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ON UNIQUENESS OF POSITIVE SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS

BY

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Abstract. It is shown that for $y'' + g(x)y^{\gamma}(x) = 0$, $\gamma > 1$, $g(x) > 0$, there is at most one positive C' solution y with $y(0) = 0$ and tending to a positive constant at infinity, under the condition that $\frac{\gamma+3}{2} + \frac{xg'}{g}$ has only finite number of zeros and there is a positive solution with positive Pohozaev function.

1. Introduction. In this paper, we are concerned about the uniqueness of positive C' solutions of the non-linear ordinary differential equation

$$y''(x) + g(x)y^{\gamma}(x) = 0, \quad 0 < x < \infty, \tag{1}$$

with $y(0) = 0$ and $\lim_{x\to\infty} y(x)$ a positive constant; where $g(x)$ is positive continuous and $\gamma > 1$.

The discussion below will cover the case in finite interval $[\alpha, \beta]$, with $0 \leq \alpha < \beta$ and boundary conditions

$$y(\alpha) = 0, \quad y'(\beta) = 0. \tag{2}$$

Notice that in this case, there were already many articles concerned about the existence and uniqueness of positive solution. [1] [2] [3] [4] [5]. Essentially, the uniqueness problem is only solved partially.

For semi-linear elliptic equations of the form

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(3) $$\Delta u + p(|x|)u^\gamma = 0, \quad x \in R^n,\ n \geq 3 \text{ and } \gamma > 1,$$

one interests in the radially symmetric solutions $u(r) = u(|x|)$, which satisfies the ordinary differential equation [7] [8]

(4) $$u'' + \frac{n-1}{r}u' + p(r)u^\gamma = 0.$$

with the change of variables $s = r^{n-2}$, $y(s) = su(r(s))$, the equation reduces to

(5) $$y''(s) + \frac{1}{(n-2)^2}\frac{r^2 p(r)}{s^{1+\gamma}}y^\gamma(s) = 0,$$

that is, of the form (1) with $g(s) = \frac{1}{(n-2)^2}\frac{r^2 p(r)}{s^{1+\gamma}}$

The uniqueness of positive solutions of order $O(r^{2-n})$ (a ground state) is of great interests and is included in the problem of equation (1).

2. A Pohozaev identity. The Pohozaev identity for solutions of (4) is [8]

(6) $$\frac{n-2}{2}r^{n-1}u(r)u'(r) + \frac{1}{2}r^n u'^2(r) + \frac{r^n}{\gamma+1}p(r)u(r)^{\gamma+1}$$
$$= \int_0^r \Big\{\Big(\frac{n}{\gamma+1} - \frac{n}{2}\Big)p(\alpha)u^{\gamma+1} + \frac{1}{\gamma+1}\alpha p'(\alpha)u^{(\gamma+1)}\Big\}\alpha^{n-1}d\alpha$$

Equivalently $G_y(s)$, the Pohozaev function of y is equal to

(7) $$\begin{aligned} G_y(s) &\equiv sy'(s)^2 - y(s)y'(s) + \frac{2}{\gamma+1}sg(s)y^{\gamma+1}(s) \\ &= \frac{2}{\gamma+1}\int_0^s \Big(\frac{tg'(t)}{g(t)} + \frac{\gamma+3}{2}\Big)g(t)y^{\gamma+1}(t)dt \\ &\equiv \int_0^s Q(t)g(t)y^{\gamma+1}(t)dt. \end{aligned}$$

we mention that this function also appeared in [6] for solutions of (1). We assume that $Q(t)$ is continuous throughout the whole interval considered and has only a finite number of zeros.

