BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 25, Number 1, March 1997

SOME MINIMAX THEOREMS ON SET FUNCTIONS

BY

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Abstract. Some minimax theorems of set functions similar to those of Terkelson [7] and Fan [6] but under different and non-comparable convexity conditions are established in this paper.

1. Introduction. Let $(X, \mathcal{A}, m)$ be a measure space. For $\Omega \in \mathcal{A}$, let χ_Ω denote the characteristic function of Ω. Morris [5] showed that if $(X, \mathcal{A}, m)$ is finite, atomless and $L_1(X, \mathcal{A}, m)$ is separable, then for any $\Omega, \Lambda \in \mathcal{A}$ and $\lambda \in I = [0,1]$, there exists a sequence $\{\Gamma_n\} \subset \mathcal{A}$ such that $\chi_{\Gamma_n} \xrightarrow{w^*} \lambda\chi_\Omega + (1-\lambda)\chi_\Lambda$, where $\xrightarrow{w^*}$ denotes weak* convergence in L_∞. The sequence $\{\Gamma_n\}$ is called a Morris-sequence associated with $\langle\lambda, \Omega, \Lambda\rangle$. A subfamily $C \subset \mathcal{A}$ is said to be convex if for every $\langle\lambda, \Omega, \Lambda\rangle \in I \times C \times C$ and every Morris sequence $\{\Gamma_n\}$ associated with it, there exists a subsequence $\{\Gamma_{n_k}\}$ in C; and a set function $F : C \longrightarrow R$ is said to be convex if $\limsup_{k\to\infty} F(\Gamma_{n_k}) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda)$. Also, a set function G is said to be concave if $-G$ is convex. For more detailed discussion of basic properties of convex set functions, the readers are referred to $[1,2,3]$.

In this note, we shall establish minimax theorems of set functions similar to those of Terkelsen [7] and Fan [4], while the convexity conditions are not comparable.

2. Minimax Theorems. When there is no danger of ambiguity, we shall identify $\Omega \in \mathcal{A}$ with χ_Ω in L_∞. It is shown in [2] that the w^*-closure

Received by the editors April 25, 1995 and in revised form April 24, 1996.
AMS subject classification:

of $\mathcal{A}$ in L_∞, $\overline{\mathcal{A}} = \{f \in L_\infty | \, |f| \leq 1\}$, is w^*-compact and is the w^*-closed convex hull of $\mathcal{A}$. Note that since $\overline{\mathcal{A}}$ is w^*-compact and L_1 is separable by assumption, $\overline{\mathcal{A}}$ is metrizable.

A set function F defined on $\mathcal{S} \subset \mathcal{A}$ is said to be w^*-lower semicontinuous (l.s.c) if $F(\Omega) = \overline{F}(\Omega)$ for all $\Omega \in \mathcal{S}$ where $\overline{F}$ is defined on $\mathcal{S} \subset L_\infty$ as

$$\overline{F}(f) = \sup_{V \in N(f)} \inf_{\Omega \in V \cap \mathcal{S}} F(\Omega) \quad \text{for} \quad f \in \overline{\mathcal{A}},$$

where $N(f)$ denotes the family of all w^*-neighborhood of f in $\overline{\mathcal{A}}$. F is said to be w^*-continuous if both F and $-F$ are w^*-l.s.c.. And if F is w^*-continuous, then $\overline{F}$ is the unique w^*-continuous extension of F on $\overline{\mathcal{S}}$.

Let $\mathcal{F}$ be a collection of w^*-continuous set functions defined on $\mathcal{S} \subset \mathcal{A}$. $\mathcal{F}$ is said to be w^*-equicontinuous on $\mathcal{S}$, if $\overline{\mathcal{F}} = \{\overline{F} | F \in \mathcal{F}\}$, the collection of w^*-continuous extension of set functions in $\mathcal{F}$, is w^*-equicontinuous on the w^*-compact subset $\overline{\mathcal{S}}$ of L_∞.

The following well-known lemma (e.g. see [7] establishes the minimax equality for collection of real-valued functions defined on a compact set.)

Lemma 2.1. *Let X be a compact space, and let $\mathcal{F}$ be a collection of l.s.c. real-valued functions defined on X. The following are equivalent:*

(i) *For any $\alpha \in R$ and any finite non-empty subset $\mathcal{G}$ of $\mathcal{F}$ such that $\alpha < \min_{x \in X} \max_{F \in \mathcal{G}} F(x)$, there exists $H \in \mathcal{F}$ with $\alpha \leq \min_{x \in X} H(x)$.*

(ii) $\sup_{F \in \mathcal{F}} \min_{x \in X} F(x) = \min_{x \in X} \sup_{F \in F} F(x).$

Lemma 2.2 below is a set-function version of Lemma 2.1.

