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# OSCILLATION OF SOLUTIONS OF HIGHER ORDER NONLINEAR DIFFERENCE EQUATIONS

BY

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**Abstract.** This paper is concerned with the study of oscillatory behavior of solutions of the nonlinear difference equation

$$\Delta^m u(n) = f(n, u(n), \dots, \Delta^{m-1}u(n)), n \in \mathcal{N}, m \geq 2,$$

where $\Delta^i u(n) = \Delta(\Delta^{i-1}u(n))(i = 1, \dots, m)$, $\Delta^0 u(n) = u(n), f : \mathcal{N} \times R^m \longrightarrow R$.

**1. Introduction.** In this paper we are concerned with the oscillatory behavior of solution of the nonlinear difference equation

$$\Delta^m u(n) = f(n, u(n), \dots, \Delta^{m-1}u(n)), \quad n \in \mathcal{N}, m \geq 2, \tag{1}$$

where $\mathcal{N} = \{1, 2, \dots\}$, $\Delta$ is the forward difference operator i.e. $\Delta u(n) = u(n+1) - u(n)$ and $\Delta^i u(n) = \Delta(\Delta^{i-1}u(n))$, $i = 1, \dots, m$, $\Delta^0 u(n) = u(n)$; $f : \mathcal{N} \times R^m \longrightarrow R$ where $R$ is the set of real numbers. Finally, $R_+ = \langle 0, \infty)$, $(r)^{(k)} = r(r-1)\cdots(r-k+1)$ is the usual factorial notation with $(r)^{(0)} = 1$. It is assumed that the function $f$ satisfies on the set $\mathcal{N} \times R^m$ either condition

(a) $f(n, x_1, \dots, x_m)x_1 \leq 0$,

or the condition

(b) $f(n, x_1, \dots, x_m)x_1 \geq 0$.

By a solution of (1) we mean any function $u : \mathcal{N} \longrightarrow R$ which satisfies (1) and such that $\sup_{n \geq k} |u(n)| > 0$ for any $k \in \mathcal{N}$. A nontrivial solution

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$u$ is called oscillatory, if for every $k \in \mathcal{N}$ there exists $n \geq k$ such that $u(n)u(n+1) \leq 0$. Otherwise it is called nonoscillatory.

The Eq. (1) has *property A* if each solution of (1) is oscillatory when $m$ is even and is either oscillatory or tends to zero monotonically as $n \longrightarrow \infty$ when $m$ is odd.

The Eq. (1) has *property B* if each solution of (1) is either oscillatory or tending monotonocally to infinity or to zero as $n \longrightarrow \infty$ when $m$ is even and is either oscillatory or monotonically tending to infinity as $n \longrightarrow \infty$ when $m$ is odd.

Recently some results concerning the oscillatory and asymptotic behavior of solutions of difference equations of higher order have been established in papers [1-3, 7-9].

The purpose of this paper is to present general oscillation theorems that give sufficient conditions under which the Eq. (1) has property $A$ and $B$. The obtained results are the discrete analogues of the well-known oscillation theorems for differential equations due to Kiguradze [5], [6, p. 288].

**2. Lemmas.** To obtain our results we need the following discrete analogue of well-known lemmas due to Kiguradze (cf. [5], [6, pp. 280-290]).

**Lemma 1.** *Let* $u : \mathcal{N} \longrightarrow R - \{0\}$, $m \in \mathcal{N}, m \geq 2$ *and*

$$u(n)\Delta^m u(n) \leq 0, \quad n \geq n_0 \tag{2}$$

*with* $u, \Delta^m u$ *of constant sign for* $n \geq n_0$ *and* $\Delta^m u(n)$ *is not identically zero for all large* $n$. *Then there exist a* $\overline{n_0} \geq n_0$ *and an integer* $l$, $0 \leq l \leq m$ *with* $m+1$ *odd such that for* $n \geq \overline{n_0}$

$$\begin{aligned} u(n)\Delta^i u(n) > 0 \quad & \textit{for} \quad i = 0, 1, \ldots, l, \\ (-1)^{l+i} u(n)\Delta^i u(n) > 0 \quad & \textit{for} \quad i = l+1, \ldots, m-1. \end{aligned} \tag{3}$$

