \partial\Omega, \\ \sum_j \sum_i a_{ij}(x) \frac{\partial u}{\partial x_j} n_i = g & \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D. \end{cases}$$

Here the matrix $\{a_{ij}(x)\}$ varies moderately and is positive definite with

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a positive uniform lower bound c for almost all x in Ω. Each $a_{ij}(x)$ is a bounded measurable function in Ω and $\vec{n}$ is the unit outward normal to $\partial\Omega$. We assume that the measure of Γ_D is strictly greater than zero. This insures a unique solution to problem (1).

We will assume, without loss of generality, that $u_0 = 0$. If not, we can always substract an arbitrary function w that equals u_0 on Γ_D from u. Let

$$V = H_D^1(\Omega) = \{u \in H^1(\Omega) |\ \gamma u = 0 \text{ on } \ \Gamma_D\}.$$

Here γ is the trace operator. The standard continuous and discrete weak formulations for the above elliptic problem (1) then consist of

$$a(u, v) = f(v), \quad \forall v \in V, \tag{2}$$

and

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in V^h, \tag{3}$$

respectively. Here

$$a(u, v) = \int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \text{ and } f(v) = \int_\Omega fv dx + \int_{\Gamma_N} gv ds.$$

The space V^h will be defined in the next few paragraphs. It is easy to see that the norm $(a(u, u))^{1/2}$ is equivalent to the seminorm $|u|_{H^1(\Omega)}$ in $H^1(\Omega)$ and the ratio of equivalence constants is not large.

To simplify the presentation, we use continuous, Lagrange finite element of type 1 and only consider homogeneous Dirichlet boundary value problems. Then we will remark how to proceed our analysis to more general mixed type boundary condition and Lagrange elements.

The space V^h is defined on a composite triangulation, which is possibly the result of a large number of successive refinements. The triangulation of Ω is given in the following way.

We first introduce a relatively coarse triangulation of Ω, also denoted by Ω_1, and denote the corresponding space of finite element functions by V^{h_1}. We can think of this space as having a relatively uniform mesh size

h_1. Let Ω_2 be a subregion where we wish to increase the resolution. We do so by subdividing the elements and introducing an additional finite element space V^{h_2}. We assure that the resulting composite space $V^{h_1}+V^{h_2}$ is conforming by having the functions of V^{h_2} vanish on $\partial\Omega_2$. We repeat this process by selecting a subregion Ω_3 of Ω_2 and introducing a further refinement of the mesh and finite element space, etc.. We denote the resulting nested subregions and subspaces by Ω_i and V^{h_i} respectively. Throughout, we have $\Omega_i \subset \Omega_{i-1}$ and $V^{h_i-1} \cap H_0^1(\Omega_i) \subset V^{h_i} \subset H_0^1(\Omega_i)$, $i = 2, \cdots, k$. The composite finite element space on the repeatedly refined mesh, is

$$V^h = V^{h_1} + V^{h_2} + \cdots + V^{h_k}.$$

We assume that all the elements are shape regular in the sense that there is a uniform bound on h_K/ρ_K. Here h_K and ρ_K are the diameter and the radius of the largest inscribed sphere of any element K, respectively. Our theoretical bounds, developed in this paper, also depend on the shapes of the subregions Ω_i.

The finite element problem is defined by equation (3) and the corresponding stiffness matrix can conveniently be computed by using a process of subassembly. Introducing subscripts to indicate the domain of integration, we write

$$a(u,v) = a_{\Omega_1\setminus\Omega_2}(u,v) + a_{\Omega_2\setminus\Omega_3}(u,v) + \cdots + a_{\Omega_k}(u,v).$$

The stiffness matrices corresponding to the regions $\Omega_i \setminus \Omega_{i+1}$, $1 \le i \le k-1$, and Ω_k are computed by working with basis functions related to the mesh size h_i. The quadratic form corresponding to the composite stiffness matrix is the sum of the quadratic forms corresponding to $\Omega_i \setminus \Omega_{i+1}$, $1 \le i \le k-1$, and Ω_k. When we refine a finite element model locally, the modified stiffness matrix is obtained by replacing the quadratic form associated with the subregion in question by the one corresponding to the refined model on the same subregion. It is therefore relatively easy to design a method which systematically generates the stiffness matrices for all the standard problems necessary while, at the same time, the stiffness matrix of the composite model is computed.

We use the framework of multilevel additive Schwarz methods, which is described in Dryja and Widlund [7], and Zhang [17] to discuss a new kind of algorithm for composite finite element problems. If we compare this kind of algorithm with the AFAC (Asynchronous FAC) methods is [3], we can see that they are both additive Schwarz methods. In the new algorithms, we decompose the problems corresponding to the refined subregions with uniform mesh size, used in AFAC, into many much smaller problems which are much easier to solve. However, this is at the expense of slower convergence of the algorithms.

We will apply quasi-interpolants and some theoretical results from FAC and AFAC methods to prove in a different way that the iteration operators of these methods have a uniform lower bound under some weaker assumptions than those before. Bornemann and Yserentant [1] have obtained another proof of the optimality based on the use of K-functionals under some restrictions on these refinement subregions. We remark that our proof can be generalized to these cases of refinement everywhere and refinement for solution singularity coming from the coefficients of the original differential equation and is different from Zhang's which was obtained by considering a decomposition based on the Galerkin projection on a larger convex domain. Therefore we conclude that our results in this paper are better than theirs.

We can also consider some multiplicative versions of above methods. These variants correspond to the FAC algorithms with inexact solvers consisting of one Gauss-Seidel or damped Jacobi iteration. We can use the similar arguments of Zhang to show that these variants have an optimal rate of convergence.

In Section 2, we describe general multilevel additive Schwarz methods and mention some theoretical results about them.

In Section 3, we describe general multilevel additive Schwarz methods with partial refinement and mention the earlier theoretical result. Then we describe quasi-interpolants and prove some lemmas about them which we need to apply in next section.

In Section 4, we develop our optimality proof based on some assumptions coming from iterative refinement methods and describe some multiplicative variants.