3. A generalized mean value theorem.

Theorem 1. *Let y be a fixed positive solution of* (1) *with $G_y(x) > 0$, and y_1 another (arbitrary) solution of* (1), *then*

$$\frac{G_{y_1}(x)}{G_y(x)} = \left(\frac{y_1}{y}\right)^{\gamma+1}(\xi), \quad \textit{some } \xi \textit{ with } 0 < \xi < x. \tag{8}$$

Proof. The proof is similar to that of Cauchy Mean Value Theorem in calculus. In fact, let

$$H(t) \equiv G_{y_1}(t) - \frac{G_{y_1}(x)}{G_y(x)} G_y(t), \tag{9}$$

then $H(x) = H(0) = 0$. At extrema (say maximum) ξ, $0 < \xi < x$, we have

$$\begin{aligned} H(\xi + h) - H(\xi) &\leq 0, \quad \text{small} \;\; h > 0 \\ &\leq 0, \qquad\qquad h < 0. \end{aligned}$$

This implies

$$\begin{aligned} \frac{1}{h}\int_{\xi}^{\xi+h} Q(t)g(t)y_1^{\gamma+1}dt &= \frac{G_{y_1}(\xi+h) - G_{y_1}(\xi)}{h} \\ &\leq \frac{G_{y_1}(x)}{G_y(x)} \cdot \frac{G_y(\xi+h) - G_y(\xi)}{h} \\ &= \frac{G_{y_1}(x)}{G_y(x)} \cdot \frac{1}{h}\int_{\xi}^{\xi+h} Q(t)g(t)y^{\gamma+1}dt \end{aligned} \tag{10}$$

for $h > 0$ and similarly for $h < 0$ (with $\geq$). Then equality (8) follows by L'Hopital Rule.

4. The Sturm-Picone like theorem for nonlinear equation (1).

Assume that y is a fixed positive solution of (1), y_1, another solution with $y_1'(0) > y'(0)$, y_1 stays positive in $(0, a)$ and $y_1(x) \geq y(x)$ in $(0, b)$, so that $b < a$. We have $y_1'y - y_1y' < 0$ in $(0, b)$ because

$$(y_1'y - y_1y')(x) = \int_0^x y_1yg[y^\gamma - y_1^\gamma]dt. \tag{11}$$

Write

(12) $$w_1(x) = \frac{xy_1'(x)}{y_1(x)}, \qquad w(x) = \frac{xy'(x)}{y(x)},$$

then $w_1 < w$ in $(0, b)$.

Theorem 2. *y, y_1 as above. We further assume $G_y(x) > 0$, then we have $w_1 < w$ in $(0, a)$. That is, as long as y_1, stays positive, then $\frac{y_1}{y}$ is decreasing.*

Proof. We want to prove that there is no way that $w_1 = w$. As stated before, in $(0, b)$, we have $w_1 < w$. If there is a first point c such that $w_1(c) = w(c)$, then $y_1(c) < y(c)$: For at first point b of crossing, $y_1(b) = y(b)$ and $y_1'(b) < y'(b)$ so that $w_1(b) < w(b)$ and at second point d of crossing (if any), we have $y_1(d) = y(d)$ and $y_1'(d) > y'(d)$ so that $w_1(d) > w(d)$. In between, there is a point c with $w_1(c) = w(c)$.

Now, by Theorem 1,

$$\begin{aligned} G_{y_1}(c) &= G_y(c)\Big(\frac{y_1(\xi)}{y(\xi)}\Big)^{\gamma+1}, \quad 0 < \xi < c \\ &> G_y(c)\Big(\frac{y_1(c)}{y(c)}\Big)^{\gamma+1}, \end{aligned}$$

because $\frac{y_1}{y}$ is decreasing in $(0, c)$ or equivalently $w_1 < w$ there. Hence

$$(k^{-1})^{\gamma+1} G_y(c) > G_{y_1}(c), \quad k = \frac{y_1(c)}{y(c)} < 1.$$

From Pohozaev identity for y and y_1, evaluated at c, we have

(13) $$(k^{-1})^{\gamma+1}[cy_1'^2(c) - y_1(c)y_1'(c)] > cy'^2(c) - y(c)y'(c).$$

At c, $w_1(c) = w(c)$ implies $\frac{y_1'(c)}{y_1(c)} = \frac{y'(c)}{y(c)}$. Or, $\frac{y_1(c)}{y(c)} = \frac{y_1'(c)}{y(c)} = k$. Therefore (13) leads to $[cy'^2(c) - y(c)y'(c)] > k^{\gamma-1}[cy'^2(c) - y(c)y'(c)]$.