*If* $l > 0$, *then for* $n \geq n_1 \geq \overline{n_0} + p$, *where* $p \in \{0, 1, 2, \ldots\}$

$$|\Delta^{l-j} u(n+j-p)| \geq \frac{i!}{j!}(n - n_1 + j)^{(j-i)} |\Delta^{l-i} u(n+i-p)| \tag{4}$$
$$(j = 1, \ldots, l, \quad i = 0, 1, \ldots, j-1)$$

*and*

$$|u(n+m-p)| \geq |u(n_1+l-1-p)| + \frac{1}{(m-1)!}\sum_{k=n_1}^{n}(k-n_1+m-1)^{(m-1)}|\Delta^m u(k)|. \tag{5}$$

**Lemma 2.** *If the inequality* (2) *is replaced by*

$$u(n)\Delta^m u(n) \geq 0, \quad n \geq n_0, \tag{6}$$

*then there exists a* $\overline{n_0} \geq n_0$ *such that for* $n \geq \overline{n_0}$

$$u(n)\Delta^i u(n) > 0 \quad \textit{for} \quad i = 1, \ldots, m-1, \tag{7}$$

*or there exists an integer* $l, 0 \leq l \leq m-2$ *with* $m+l$ *even such that the inequalities* (3) *hold. If* $l > 0$, *then the inequalities* (4) *and* (5) *hold.*

*Proof of Lemma 1 and Lemma 2.* For a proof of the inequalities (3) and (7) we refer to [3] or [4].

Now, let $l > 0$ and $n \geq n_1 \geq \overline{n_0} + p$, where $p \in \{0, 1, 2, \ldots\}$. We prove that for $i = 1, \ldots, l$ and all $n \geq n_1$

$$i\Delta^{l-i}u(n+i-p) \geq (n-n_1+i)\Delta^{l-i+1}u(n+i-p-1). \tag{8}$$

Since $\Delta^l u(n)$ is decreasing, we see that

$$\Delta^{l-1}u(n-p+1) - \Delta^{l-1}u(n_1-p) = \sum_{k=n_1}^{n}\Delta^l u(k-p) \geq \Delta^l(n-p)(n-n_1+1)$$

and thus (8) is true for $i = 1$.

Now assume that (8) holds for $i-1 (i = 2, \ldots, l)$ i.e.

$$(i-1)\Delta^{l-i+1}u(n+i-1-p) \geq (n-n_1+i-1)\Delta^{l-i+2}u(n+i-2-p), \; n \geq n_1.$$

Summing the above inequality from $n_1$ to $n$ yields

$$(i-1)\sum_{k=n_1}^{n}\Delta^{l-i+1}u(k+i-1-p) \geq \sum_{k=n_1}^{n}(k-n_1+i-1)\Delta^{l-i+2}u(k+i-2-p).$$

According to summation by parts formula we may write

$$\begin{aligned}
&(i-1)\Big[\Delta^{l-i}u(n+i-p)-\Delta^{l-i}u(n_1+i-1-p)\Big]\\
\geq&(n-n_1+i)\Delta^{l-i+1}u(n+i-1-p)-(i-1)\Delta^{l-i+1}u(n_1+i-2-p)\\
&-\sum_{k=n_1}^{n}\Delta^{l-i+1}u(k+i-1-p).
\end{aligned}$$

Hence

$$\begin{aligned}
&i\Big[\Delta^{i-1}u(n+i-p)-\Delta^{l-i}u(n_1+i-1-p)\Big]\\
\geq&(n-n_1+i)\Delta^{l-i+1}u(n+i-1-p)-(i-1)\Delta^{l-i+1}u(n_1+i-2-p),
\end{aligned}$$

which yields

$$\begin{aligned}
i\Delta^{l-i}u(n+i-p)\geq&(n-n_1+i)\Delta^{l-i+1}u(n+i-1-p)\\
&+i\Big[\Delta^{l-i}u(n_1+i-1-p)-\Delta^{l-i+1}u(n_1+i-2-p)\Big]\\
&+\Delta^{l-i+1}u(n_1+i-2-p),
\end{aligned}$$

and so

$$i\Delta^{l-i}u(n+i-p)\geq(n-n_1+i)\Delta^{l-i+1}u(n+i-1-p).$$

Thus, by induction, the inequality (8) holds for $i=1,\ldots,l$. From (8) we conclude that