2. General multilevel additive Schwarz methods. We now give a description of general multilevel additive Schwarz methods which is developed in Dryja and Widlund [7]. We define a sequence of nested triangulations $\{\Upsilon_{l=1}^k\}$. We start with a coarse triangulation $\Upsilon^1 = \{\tau_i^1\}_{i=1}^{N_1}$ with quasi-uniform mesh size h_1, where τ_i^1 represent an individual triangle. The successively finer triangulations $\Upsilon^l = \{\tau_i^l\}(l = 2, \dots, L)$ are defined by dividing each triangle in the triangulation Υ^{l-1} into several triangles, i.e.

$$\Upsilon^1 = \{\tau_i^1\}_1^{N_1} \overset{\text{refinement}}{\Longrightarrow} \Upsilon^2 = \{\tau_i^1\}_i^{N_2} \overset{\text{refinement}}{\Longrightarrow} \dots \overset{\text{refinement}}{\Longrightarrow} \Upsilon^k = \{\tau_i^k\}_1^{N_k}.$$

We assume that the triangulations Υ^l have quasi-uniform mesh size h_{l-1} for each l.

Let $V^{h_l}, l = 1, \cdots, k$, be the space of continuous piecewise linear element associated with the triangulation Υ^l. The finite element solution, $u_h \in V^h = V^{h_k}$, satisfies

$$a(u_h, \phi_h) = f(\phi_h), \quad \forall \phi_h \in V^h = V^{h_k}. \tag{4}$$

We assume that there are $k-1$ sets of overlapping subdomains $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, $l = 2, 3, \cdots, k$. On each level, we have an overlapping decomposition

$$\Omega = \cup_{i=1}^{N_l} \tilde{\Omega}_i^l.$$

We assume that the sets $\{\tilde{\Omega}_i^l\}$ satisfy the following assumption.

Assumption 1. *The decomposition* $\Omega = \cup_{i=1}^{N_l} \tilde{\Omega}_i^l$ *satisfies*

(1) $\partial\tilde{\Omega}_i^l$ *aligns with the boundaries of level l triangles, i.e.* $\tilde{\Omega}_i^l$ *is the union of level l triangles. Diameter* $(\tilde{\Omega}_i^l) = O(h_{l-1})$.

(2) *On each level, the subdomains* $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$ *form a finite covering of* Ω, *with a covering constant* N_c, *i.e. we can color* $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, *using at most* N_c *colors in such a way that subdomains of the same color are disjoint.*

(3) *On each level, associated with* $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, *there exists a partition of unity* $\{\theta_i^l\}$ *satisfying*

$$\sum_i \theta_i^l = 1, \text{ with } \theta_i^l \in H_0^1(\tilde{\Omega}_i^l) \cap C^0(\tilde{\Omega}_i^l), 0 \leq \theta_i^l \leq 1 \text{ and } |\nabla \theta_i^l| \leq C/h_{l-1}.$$

(4) h_l/h_{l+1} *is uniformly bounded.*

One way of constructing subdomains $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}, l = 2, \cdots, k$, with the above properties is described in Dryja and Widlund [4], [6]. Each triangle $\tau_i^{l-1}, i = 1, \cdots, N_l, l = 2, \cdots, k$, is extended to a larger region $\tilde{\tau}_i^{l-1}$ so that $ch_{l-1} \leq \text{dist}(\partial\tilde{\tau}_i^{l-1}, \partial\tau_i^{l-1}) \leq Ch_{l-1}$, aligning $\partial\tilde{\tau}_i^{l-1}$ with the boundaries of level l triangles. We cut off the part of $\tilde{\tau}_i^{l-1}$ that is outside Ω. We use $\tilde{\tau}_i^{l-1}$ as the subdomains $\tilde{\Omega}_i^l$. Another way of constructing $\{\tilde{\Omega}_i^l\}$ is given in the next section.

Let $N_1 = 1, V_1^{h_1} = V^{h_1}$ and $V_i^{h_l} = V^{h_l} \cap H_0^1(\tilde{\Omega}_i^l)$ for $i = 1, \cdots, N_l, l = 2, \cdots, k$. The finite element space $V^h = V^{h_k}$ is represented by

$$V^h = \sum_{l=1}^{k} V^{h_l} = \sum_{l=1}^{k}\sum_{i=1}^{N_l} V_i^{h_l}.$$

Let $P_i^l : V^h \to V_i^{h_l}$, be the projection defined by

$$a(P_i^l u, \phi) = a(u, \phi), \quad \forall \phi \in V_i^{h_l}.$$

The k-level additive Schwarz operator P is defined by

$$P = \sum_{l=1}^{k}\sum_{i=1}^{N_l} P_i^l. \tag{5}$$

Instead of solving the original finite element equation (4), we use the following algorithm.

Algorithm 1. *Let P be the operator defined by* (5). *Apply the conjugate gradient method to the following symmetric and positive definite system*

$$Pu_h = g_h,$$

with respect to the inner product $a(\cdot,\cdot)$ *for an appropriate* g_h *such that the solution* u_h *is the same as that of* (4).

The following theorem, which is given in Zhang [17], proves that this multilevel additive Schwarz method has an optimal rate of convergence.

Theorem 1. *For* P *defined above, the following inequalities hold*

$$C_1 a(u_h, u_h) \le a(Pu_h, u_h) \le C_2 a(u_h, u_h) \quad \forall u_h \in V^h.$$

Thus $\kappa(P) \le C_2 C_1^{-1}$. *Here the constants* C_1 *and* C_2 *are independent of the mesh sizes* $\{h_l\}$ *and* k.

3. Description of multilevel additive Schwarz methods with partial refinement and some properties of quasi-interpolants. We can modify the general multilevel additive Schwarz methods such that they can handle the finite element problems (3) with composite mesh sizes.

We now give a description of multilevel additive Schwarz methods with partial refinement. Like the procedure in last section, we define a sequence of nested triangulations $\{\Upsilon_{l=1}^k\}$. We start with a coarse triangulation $\Upsilon^1 = \{\tau_i^1\}_{i=1}^{N_1}$ with quasi-uniform mesh size h_1, where τ_i^1represent an individual triangle. The successively finer triangulations $\Upsilon^l = \{\tau_i^l\}(l = 2, \cdots, k)$ are defined by dividing each triangle in the triangulation Υ^{l-1} into several triangles, i.e.

$$\Upsilon^1 = \{\tau_i^l\}_1^{N_1} \overset{\text{refinement}}{\Longrightarrow} \Upsilon^2 = \{\tau_i^1\}_1^{N_2} \overset{\text{refinement}}{\Longrightarrow} \cdots \overset{\text{refinement}}{\Longrightarrow} \Upsilon^k = \{\tau_i^k\}_1^{N_k}.$$

We assume that the triangulations Υ^l have quasi-uniform mesh sizes h_{l-1} for each l.