That is not the case because $k < 1$ and $cy'^2(c) - y(c)y'(c)$ is always negative. (It is well known $y'(x) > 0$, $0 < xy'(x) < y(x)$) The proof is completed.

5. The main uniqueness theorem.

Theorem 3. *Assume that there is a positive solution z of (1) with $z(0)=0$ and $G_z(x)>0$. Also assume that $\frac{\gamma+3}{2}+\frac{xg'(x)}{g(x)}$ is continuous and has only finite number of zeros. Then (1) has at most one positive solution y with $y(0)=0$ and tending to a positive constant at infinity.*

Proof. Assume there are two positive solution y and y_1 as stated. We may assume $y_1'(0)>y'(0)$. Hence

$$\left(\frac{y_1}{y}\right)'(x)=\frac{y_1'(x)y(x)-y_1(x)y'(x)}{y^2(x)}<0,\quad \text{(by Theorem 2)} \tag{14}$$

And it is well known (by (11)) that there is a point a such that $y_1(a)=y(a)$.

Hence $\lim_{x\to\infty} y_1'(x)=0=\lim_{x\to\infty} y'(x)$ and of course $\lim_{x\to\infty} y_1(x)<\lim_{x\to\infty} y(x)$. Also, it is clear that $G_y(x)>0$ by generalized mean value theorem applied to z and y.

Let b be the last zero of $Q(x)$, then

$$G_{y_1}(x)-G_{y_1}(b)=\int_b^x Q(t)g(t)y_1^{\gamma+1}(t)dt, \tag{15}$$

$$G_y(x)-G_y(b)=\int_b^x Q(t)g(t)y^{\gamma+1}(t)dt,\quad b<x<\infty \tag{16}$$

$$\begin{aligned}\frac{G_{y_1}(b)}{G_y(b)}\equiv k&=\left(\frac{y_1}{y}(\xi)\right)^{\gamma+1},\quad 0<\xi<b \text{ (Theorem 1)}\\ &>\left(\frac{y_1}{y}(t)\right)^{\gamma+1},\quad t\ge b.\ \text{(Theorem 2)}\end{aligned} \tag{17}$$

Consider $(15)-k\times(16)$ and letting $x\to\infty$, then the left hand side tends to zero because $0<xy'(x)<y(x)$, $0<xy_1'(x)<y_1(x)$ and it is well known that $xg(x)$ is integrable (Atkinson Theorem). Now, $y_1^{\gamma+1}(t)<ky^{\gamma+1}(t)$, $t\ge b$, so that the integrand of right hand side of $(15)-k\times(16)$ is everywhere positive (or everywhere negative) and contradiction follows. Therefore the theorem is proved.

Remark. For positive solution z with $z(0)=0$ and tending to a positive constant at infinity, the condition $G_z(x)>0$ is satisfied if $Q(x)>0$

everywhere or $Q(x)$ is positive at beginning and across zero only at once. (cf. The right hand side of Pohozeav identity)

6. Finite interval $[\alpha, \beta]$. On $[\alpha, \beta]$, with $\alpha \geq 0$, we also assume $\frac{\gamma+3}{2} + \frac{sg'}{g}$ is continuous and has only finite number of zeros. The existence of a positive solution of (1) with $y(\alpha) = 0$, $y'(\beta) = 0$ is asserted in [4] for $\alpha > 0$. For $\alpha = 0$, of course we need some integrable condition on g. cf. [1].

It is also true that for two such solutions y and y_1, there is a point a in (α, β) with $y(a) = y_1(a)$. The decreasing of $\frac{y_1(x)}{y(x)}$ is proved in the same way as before under the condition that $y_1'(0) > y'(0)$ and that there is positive solution z with $z(0) = 0$ and $G_z(x) > 0$.