$$\begin{aligned}
u(n+1-p)&\geq\frac{1}{l!}(n-n_1+l)^{(l)}\Delta^{l}u(n-p),\\
u(n+l-p)&\geq\frac{i!}{l!}(n-n_1+l)^{(l-i)}\Delta^{(l-i)}u(n+i-p),\\
&\qquad i=0,1,\ldots,l-1,\\
\Delta u(n+l-1-p)&\geq\frac{i!}{(l-1)!}(n-n_1+l-1)^{(l-1-i)}\Delta^{l-i}u(n+i-p),\\
&\qquad i=0,1,\ldots,l-2,
\end{aligned}$$

and consequently we have

$$\begin{aligned}
&\Delta^{l-j}u(n+j-p)\geq\frac{i!}{j!}(n-n_1+j)^{(j-i)}\Delta^{l-i}u(n+i-p),\ n\geq n_1,\\
&j=1,\ldots,l,\ i=0,1,\ldots,j-1,\ p\in\{0,1,2,\ldots\},
\end{aligned}$$

i.e. the inequality (4).

From (4) for $j = l-1$, $i = 0$ we have

$$\Delta u(n+l-1-p) \geq \frac{1}{(l-1)!}(n-n_1+l-1)^{(l-1)}\Delta^l u(n-p), \; n \geq n_1.$$

Summing the above inequality from $n_1$ to $n+m-l$ yields

$$\begin{aligned} &u(n+m-p) - u(n_1+l-1-p) \\ \geq &\frac{1}{(l-1)!}\sum_{k=n_1}^{n+m-l}(k-n_1+l-1)^{(l-1)}\Delta^l u(k). \end{aligned}$$

Using the summation by parts formula $m-l$ times to the right-hand side of (9) we obtain

$$\begin{aligned} &\frac{1}{(l-1)!}\sum_{k=n_1}^{n+m-l}(k-n_1+l-1)^{(l-1)}\Delta^l u(k) \\ =&\sum_{j=1}^{m-1}\frac{(-1)^{m-l-j}}{(m-j)!}\Delta^{m-j}u(n+j)(n-n_1+m)^{(m-j)} \\ &+\frac{(-1)^{m-l}}{(m-1)!}\sum_{k=n_1}^{n}\Delta^m u(k)(k-n_1+m-1)^{(m-1)}. \end{aligned}$$

Thus (9) and (3) imply that

$$\begin{aligned} &u(n+m-p) - u(n_1+l-1-p) \\ \geq &\frac{(-1)^{m-l}}{(m-1)!}\sum_{k=n_1}^{n}\Delta^m u(k)(k-n_1+m-1)^{(m-1)}, \end{aligned}$$

which means that (5) is true. This completes the proof.

## 3. Main Results.

**Theorem.** *Suppose the following conditions hold*

$1^0$ *condition* (a) [*condition* (b)],

$2^0$ *for every* $c > 0$ *there exists a function* $\varphi_c : \mathcal{N} \times R_+ \longrightarrow R_+$, *where* $\varphi_c(n,x)$ *is continuous and nondecreasing with respect to* $x \in R_+$ *such that*

$$|f(n,x_1,\ldots,x_m)| \geq \varphi_c(n,|x_1|), \tag{10}$$

*for*

$$(D_c) \qquad n \in \mathcal{N},\ \frac{1}{c} \le |x_1| \le cn^{m-1},\ (x_2,\dots,x_m) \in R^{m-1},$$

$3^0$ *for an* $n_0 \in \mathcal{N}$ *the difference equation*

$$(11)\ \ \Delta x(n-1) = \frac{1}{(m-1)!}(n-n_0+m-1)^{(m-1)}\varphi_c(n,x(n)),\ \ n \ge n_0$$

*has no eventually positive solution.*

*Then Eq.* (1) *has property A* [*property B*].