Let us define $\Omega_1 = \Omega$. Then for each $2 \le l \le k$, we choose Ω_l,which is a subregion of Ω_{l-1}, such that $\partial\Omega_l$ aligns with boundaries of level $l-1$ triangles. Let $\tilde{V}^{h_k}, l = 1, \cdots, k$, be the subspace of continuous piecewise linear element associated with the triangulation Υ^l of $H_0^1(\Omega)$. We also set V^{h_l} to be $\tilde{V}^{h_l} \cap H_0^1(\Omega_i)$. The finite element problem is to find $u_h \in V^h =$

$V^{h_1}+\cdots+V^{h_k}$ satisfying

$$a(u_h,\phi_h)=f(\phi_h),\quad \forall\phi_h\in V^h=V^{h_1}+\cdots+V^{h_k}. \tag{6}$$

We assume that there are $k-1$ sets of overlapping subdomain $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, $l=2,3,\cdots,k$. On each level, we have an overlapping decomposition

$$\Omega_l=\cup_{i=1}^{N_l}\tilde{\Omega}_i^l.$$

We also assume that there are another $k-1$ sets of overlapping subdomains $\{\tilde{\Omega}_i^l\}_{i=N_l+1}^{N_l+M_l}$ such that we have

$$\Omega=\cup_{i=1}^{N_l+M_l}\tilde{\Omega}_i^l.$$

We can now make the following assumptions similar to Assumption 1.

Assumption 2. *Let us assume that*

(1) *The mesh sizes h_l are bounded from above and below by const.q^l uniformly for all l. Here q is a positive constant less than* 1.

(2) $(\partial\Omega_{l-1}\cap\partial\Omega_l)\setminus\partial\Omega=\emptyset$ *for $l=2,3,\cdots,k$.*

(3) $\partial\tilde{\Omega}_i^l$ *aligns with boundaries of level l triangles, i.e. $\tilde{\Omega}_i^l$ is the union of level l triangles. Diameter $(\tilde{\Omega}_i^l)=O(h_{l-1})$.*

(4) *On each level, the subdomains $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l+M_l}$ form a finite covering of Ω, with a covering constant N_c, i.e. We can color $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l+M_l}$, using at most N_c colors in such a way that subdomains of the same color are disjoint.*

(5) *On each level, associated with $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, there exists a partition of unity $\{\theta_i^l\}$ satisfying*

$$\sum_i\theta_i^l=1,\ \text{with}\ \theta_i^l\in H_0^1(\tilde{\Omega}_i^l)\cap C^o(\tilde{\Omega}_i^l), 0\le\theta_i^l\le 1\ \text{and}|\nabla\theta_i^l|\le C/h_{l-1}.$$

One way of constructing subdomains $\{\tilde{\Omega}_{i=1}^{N_l+M_l}\}$, $l=2,\cdots,k$, with the above properties is mentioned in last section. Let $N_1=1$, $V_1^{h_1}=V^{h_1}$ and $V_i^{h_l}=V^{h_l}\cap H_0^1(\tilde{\Omega}_i^l)$ for $i=1,\cdots,N_l+M_l$, $l=2,\cdots,k$. The finite element space V^h is represented by

$$V^h = \sum_{l=1}^{k} V^{h_l} = \sum_{l=1}^{k}\sum_{i=1}^{N_l} V_i^{h_l}.$$

Let us define P_i^l as the orthogonal projection from V^h onto $V_i^{h_l}$ with respect to $a(\cdot,\cdot)$ which is the same as those in last section. The k-level additive Schwarz operator P is defined by

$$P = \sum_{l=1}^{k}\sum_{i=1}^{N_l} P_i^l. \tag{7}$$

Let us denote the multilevel additive Schwarz methods by MAS. Then we have the following algorithm.

Algorithm 2 (MAS with partial refinement). *Let P be the operator defined by* (7). *Apply the conjugate gradient method to the following symmetric and positive definite system*

$$Pu_h = g_h,$$

with respect to the inner product $a(\cdot,\cdot)$ for appropriate g_h such that the solution u_h is the same as that of (6).

Bornemann and Yserentant [1] have established the following optimality theorem for MAS under Assumption 2. Our main purpose of this paper is to prove the following theorem under some weaker assumptions.

Theorem 2. *For P defined by* (7) *and under Assumption* 2, *the following inequalities hold*

$$C_1 a(u_h, u_h) \le a(Pu_h, u_h) \le C_2 a(u_h, u_h), \quad \forall u_h \in V^h.$$

Thus $k(P) \le C_2 C_1^{-1}$. Here the constants C_1 and C_2 are independent of the mesh sizes $\{h_l\}$ and k.

We now mention a special decomposition of the domain Ω in Zhang [17]. It is called MDS (the multilevel diagonal scaling). Let ϕ_i^l be a nodal basis function of V^{h_l}, and associate with each ϕ_i^l the subdomain $\tilde{\Omega}_i^l = supp\{\phi_i^l\}$.

We may choose $V_i^l = span\{\phi_i^l\} = V^{h_l} \cap H_0^1(\tilde{\Omega}_i^l)$ and obtain the decomposition

$$V^h = \sum_{l=1}^{k} \sum_{i=1}^{N_l} V_i^{h_l}$$

and the Galerkin projection P_i^l corresponding to $V_i^{h_l}$. Let $P' = \sum_{l=1}^{k} \sum_{i=1}^{N_l} P_i^l$. It is easy to see that the above construction satisfies Assumption 2. Therefore we have another variant of Algorithm 2 whose optimality follows from Theorem 2.

Algorithm 3 (MDS with partial refinement). *Let P' be the operator defined above. Apply the conjugate gradient method to the following symmetric and positive definite system*

$$P'u_h = g_h,$$

with respect to the inner product $a(\cdot,\cdot)$ for an appropriate g_h such that the solution u_h is the same as that of (6).