Theorem 4. *In $[\alpha, \beta]$, we assume* (1) *has a positive solution z with $z(\alpha) = 0$, $G_z(x) > 0$ and Q has only finite number of zeros. Then* (1) *has at most one positive solution satisfying $y(\alpha) = 0$, $y'(\beta) = 0$.*

Proof. Assume there are two such solutions y and y_1, then as before $G_y(x) > 0$ and also $\frac{y_1}{y}$ is decreasing. Using Pohozaev identities (15) (16) as before at b, where b is the last zero of Q if $Q(\beta) \neq 0$ or the zero prior to the last if $Q(\beta) = 0$. Now, $k \equiv (\frac{y_1}{y})^{\gamma+1}(b) > (\frac{y_1}{y})^{\gamma+1}(t) > (\frac{y_1}{y})^{\gamma+1}(\beta) \equiv m$, $b < t < \beta$. When $Q(t) < 0$ in (b, β), we consider (15) $- k \times$ (16), evaluated at β. When $Q(t) > 0$ in (b, β), we consider (15) $- m \times$ (16), also evaluated at β. In both cases, the left hand side is negative, while the right hand side is positive. The contradictions proved the theorem.

Corollary 5. (Moroney) *If $g'(x) > 0$ everywhere then uniqueness follows.*

Proof. For then $Q(x) = \frac{\gamma+3}{2} + \frac{xg'(x)}{g(x)} > 0$ and $b = \alpha$ in the proof of theorem 4. Notice that in this case $G_y(x) > 0$ for positive solution y. The Remark following theorem 3 is still hold in this finite interval case.

Remark. Moroney theorem can be eased to $Q(x) > 0$.

Remark. When $Q(x) \equiv 0$, the theorem is still hold. For $G_y(x) \equiv 0 \equiv G_{y_1}(x)$, when evaluated at β, leads to $y(\beta) = y_1(\beta)$. Adding to the condition $y'(\beta) = y_1'(\beta) = 0$, the backward initial value problem implies the uniqueness.

7. Examples.

(a) In [2], the equation considered is

$$y'' + (x\text{csch}^2(x))^2 y^3 = 0 \tag{18}$$

$$\frac{\gamma+3}{2} + \frac{xg'}{g} = 3 + 2[1 - 2x\coth x].$$

Notice that $1 - 2x\coth x < 0$ and is actually decreasing to $-\infty$. Also, the existence of a positive solution tending to a positive constant is asserted [2] or can be inferred from [1]. So that the Remark of Theorem 3 applied to this case and Uniqueness hold.

(b) In [8], the Matukuma equation was considered,

$$\Delta u + \frac{1}{1+r^2} u^\gamma = 0, \quad 1 < \gamma < \frac{n+2}{n-2}. \tag{19}$$

For radial solution $u(r) = u(|x|)$, this reduces to

$$u_{rr} + \frac{n-1}{r} u_r + \frac{1}{1+r^2} u^\gamma = 0, \quad r > 0. \tag{20}$$

Write $y(s) = su(r)$, $r = s^{\frac{1}{n-2}}$, then

$$y'' + g(s) y^\gamma(s) = 0, \quad g(s) = \frac{1}{(n-2)^2} \frac{\frac{r^2}{1+r^2}}{s^{1+\gamma}}. \tag{21}$$

So, $\frac{\gamma+3}{2} + \frac{sg'}{g} = \frac{1-\gamma}{2} + \frac{2}{(n-2)(1+r^2)}$, and $Q(0) > 0$ because $\gamma < \frac{n+2}{n-2}$. Also $Q(\infty) < 0$ because $\gamma > 1$. The existence of a positive solution tending to a positive constant can be inferred from [1]. Hence the Remark of Theorem 3 applies to this case too. (The decreasing of Q is obvious.)

(c) For the semi-linear elliptic equation

$$\Delta u + u^{\frac{n+2}{n-2}} = 0, \tag{22}$$

we have infinite positive solutions tending to zero [7]. Or,

$$y'' + g(s)y^{\frac{n+2}{n-2}} = 0, \quad g(s) = \frac{1}{s^{2+\frac{2}{n-2}}}, \tag{23}$$

has infinite many positive solution tending to positive constants. (with 0 initials) In this case, we have $\frac{\gamma+3}{2} + \frac{sg'}{g} \equiv 0$ and the condition of theorem 3 is not fulfilled.

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