*Proof.* First, we show that for every $c > 0$ and $\eta > 0$

$$\sum_n^\infty \varphi_c(n, \eta n^{m-1}) = \infty.$$

Suppose that it is not true. Choose $n_0 \ge m$ so large that

$$\sum_{k=n_0}^\infty \varphi_c(k, \eta k^{m-1}) < \frac{\eta}{2}.$$

Consider the solution $x$ of Eq. (11) with initial condition

$$(12) \qquad x(n_0-1) = \frac{\eta}{2},$$

and the continuous function defined as follows

$$h(t) = \begin{cases} t - \frac{\eta}{2} - \frac{1}{(m-1)!}\Big[\sum_{p=n_0}^k \varphi_c(p,x(p))(p-n_0+m-1)^{(m-1)} \\ \quad +\varphi_c(k+1,t)(k-n_0+m)^{(m-1)}\Big],\ \ k \ge n_0, \\ t - \frac{\eta}{2} - \varphi_c(n_0,t), \quad k = n_0 - 1. \end{cases}$$

We note that

$$h[x(n_0-1)] = x(n_0-1) - \frac{\eta}{2} - \varphi_c(n_0, x(n_0-1)) \le 0,$$
$$h\Big[\eta n_0^{m-1}\Big] = \eta n_0^{m-1} - \frac{\eta}{2} - \varphi_c(n_0, \eta n^{m-1}) > \eta n_0^{m-1} - \eta > 0.$$

Hence there exists $v_0 \in \left\langle \frac{\eta}{2}, \eta n_0^{m-1}\right)$ such that $h(v_0) = 0$ i.e. $v_0 = \frac{\eta}{2} + \varphi_c(n_0, v_0)$ that is $v_0 = x(n_0)$ and $x(n_0) < \eta n_0^{m-1}$

Now, assume that $x(n)$ is defined for $n = n_0-1, n_0, \dots, k$, where $k \in \{n_0-1, n_0, \dots\}$ and $x(n) < \eta n^{m-1}$ for $n = n_0-1, n_0, \dots, k$. We see that

$$h[x(k)] = x(k) - \frac{\eta}{2} - \frac{1}{(m-1)!}\Big[\sum_{p=n_0}^{k} \varphi_c(p, x(p))(p - n_0 + m - 1)^{(m-1)} + \varphi_c(k+1, x(k))(k - n_0 + m)^{(m-1)}\Big].$$

From Eq. (11) we get

$$x(k) = x(n_0 - 1) + \frac{1}{(m-1)!}\sum_{p=n_0}^{k} \varphi_c(p, x(p))(p - n_0 + m - 1)^{(m-1)},$$

hence

$$h[x(k)] = -\frac{1}{(m-1)!}\varphi_c(k+1, x(k))(k - n_0 + m)^{(m-1)} \leq 0.$$

Next we have

$$\begin{aligned} & h\Big[\eta k^{m-1}\Big] \\ & \geq \eta k^{m-1} - \frac{\eta}{2} - \frac{1}{(m-1)!}\sum_{p=n_0}^{k+1} \varphi_c(p, \eta p^{m-1})(p - n_0 + m + 1)^{(m-1)} \\ & \geq \eta k^{m-1} - \frac{\eta}{2} - \frac{(k - n_0 + m)^{(m-1)}}{(m-1)!}\sum_{p=n_0}^{k+1} \varphi_c(p, \eta p^{m-1}) \\ & \geq \eta k^{m-1} - \frac{\eta}{2} - k^{m-1}\frac{\eta}{2} > 0. \end{aligned}$$

Therefore there exists $v_0 \in \Big\langle x(k), \eta k^{m-1}\Big)$ such that $h(v_0) = 0$ i.e.

$$v_0 = \frac{\eta}{2} + \frac{1}{(m-1)!}\sum_{p=n_0}^{k} \varphi_c(p, x(p))(p - n_0 + m - 1)^{(m-1)} + \frac{1}{(m-1)!}\varphi_c(k+1, v_0)(k + 1 - n_0 + m - 1)^{(m-1)}.$$

Thus the solution $x$ of Eq. (11) is defined for $n = k+1$ and $x(k+1) = v_0 < \eta k^{m-1}$. Hence, by induction, the solution $x$ of (11) with initial condition (12) is defined for all $n \geq n_0 - 1$ and $\eta/2 \leq x(n) < \eta n^{m-1}$. But this contradicts assumption $3^0$.