Let K_l be the stiffness matrix associated with V^{h_l}, let K_h be the stiffness matrix associated with V^h and let $D_l = diag(K_l)$. Let $I_l : V^{h_l} \to V^h$, $1 \leq l \leq k$, be the standard inclusion operator, and let $I_l^t : V^h \to V^{h_l}$ be an operator related to I_l in the following way:

$$(I_l^t u_h, v^l)_l = (u_h, I_l v^l)_{L^2(\Omega)}, \quad \forall v^l \in V^{h_l}.$$

Here $(\cdot,\cdot)_l$ is the discrete inner product in V^{h_l}, which is equivalent to $L^2(\Omega)$, defined by

$$(u^l, v^l)_l = h_l^n \sum_{x \in \mathcal{N}_l} u^l(x) v^l(x), \quad \forall u^l, v^l \in V^{h_l}.$$

Here $\mathcal{N}_l$ is the set of nodes of the degrees of freedom in V^{h_l}. Algorithm 3 can then be written as: Find the solution of $K_h x = b$ by solving the preconditioned system

$$B_h^{-1} K_h x = B_h^{-1} b,$$

where

$$B_h^{-1} = h_1^n \cdot I_1 K_1^{-1} I_1^t + \cdots + h_{k-1}^n \cdot I_{k-1} D_{k-1}^{-1} I_{k-1}^t + h_k^n \cdot D_k^{-1}.$$

We remark that if we replace the matrices D_l by identity matrices, we obtain the BPX algorithm with partial refinement.

In order to prove Theorem 2 under some weaker assumptions, we need to introduce the concept of quasi-interpolants and prove some of their properties which we need later.

Definition 1. *Given a triangulation Υ of Ω, we associate with Υ a finite element subspace $V(\Upsilon)$ of $L^2(\Omega)$ which consists of piecewise polynomials of degree less than or equal to m. A linear mapping*

$$Q : L^2(\Omega) \to V(\Upsilon)$$

is called a quasi-interpolant of order m if it satisfies the properties

$$Qu = u, \quad \forall u \in V(\Upsilon),$$

and for a constant C depending only on the shape regularity such that

$$\|Qu\|_{L^2(K)} \leq C \cdot \|u\|_{L^2(\overline{K})}, \quad \forall K \in \Upsilon, \quad \forall u \in L^2(\Omega).$$

Here $\overline{K}$ denotes the union of the neighbouring elements of K.

The following example is similar to one given in Oswald [12].

Example. We construct a quasi-interpolant for linear elements in two dimensions with zero boundary data. The procedure can be generalized to the cases of more general Lagrange elements and higher-dimensional spaces. Consider an arbitrary nodal point P_i(=vertex) of Υ which in not on $\partial\Omega$ and its adjacent triangles. Define on the region a piecewise linear and continuous function with value 3 at P_i and -1 at the other vertices of this region. Extend this function by zero to Ω and scale it by the factor $3/|A_i|$, where A_i denotes the support of this function. Denote the nodal basis function corresponding to P_i by ϕ_i and this L^∞ function by ϕ_i^*. Now we may take

$$Qu = \sum_i \int_\Omega (u, \phi_i^*)_{L^2(\Omega)} \cdot \phi_i.$$

If we look at the quadrature rule for triangles which uses the side midpoints as integration points and is exact for polynomials of degree 2, it is easy to see that Q satisfies the first condition in Definition 1. There remains to verify the second condition. We first consider these elements K which satisfy $K \cap \partial\Omega = \emptyset$. For such element K with area $|K|$, we have $|A_i| \geq |K|$, for each of its three vertices P_i, and

$$|(u, \phi_i^*)_{L^2(\Omega)}| \leq \|u\|_{L^2(A_i)} \cdot \|\phi_i^*\|_{L^2(A_i)} \leq C \cdot |A_i|^{-1/2} \cdot \|u\|_{L^2(\bar{K})}$$

for the corresponding coefficients. Therefore

$$\|Qu\|_{L^2(K)} \leq |K|^{1/2} \cdot \max |(u.\phi_i^*)_{L^2(\Omega)}| \leq C \cdot \|u\|_{L^2(\overline{K})}.$$

For these elements which intersect $\partial\Omega$, we can use similar arguments to prove the same inequality as above. Finally we remark that the value of Qu at an arbitrary interior vertex P only depends upon the values of u in the elements which have P as a vertex.

Let $\tilde{Q}_l$ be a quasi-interpolant of order 1 from $H_0^1(\Omega) \subset L^2(\Omega)$ onto the space $\tilde{V}^{h_l}$, for $l = 1, 2, \cdots, k$. We construct $\tilde{Q}_l u$ by the rules of the above example. In order to prove our optimality result in next section, we need the following three lemmas about quasi-interpolants $\tilde{Q}_l u$. The proof of the first lemma is based on using smooth functions to approximate elements in $H^1(\Omega)$ and applying the fundamental theorem of calculus to the region. Proofs of Lemma 2 may be found in [10] and [11]. Proofs of the boundness of the L^2 projection in $H_0^1(\Omega)$ are given in Scott and Zhang [13], and in Bramble and X_u [2]. Lemma 3 just states the corresponding result for quasi-interpolants.

Lemma 1. *Let Ω be a domain in R^2, which has the following special form : $\{(x_1, x_2)|a < x_1 < b,\ g(x_1) < x_2 < f(x_1)\}$. Here $f(x)$ and $g(x)$ are piecewise C^1, continuous functions on $[a, b]$ such that $g(x) < f(x)$ on (a, b). Then for all $u \in H^1(\Omega)$ vanishing on $\{(x_1, x_2)|a \leq x_1 \leq b,\ x_2 = g(x_1)\}$, we have*

$$\|u\|_{L^2(\Omega)} \leq \max_{a\leq x\leq b} |f(x) - g(x)| \cdot |u|_{H^1(\Omega)}.$$

We can also get similar inequalities in R^n for $n > 2$.

Lemma 2 (Poincaré's inequality). *Let*

$$\{u\}_\Omega = \frac{1}{|\Omega|} \int_\Omega u.$$

Then there exists a constant $C(\Omega)$, which depends only on the Lipschitz constant of $\partial\Omega$, such that for all $u \in H^1(\Omega)$ we have

$$\|u - \{u\}_\Omega\|_{L^2(\Omega)} \leq C(\Omega) H_\Omega |u|_{H^1(\Omega)}.$$

Here H_Ω is the diameter of Ω.

Lemma 3. *There exists a constant C, which depends only on the shape regularity, such that*

$$|\tilde{Q}_l u|_{H^1(\Omega)} \leq C|u|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Proof. Let us first consider the elements K that satisfy $\bar{K} \cap \partial\Omega = \emptyset$. Then we have

$$\begin{aligned}|\tilde{Q}_l u|_{H^1(K)} &= |\tilde{Q}_l u - c|_{H^1(K)} = |\tilde{Q}_l(u - u_c)|_{H^1(K)}\\ &\leq C h_l^{-1} \|\tilde{Q}_l(u - u_c)\|_{L^2(K)} \leq C h_l^{-1} \|u - c\|_{L^2(\bar{K})}\end{aligned}$$

for any constant c. Here $\overline{K}$ is the union of the neighbouring elements of K and $u_c \in \tilde{V}^{h_l}$ is equal to c at the interior nodes and 0 at the boundary nodes..

Take c to achieve the infimum. By Lemma 2, we have

$$|\tilde{Q}_l u|_{H^1(K)} \leq C h_l^{-1} \cdot \inf_c \|u - c\|_{L^2(\overline{K})} \leq C h_l^{-1} \cdot C' h_l |u|_{H^1(\overline{K})} \equiv C|u|_{H^1(\overline{K})}.$$

Let Ω_0 denote the union of such elements K. By shape regularity, we obtain

$$|\tilde{Q}_l u|_{H^1(\Omega_0)} \leq C|u|_{H^1(\Omega)}$$

for some constant C. Now it is sufficient to prove that

$$|\tilde{Q}_l u|_{H^1(\Omega\setminus\Omega_0)} \leq C|u|_{H^1(\Omega)}.$$

It is obvious that we can write $\overline{\Omega}\setminus\Omega_0 = \cup_{i=1}^{N_0}\overline{\Omega}_{0i}$ as a nonoverlapping union. Let $\Omega'_{0i} = \cup_{K\in\Omega_{0i}}\overline{K}$ where $\overline{K}$ is the union of the neighbouring elements of K. It is obvious that each region Ω'_{0i} and each function $u \in H_0^1(\Omega)$ satisfy the conditions of Lemma 1 and that the constant of this lemma is $O(h_l)$. Then we have

$$\begin{aligned}|\tilde{Q}_l u|_{H^1(\Omega_{0i})} &\leq Ch_l^{-1}\|\tilde{Q}_l u\|_{L^2(\Omega_{0i})}\\ &\leq Ch_l^{-1}\|u\|_{L^2(\Omega'_{0i})} \leq Ch_l^{-1}\cdot h_l|u|_{H^1(\Omega'_{0i})} = C|u|_{H^1(\Omega'_{0i})}.\end{aligned}$$

By combining the above results, the proof of the lemma easily follows.

Besides the above lemmas, we also need the following two lemmas which are not directly related to quasi-interpolants. The first lemma is often used in finite element approximation theory and its proof can be carried out by using a standard duality argument; cf. [3]. The proof of the second lemma may be found in [9] and [15]. It means that we can estimate the lower bound of the operator P defined by (7) through constructing a good decomposition of finite element functions.

Lemma 4. *Let $\tilde{P}_l$ be the orthogonal projection onto the space $\tilde{V}^{h_l}$ with respect to $a(\cdot,\cdot)$ and suppose that the coefficients $\{a_{ij}(x)\}$ of elliptic problem (1) are in $W^{1,\infty}(\Omega)$. Then there exists a $s \in (1/2, 1]$ and a constant C such that*

$$\|(I-\tilde{P}_l)u\|_{H^{1-s}(\Omega)} \leq Ch_l^s\|(I-\tilde{P}_l)u\|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Lemma 5. *Let V be a finite-dimensional Hilbert space with the inner product $a(\cdot,\cdot)$ and let V_i be subspaces of V so that $V = V_1 + \cdots + V_N$. We define P_i as the orthogonal projection onto V_i and $P = P_1 + \cdots + P_N$. If a decomposition of $u, u = \sum_i u_i$ where $u_i \in V_i$, can be found such that*

$$\sum_i a(u_i,u_i) \leq C_1 a(u,u), \quad \forall u \in V^h,$$

then

$$\lambda_{\min}(P) \geq C_1^{-1}.$$

4. The main optimality result and some multiplicative variants. In this section, we will construct another proof of our main theorem by using the approach of iterative refinement methods. For convenience, we use the same notations as in last section. The typical assumption for iterative refinement methods is related to the extension theorem for finite element functions with respect to the $a(\cdot,\cdot)$.

Assumption 3. *For each j, there exists a bounded Lipschitz polyhedral region $\tilde{\Omega}_j$ such that $\Omega_j \subset \tilde{\Omega}_j$, $(\tilde{\Omega}_j \setminus \Omega_j) \cap \Omega = \emptyset$, $\partial\tilde{\Omega}_j \cap \partial\Omega_{j+1} = \emptyset$ and the Lipschitz constants of $\tilde{\Omega}_j \setminus \Omega_{j+1}$ are uniformly bounded.*

The above assumption can usually be weakened to Assumption 4.

Assumption 4. *For each j, either $\Omega_j = \Omega_{j+1}$ or there exists a bounded Lipschitz polyhedral region $\tilde{\Omega}_j$ such that $\Omega_j \subset \tilde{\Omega}_j$, $(\tilde{\Omega}_j \setminus \Omega_j) \cap \Omega = \emptyset$, $\partial\tilde{\Omega}_j \cap \partial\Omega_{j+1} = \emptyset$ and the Lipschitz constants of $\tilde{\Omega}_j \setminus \Omega_{j+1}$ are uniformly bounded.*

Let us define $P_j^i, i \leq j$, as the orthogonal projections onto the spaces $V^{h_i} \cap H_0^1(\Omega_j)$ with respect to the inner product $a(\cdot,\cdot)$. Now we recall a result from Cheng [3] and Dryja and Widlund [5].

Lemma 6. *Under Assumption 4, there is an absolute constant C which depends on the Lipschitz constant in Assumption 4 and shape regularity such that for any $u \in V^h$ we can decompose u into $u = \sum_{i=1}^k u_i$, where $u_i \in Range(P_i^i - P_i^{i-1})$, and*

$$\sum_{i=1}^{k} a(u_i, u_i) \leq C a(u, u).$$

We remark that the proof of this lemma can be done by first considering the case under Assumption 3 and then doing a further decomposition of u under Assumption 4.