Similarly as above one can show that for every $c > 0$ and $\eta > 0$

$$\sum_{n}^{\infty} n^{m-1}\varphi_c(n, \eta) = \infty. \tag{13}$$

Now, suppose that theorem is not true. Let $u$ be a nonoscillatory solution of (1) and assume that $u(n) > 0$ for $n \geq n_0$. Then from Lemma 1 it follows that one of the following two cases holds:

(i) $m$ is odd and the inequalities (3) hold for $l = 0$,

(ii) the inequalities (3) hold for $l > 0$ with $m + l$ odd.

*Case* (i). We show that $\lim_{n\to 0} u(n) = 0$.

Suppose that $\lim_{n\to\infty} u(n) = \gamma > 0$. Then $u(n) \geq \frac{1}{c}$ for $n \geq n_0$. From Eq. (1) and (10) we have

$$-\Delta^m u(n) \geq \varphi_c(n, u(n)) \geq \varphi_c(n, 1/c), \;\; n \geq n_0.$$

Using the equality [3]

$$u(n_0) = \sum_{j=0}^{m-1} \frac{(-1)^j (n - n_0 + j - 1)^{(j)}}{j!} \Delta^j u(n) + \frac{(-1)^m}{(m-1)!} \sum_{k=n_0}^{n-1} (k - n_0 + m - 1)^{(m-1)} \Delta^m u(k),$$

we see, by (3), that

$$u(n_0) \geq -\frac{1}{(m-1)!} \sum_{k=n_0}^{n-1} (k - n_0 + m - 1)^{(m-1)} \Delta^m u(k),$$

and this implies

$$u(n_0) \geq \frac{1}{(m-1)!} \sum_{k=n_0}^{n-1} (k - n_0 + m - 1)^{(m-1)} \varphi_c(k, 1/c), \;\; n > n_0,$$

which contradicts (13).

*Case* (ii). Let $l \geq 1$. Thus $u$ is increasing. From (1), by the assumptions. We have

$$-\Delta^m u(n) \geq \varphi_c(n, u(n)), \quad n \geq \overline{n_0}.$$

Putting $p = m$ in (5) we get for $n \geq n_1 \geq \overline{n_0} + m$

$$u(n) \geq u(n_1 - m + l - 1) + \frac{1}{(m-1)!} \sum_{k=n_1}^{n} (k - n_1 + m - 1)^{(m-1)} \varphi_c(k, u(k)).$$

Consider the solution $x$ of Eq. (11) with initial condition $x(n_1-1) = u(n_1-m+l-1) > 0$ and assume that $x$ is defined for $n = n_1-1, n_1, \dots, k$, $k \in \{n_1-1, n_1 \dots\}$ and $x(n) \le u(n)$ for $n = n_1-1, n_1, \dots, k$.

For the function defined as follows

$$h(t) = \begin{cases} t-u(n_1-m+l-1)-\frac{1}{(m-1)!}\Big[\sum\limits_{p=n_1}^{n} \varphi_c(p,x(p))(p-n_1+m-1)^{(m-1)} \\ \quad +\varphi_c(k+1,t)(k-n_1+m)^{(m-1)}\Big], \ k \ge n_1, \\ t-u(n_1-m+l-1)-\varphi_c(n_1,t), \ k = n_1-1, \end{cases}$$

in exactly the same way as the previous one we can show that

$$h[x(k)] \le 0, \quad h[u(k+1)] \ge 0.$$

Hence there exists $v_0 \in \langle x(k), u(k+1)\rangle$ such that $h(v_0) = 0$ i.e. the solution $x$ of Eq. (11) is defined for $n = k+1$ and $x(k+1) = v_0 \le u(k+1)$. Thus by induction, the solution $x$ of (11) with initial condition $x(n_1-1) = u(n_1-m+l-1)$ is defined for all $n \ge n_1-1$ and $x(n) \le u(n)$ which contradicts assumption $3^0$.

*Proof of property B.*

Let $u$ be a nonoscillatory solution of (1) and let $u(n) > 0$ for $n \ge n_0$. Then, by Lemma 2, it follows that one of the following three cases holds:

(i) $m$ is even and the inequalities (3) hold for $l = 0$,

(ii) the inequality (3) hold for $l > 0$ with $m+l$ even,

(iii) the inequalities (7) hold.