Now let us make the remaining assumption used in the main theorem in this section and then state the main theorem.

Assumption 5. *Let us assume that*

(1) *These mesh sizes h_l are bounded from above and below by const.q^l uniformly for all l. Here q is a positive constant less than 1.*

(2) *$\partial\tilde{\Omega}_i^l$ aligns with boundaries of level l triangles, i..e. $\tilde{\Omega}_i^l$ is the union of level l triangles. Diameter $(\tilde{\Omega}_i^l) = O(h_{l-1})$.*

(3) *On each level, the subdomains $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l+M_l}$ form a finite covering of Ω, with a covering constant N_c, i.e. we can color $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l+M_l}$, using at most N_c colors in such a way that subdomains of the same color are disjoint.*

(4) *On each level, associated with $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, there exists a partition of unity $\{\theta_i^l\}$ satisfying*

$$\sum_i \theta_i^l = 1, \text{ with } \theta_i^l \in H_0^1(\tilde{\Omega}_i^l) \cap C^0(\tilde{\Omega}_i^l), 0 \le \theta_i^l \le 1 \text{ and } |\nabla \theta_i^l| \le C/h_{l-1}.$$

Theorem 3. *Under Assumptions* 4 *and* 5, *there exist absolute constants C_1 and C_2 such that*

$$C_1 a(u_h, u_h) \le a(Pu_h, u_h) \le C_2 a(u_h, u_h), \quad \forall u_h \in V^h.$$

Here P is defined by (7). *Thus $\kappa(P) \le C_2 C_1^{-1}$. Here the constants C_1 and C_2 are independent of the mesh sizes $\{h_l\}$ and k.*

Before proceeding the proof of Theorem 3, let us compare the assumptions used in our proof with those used in Theorem 2. Our assumptions are weaker than theirs in some sense. In their proof, they need to assume that $(\partial\Omega_{l-1} \cap \partial\Omega_l) \setminus \partial\Omega = \emptyset$ for $l = 2, 3, \cdots, k$. It means that we cannot allow the next-level refinement subregion Ω_l, chosen from a given Ω_{l-1}, whose boundary has a nonempty intersection with that of Ω_{l-1} in the interior of the whole domain Ω. This condition restricts the choice of Ω_l in two ways. First, we cannot choose Ω_l to be the same as Ω_{l-1}. However, we cannot allow the consecutive mesh size ratio h_l/h_{l+1} to be arbitrarily large in general multilevel Schwarz methods. Therefore in order to get the prescribed accuracy of our solution of partial differential equation, it is possible that we need to refine the subregion Ω_{l-1} everywhere because we cannot control the size of Ω_l as small as we can. The second way happens when we consider a boundary value problem of an elliptic partial differential equation whose coefficients are not smooth enough. In such a case, a solution singularity maybe happen around an interior point of Ω and therefore we need to do the mesh refinement near this singularity point. It is possible that we have chosen Ω_{l-1} such that this singularity point is on its boundary from a given error estimate criteria. Then we cannot continue the refinement process according to their assumptions.

The main idea of proving Theorem 3 is that constructing a good decomposition of $u \in V^h$ satisfies the condition of Lemma 5. Let us define the operators $R_l : V^h \to \tilde{V}^{h_l}$ by

$$R_l u(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega} \setminus \Omega_{l+1} \\ \tilde{Q}_l(x) & \text{if } x \in \Omega_{l+1} \end{cases}$$

for $l = 1, 2, \cdots, k-1$ and $R_k u = u$. It is obvious that $R_m R_n = R_n$ for $1 \le n \le m \le k$. It is also clear that there exists an absolute constant C such that

$$\|R_l u\|_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega)}.$$

There are some other important properties of R_l which we need. They are stated below.

Lemma 7. *There is an absolute constant C such that*

$$\|u - R_l u\|_{L^2(\Omega)} \le C h_l |u|_{H^1(\Omega)}.$$

Proof. Let us denote the union of the element K of level l in Ω_{l+1} which satisfies $K \cap (\Omega \setminus \Omega_{l+1}) \ne \emptyset$ by Ω_{0l}. Then

$$\begin{aligned}\|u - R_l u\|_{L^2(\Omega)} &\le \|u - \tilde{Q}_l u\|_{L^2(\Omega)} + \|\tilde{Q}_l u - R_l u\|_{L^2(\Omega)} \\ &\le C h_l |u|_{H^1(\Omega)} + \|u - \tilde{Q}_l u\|_{L^2(\Omega \setminus \Omega_{l+1})} + \|w\|_{L^2(\Omega_{0l})}.\end{aligned}$$

Here $w \in \tilde{V}^{h_l}$ in Ω_{l+1} and

$$w(x) = \begin{cases} (u - \tilde{Q}_l u)(x) & \text{if } x \in \partial\Omega_{l+1} \\ 0 & \text{if } x \text{ is a node in } \Omega_{l+1}. \end{cases}$$

By considering a discrete norm of w which is equivalent to the L^2 norm, it is easy to see that

$$\|w\|_{L^2(\Omega_{0l})} \leq C\|u - \tilde{Q}_l u\|_{L^2(\Omega\setminus\Omega_{l+1})}.$$

Therefore

$$\|u - R_l u\|_{L^2(\Omega)} \leq Ch_l|u|_{H^1(\Omega)} + C\|u - \tilde{Q}_l u\|_{L^2(\Omega)} \leq Ch_l|u|_{H^1(\Omega)}.$$

In order to proceed with the proof of the next lemma, we need to introduce the operators $H_l : V^h \to \tilde{V}^{h_l}$ by

$$\begin{aligned} H_l u(x) &\in V^{h_l} + \cdots + V^{h_l}, \\ H_l u(x) &= u(x), \quad \forall x \in \Omega \setminus \Omega_{l+1}, \\ a(H_l u, w_h) &= 0, \quad \forall w_h \in V^{h_l} \cap H_0^1(\Omega_{l+1}). \end{aligned}$$

It is natural to call $H_l u$ the h_l-harmonic extension of u to Ω_{l+1}. Let us recall a result from Cheng [3]; cf. Widlund [14]. There exists an absolute constant C such that

$$a(H_l u, H_l u) \leq Ca(u, u), \quad \forall u \in V^h. \tag{8}$$

By using this inequality, we can prove that R_l is a bounded operator from V^h into $\tilde{V}^{h_l}$ in the H_0^1-norm.

Lemma 8. *There exists an absolute constant C such that*

$$|R_l u|_{H^1(\Omega)} \leq C|u|_{H^1(\Omega)}, \quad \forall u \in V^h.$$

Proof. We observe that

$$\begin{aligned} |R_l u|^2_{H^1(\Omega)} &= |u|^2_{H^1(\Omega\setminus\Omega_{l+1})} + |R_l u|^2_{H^1(\Omega_{l+1})} \\ &\leq |u|^2_{H^1(\Omega)} + C(|R_l u - H_l u|^2_{H^1(\Omega_{l+1})} + |H_l u|^2_{H^1(\Omega_{l+1})}) \\ &= |u|^2_{H^1(\Omega)} + C(|\tilde{Q}_l(u - H_l u)|^2_{H^1(\Omega_{l+1})} + |H_l u|^2_{H^1(\Omega_{l+1})}) \end{aligned}$$

and that $u - H_l u = 0$ on $\partial\Omega_{l+1}$. Therefore we can apply Lemma 3 to conclude that

$$|R_l u|^2_{H^1(\Omega)} \leq |u|^2_{H^1(\Omega)} + C(|u - H_l u|^2_{H^1(\Omega_{l+1})} + |H_l u|^2_{H^1(\Omega_{l+1})}) \leq C|u|^2_{H^1(\Omega)}.$$

The last step follows from equation (8).

The proof of the following lemma is similar to one that appears in Xu [16] after replacing the L^2 projection by the operators R_l. However, in our proof, we do not need to use the fact that the R_l are bounded linear mappings in the space $H^1_0(\Omega)$.

Lemma 9. *There exists an absolute constant C, which depends only on these Lipschitz constants that appear in Assumption 4 and the shape regularity, such that*

$$\sum_{l=2}^{k} \|(R_l - R_{l-1})u\|^2_{L^2(\Omega)} \cdot \frac{1}{h_l^2} \leq C|u|^2_{H^1(\Omega)}, \quad \forall u \in V^h.$$

Proof. Let us first decompose u into $u = \sum_{i=1}^{k} u_i$, where $u_i \in Range(P_i^i - P_i^{i-1})$ is the same as in Lemma 6. We observe that

$$\|(R_l - R_{l-1})u_i\|_{L^2(\Omega)} \leq C\|u_i\|_{L^2(\Omega)},$$

by the shape regularity assumption, and that

$$\begin{aligned}\|(R_l - R_{l-1})u_i\|_{L^2(\Omega)} =& \|R_l(u_i - R_{l-1}u_i)\|_{L^2(\Omega)} \\ &\leq C\|u_i - R_{l-1}u_i\|_{L^2(\Omega)} \leq Ch_l\|u_i\|_{H^1(\Omega)},\end{aligned}$$

by using Lemma 7. By using an interpolation theorem of Hilbert scales; cf. [3] and [8], we have

$$\|(R_l - R_{l-1})u_i\|_{L^2(\Omega)} \leq Ch_l^{1-s} \cdot \|u_i\|_{H^{1-s}(\Omega)}, \quad \forall s \in (0,1).$$

We choose s as in Lemma 4. Then

$$\|(R_l - R_{l-1})u_i\|_{L^2(\Omega)} \leq Ch_l^{1-s}\|u_i\|_{H^{1-s}(\Omega)} \leq Ch_l^{1-s}h_i^s\|u_i\|_{H^1(\Omega)}.$$

With $i \wedge j = \min(i,j)$ and the observation that $(R_l - R_{l-1})u_i = 0$ for $i < l$, we have

$$
\begin{aligned}
&\sum_{l=1}^{k} \|(R_l - R_{l-1})u\|^2_{L^2(\Omega)} \cdot \frac{1}{h_l^2} \\
=&\sum_{l=1}^{k} \sum_{i,j=l}^{k} ((R_l - R_{l-1})u_i, (R_l - R_{l-1})u_j)_{L^2(\Omega)} \cdot \frac{1}{h_l^2} \\
=&\sum_{i,j=1}^{k} \sum_{l=1}^{i\wedge j} ((R_l - R_{l-1})u_i, (R_l - R_{l-1})u_j)_{L^2(\Omega)} \cdot \frac{1}{h_l^2} \\
\le& C \sum_{i,j=1}^{k} \sum_{l=1}^{i\wedge j} \frac{1}{h_l^2} h_l^{2(1-s)} \|u_i\|_{H^{1-s}(\Omega)} \|u_j\|_{H^{1-s}(\Omega)} \\
\le& C \sum_{i,j=1}^{k} \sum_{l=1}^{i\wedge j} \frac{1}{h_l^2} h_L^{2(1-s)} h_i^s h_j^s \|u_i\|_{H^1(\Omega)} \|u_j\|_{H^1(\Omega)} \\
=& C \sum_{i,j=1}^{k} h_i^s h_j^s \|u_i\|_{H^1(\Omega)} \|u_j\|_{H^1(\Omega)} \sum_{l=1}^{i\wedge j} h_l^{-2s} \\
\le& C \sum_{i,j=1}^{k} h_{i\wedge j}^{-2s} h_i^s h_j^s \|u_i\|_{H^1(\Omega)} \|u_j\|_{H^1(\Omega)} \\
\le& C \sum_{i,j=1}^{k} q^{s|i-j|} \|u_i\|_{H^1(\Omega)} \|u_j\|_{H^1(\Omega)} \\
\le& C \sum_{i=1}^{k} \|u_i\|^2_{H^1(\Omega)} \le C(1 + C'(d)) |u|^2_{H^1(\Omega)}
\end{aligned}
$$

by using Lemma 6 and Friedrichs' inequality. Here $C'(d)$ is a constant which only depends upon the diameter d of the domain Ω. By using a simple dilation argument, we can completely remove the dependence of this constant upon the diameter of Ω and complete the proof.

We now return to the proof of Theorem 3 by using those previous lemmas.

Proof of Theorem 3. Let us first prove that the operator P has an uniform upper bound. We define $\tilde{P}$ by

$$\tilde{P} = \sum_{l=1}^{k} \sum_{i=1}^{N_l+M_l} P_i^l$$

We observe that $\tilde{P}$ has a uniform upper bound by Theorem 1 and $P \leq \tilde{P}$. Therefore P has a uniform upper bound.