In the cases (i) and (ii) the proofs are the same as the previous one.

*Case* (iii). We show that $\lim_{n\to\infty} \Delta^i u(n) = \infty$ for $i = 0, 1, \dots, m-1$. By (7), there exists $\eta > 0$ such that $u(n) \ge \eta n^{m-1}$, $n \ge n_1$. On the other hand, it follows from (10) that

$$\Delta^m u(n) \ge \varphi_c(n, u(n)) \ge \varphi_c(n, \eta n^{m-1}), \ n \ge n_1.$$

This in turn implies

$$\Delta^{m-1}u(n) - \Delta^{m-1}u(n_1) \geq \sum_{k=n_1}^{n-1} \varphi_c(k, \eta k^{m-1}) \longrightarrow \infty \text{ as } n \longrightarrow \infty,$$

which gives our assertion. This complies the proof.

**Corollary 1.** *Suppose that the following conditions hold*

$1^0$ *condition (a) [condition (b)],*

$2^0$ *for every $c > 0$ there exist a nondecreasing continuous function $\varphi_c : (0,\infty) \longrightarrow (0,\infty)$ and $a_c : \mathcal{N} \longrightarrow R_+$ such that*

$$|f(n, x_1, \dots, x_m)| \geq a_c(n)\varphi_c(|x_1|) \text{ on } (D_c) \tag{14}$$

*and*

$$\sum^{\infty} n^{m-1}a_c(n) = \infty, \quad \int^{\infty} \frac{ds}{\varphi_c(s)} < \infty. \tag{15}$$

*Then Eq.* (1) *has property A [property B].*

*Proof.* It suffices to show that for any $n_0 \in \mathcal{N}$ the equation

$$\Delta x(n-1) = \frac{1}{(m-1)!}(n - n_0 + m - 1)^{(m-1)}a_c(n)\varphi_c(x(n))$$

does not have a positive solution for sufficiently large $n$. Suppose that this equation has a positive solution for $n \geq n_1 \geq n_0$. Then we have

$$\frac{1}{(m-1)!}(n - n_0 + m - 1)^{(m-1)}a_c(n) = \frac{x(n) - x(n-1)}{\varphi_c(x(n))} \leq \int_{x(n-1)}^{x(n)} \frac{ds}{\varphi_c(s)}.$$

Summing the above inequality leads to a contradiction.

**Corollary 2.** *Suppose there exists a function $a : \mathcal{N} \longrightarrow R_+$ such that*

$$f(n, x_1, \dots, x_m) \quad \text{sign} \quad x_1 \leq -a(n)|x_1|$$
$$[f(n, x_1, \dots, x_m) \quad \text{sign} \quad x_1 \geq a(n)|x_1|],$$

*on $\mathcal{N} \times R^m$ and there exists a nondecreasing function $w : \mathcal{N} \longrightarrow (0,\infty)$ such that*

$$\sum^{\infty} \frac{1}{nw(n)} < \infty, \quad \sum^{\infty} \frac{n^{m-1}a(n)}{w(n)} = \infty. \tag{16}$$

*Then* Eq. (1) *has property A [property B].*

*Proof.* Note, that the inequality (14) holds, where the functions

$$a_c(n) = a(n)/w(n), \quad \varphi_c(x) = xw\Big[(\frac{x}{c})^{\frac{1}{m-1}}\Big]$$

satisfy the assumption (15).

In fact, for $0 < x_1 < cn^{m-1}$ we have

$$\begin{aligned} f(n, x_1, \ldots, x_m) &\le -a(n)x_1 = -\frac{a(n)}{w(n)}x_1 w(n) \\ &\le -a_c(n)x_1 w\Big[(\frac{x_1}{c})^{\frac{1}{m-1}}\Big], \end{aligned}$$

i.e. the inequality (14) and, by (16), we obtain

$$\sum^{\infty} a_c(n)n^{m-1} = \sum^{\infty} \frac{a(n)}{w(n)} n^{m-1} = \infty,$$

$$\int^{\infty} \frac{ds}{sw\Big[(\frac{s}{c})^{\frac{1}{m-1}}\Big]} = (m-1)\int^{\infty} \frac{dt}{tw(t)} < \infty.$$

Thus the condition (15) is satisfied.

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