To establish a uniform lower bound, we will apply Lemma 5. We note that it is sufficient to find a good decomposition of $u \in V^h$ such that the constant is uniformly bounded from above. Let us first decompose u as

$$u = R_1 u + \sum_{l=2}^{k} (R_l - R_{l-1}) u \equiv \sum_{l=1}^{k} u^l.$$

It is easy to see that $u^l \in V^{h_l}$. We need to further decompose u^l, for $l \geq 2$, as

$$u^l = \sum_{i=1}^{N_l} u_i^l, \quad \text{with } u_i^l \equiv \tilde{I}_l(\theta_i^l u^l) \in V_i^{h_l}.$$

Here $\{\theta_i^l\}$ is a partition of unity as in Assumption 5 and $\tilde{I}_l$ is the standard nodal interpolants into $\tilde{V}^{h_l}$. It can be shown that

$$\begin{aligned}
|u_i^l|^2_{H^1(\tilde{\Omega}_i^l)} &= |\tilde{I}_l(\theta_i^j u^l)|^2_{H^1(\tilde{\Omega}_i^l)} \\
&\leq C(|\theta_i^l|^2_{L^\infty(\Omega)} |u^l|^2_{H^1(\tilde{\Omega}_i^l)} + |\theta_i^l|^2_{W^{1,\infty}(\Omega)} \|u^l\|^2_{L^2(\tilde{\Omega}_i^l)}) \\
&\leq C(|u^l|^2_{H^1(\tilde{\Omega}_i^l)} + (1/h_{l-1}^2) \|u^l\|^2_{L^2(\tilde{\Omega}_i^l)}).
\end{aligned}$$

Summing over i and using the finite covering property of $\{\tilde{\Omega}_i^l\}$, we obtain

$$\begin{aligned}
\sum_i |u_i^l|^2_{H^1(\Omega)} = \sum_i |u_i^l|^2_{H^1(\tilde{\Omega}_i^l)} &\leq C \sum_i (|u^l|^2_{H^1(\tilde{\Omega}_i^l)} + \frac{1}{h_{l-1}^2} \cdot \|u^l\|^2_{L^2(\tilde{\Omega}_i^l)}) \\
&\leq C(|u^l|^2_{H^1(\Omega)} + \frac{1}{h_l^2} \|u^l\|^2_{L^2(\Omega)}) \leq C \frac{1}{h_l^2} \|u^l\|^2_{L^2(\Omega)}.
\end{aligned}$$

Summing over l, for $1 \leq l \leq k$, and using Lemmas 8 and 9, we get

$$\sum_{l=1}^{k} \sum_i |u_i^l|^2_{H^1(\Omega)} \leq C(|u^1|^2_{H^1(\Omega)} + \sum_{l=2}^{k} \frac{1}{h_l^2} \|u^l\|^2_{L^2(\Omega)}) \leq C |u|^2_{H^1(\Omega)}.$$

The lower bound of P now follows.

Then we discuss some multiplicative variants of the MDS algorithm with partial refinement. In particular, we can estimate the energy norm of the following operators

$$E_G = \prod_{l=1}^{k} \prod_{i=1}^{N_l} (I - P_i^l),$$

$$E_J = \prod_{l=1}^{k} (I - T_l) \equiv \prod_{l=1}^{k} (I - \beta \sum_{i=1}^{N_l} P_i^l),$$

where β is a damping factor such that $\|T_l\|_a \leq w < 2$. The operators E_G and E_J correspond to the FAC algorithms with inexact solvers consisting of one Gauss-Seidel and damped Jacobi iteration, respectively, except for the coarsest space V^{h_1}. We can use the techniques in Zhang [17] and the fact that the multilevel additive Schwarz operator P has a uniform lower bound to prove the following theorem.

Theorem 4. *There exist absolute constants η_G and η_J,which depend only on the Lipschitz constants appearing in Assumption 4 and the shape regularity, such that*

$$\|E_G\|_a \leq \eta_G < 1 \quad \text{and} \quad \|E_J\|_a \leq \eta_J < 1.$$

In order to prove Theorem 4, we need the following lemma, which is given in Zhang [17].

Lemma 10. *Let $T_i, i = 1, \cdots, N$, be symmetric, semi-positive definite operators with respect to the $a(\cdot,\cdot)$ and let $\|T_i\|_a \leq w < 2$. Let $T = \sum_{i=1}^{N} T_i$ and $E = (I - T_1)(I - T_2) \cdots (I - T_N)$. Then*

$$\|E\|_a \leq \sqrt{1 - (2 - \omega) \frac{\lambda_{\min}(T)}{\|\Theta_T\|_2^2}}.$$

Here $\Theta_T = \{\theta_T^{ij}\}$, where $\theta_T^{ii} = 1$ and θ_T^{ij}, $i \neq j$, are given by

$$\theta_T^{ij} = \sup_{u,v} \frac{a(T_i u, T_j v)}{a(T_i u, u)^{1/2} a(T_j v, v)^{1/2}}.$$

Proof of Theorem 4. We first estimate E_G. In this case, $T_i = P_i^l$ for each subspace $V_i^{h_l}$. Let us denote by $\tilde{T}$ the operator corresponding to the case of refinement everywhere. In [17], Zhang established that $\|\Theta_{\tilde{T}}\|_2^2$ is uniformly bounded. We note that each space corresponding to T_i is a space corresponding to a $\tilde{T}_j$ for some j. Therefore $\|\Theta_T\|_2^2$ is uniformly bounded. By Lemma 10 and Theorem 3, the first part follows easily.

As for the case of E_J, we take $T_l = \beta \sum_i P_i^l$ and use an argument similar to the above one used.

Finally we discuss the extension to the cases of general mixed type boundary condition and more general Lagrange elements. For general Lagrange elements in higher-dimensional space, it is possible to construct high order quasi-interpolants by looking at quadrature rules preserving high order polynomials which are similar to the example given in Section 3. For general boundary condition, it is sufficient to prove counterparts of Lemmas 3, 4, and 6. The modification of Lemmas 4 and 6 have been discussed for the work of iterative refinement methods in [3]. However, we can construct the counterpart of Lemma 3 by separately considering two cases of the elements K satisfying $\overline{K} \cap \Gamma_D = \emptyset$ and those who do not.

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