Lemma 2.2. *Let $\mathcal{F}$ be a collection of w^*-equicontinuous real-valued set functions defined on $\mathcal{S} \subset \mathcal{A}$. The following are equivalent:*

(i) *For any $\alpha \in R$ and any finite non-empty subset $\mathcal{G}$ of $\mathcal{F}$ such that $\alpha < \inf_{\Omega \in \mathcal{S}} \max_{F \in \mathcal{G}} F(\Omega)$, there exists $H \in \mathcal{F}$ with $\alpha \leq \inf_{\Omega \in \mathcal{S}} H(\Omega)$.*

(ii) $\sup_{F \in \mathcal{F}} \inf_{\Omega \in \mathcal{S}} F(\Omega) = \inf_{\Omega \in \mathcal{S}} \sup_{F \in \mathcal{F}} F(\Omega)$

Proof. Let $\overline{\mathcal{F}}$ be the w^*-continuous extension of $\mathcal{F}$. Then $\overline{\mathcal{F}}$ is w^*-equicontinuous on $\overline{\mathcal{S}}$. Note that for any non-empty subset $\mathcal{G} \subset \mathcal{F}$, the w^*-

equicontinuity asserts that the function $f \longmapsto \sup_{F\in\mathcal{G}} \overline{F}(f)$ is w^*-continuous on $\overline{\mathcal{S}}$. If $\mathcal{G}$ is also finite, then we have

$$\min_{f\in\overline{\mathcal{S}}} \max_{F\in\mathcal{G}} F(f) = \inf_{\Omega\in\mathcal{S}} \max_{F\in\mathcal{G}} F(\Omega).$$

Therefore, if condition (i) holds for $\mathcal{F}$ and $\mathcal{G}$, then condition (i) of Lemma 2.1 holds for $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$. It follows from Lemma 2.1 that (i) implies $\sup_{F\in\mathcal{F}} \min_{f\in\overline{\mathcal{S}}} \overline{F}(f) = \min_{f\in\overline{\mathcal{S}}} \sup_{F\in\mathcal{F}} \overline{F}(f)$, hence $\sup_{F\in\mathcal{F}} \inf_{\Omega\in\mathcal{S}} F(\Omega) = \inf_{\Omega\in\mathcal{S}} \sup_{F\in\mathcal{F}} F(\Omega)$. This shows that (i)$\Longrightarrow$(ii). The converse is trivial.

As an immediate consequence, we have

Theorem 2.1. *Let $\mathcal{F}$ be a collection of w^*-equicontinuous real-valued set functions defined on $\mathcal{S} \subset \mathcal{A}$. Furthermore, if $\mathcal{F}$ is directed with respect to the relation $\leq$, i.e., if for any $F, G \in \mathcal{F}$ there exists an $H \in \mathcal{F}$ with $F \leq H$ and $G \leq H$. Then*

$$\sup_{F\in\mathcal{F}} \inf_{\Omega\in\mathcal{S}} F(\Omega) = \inf_{\Omega\in\mathcal{S}} \sup_{F\in\mathcal{F}} F(\Omega).$$

Corollary 2.1. *Let $\{F_n\}$ be an ascending sequence of w^*-equicontinuous set functions on $\mathcal{S} \subset \mathcal{A}$. Then*

$$\lim_{n\to\infty} \inf_{\Omega\in\mathcal{S}} F_n(\Omega) = \inf_{\Omega\in\mathcal{S}} \lim_{n\to\infty} F_n(\Omega).$$

Example 2.1. Let $\{f_n\}$ be an ascending sequence of equicontinuous functions on $[0,1]$, and let $F_n : \mathcal{S} \longrightarrow R$ be defined by $F_n(\Omega) = \int_\Omega f_n$ where $\mathcal{S}$ is a subfamily of Lebesgue-measurable sets in $[0,1]$. Then since $\{F_n\}$ satisfies the hypothesis of Corollary 2.1, we have

$$\lim_{n\to\infty} \inf_{\Omega\in\mathcal{S}} \int_\Omega f_n = \inf_{\Omega\in\mathcal{S}} \lim_{n\to\infty} \int_\Omega f_n.$$

When convexity condition is present, the directed order condition can be weakened.

Theorem 2.2. *Let $\mathcal{F}$ be a collection of w^*-equicontinuous real-valued convex set functions on a convex subfamily $\mathcal{S} \subset \mathcal{A}$. Then* $\sup_{F\in\mathcal{F}} \inf_{\Omega\in\mathcal{S}} F(\Omega) =$

$\inf_{\Omega \in \mathcal{S}} \sup_{F \in \mathcal{F}} F(\Omega)$, *if for any* $F, G \in \mathcal{F}$, *there exists* $H \in \mathcal{F}$ *such that* $F + G \leq 2H$.

The next minimax theorem on set functions is free of topological structures, which is an application of Fan's minimax theorem (Theorem 3 [4]) dealing with almost periodic functions on product sets. A real-valued function F defined on the product set $X \times Y$ of two arbitrary sets X, Y is said to be right almost periodic, if F is bounded on $X \times Y$ and if, for any $\epsilon > 0$, there exists a finite convering $Y = \cup_{k=1}^{m} Y_k$ of Y such that $|F(x, y') - F(x, y'')| < \epsilon$ for all $x \in X$, whenever y', y'' belong to the same Y_k. Left almost periodic functions are defined similarly. Since every right almost periodic function on $X \times Y$ is also left almost periodic and vice versa, we may simply use the term almost periodic.

Theorem 2.3. *Let F be a real-valued almost periodic function defined on the product of $\mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are convex subfamilies of some finite, atomless measure spaces with L_1-separable. Then*

$$F(\Gamma_\ell^1, \Lambda_1) \leq \lambda F(\Omega_1, \Lambda_1) + (1 - \lambda) F(\Omega_2, \Lambda_1) + \epsilon.$$

Since $\limsup_{\ell \to \infty} F(\Gamma_\ell^1, \Lambda_2) < \lambda F(\Omega_1, \Lambda_2) + (1 - \Lambda) F(\Omega_2, \Lambda_2)$, we may find a subsequence $\{F_\ell^2\}$ of $\{F_\ell^1\}$ such that $F(\Gamma_\ell^2, \Lambda_1) \leq \lambda F(\Omega_1, \Lambda_2) + (1 - \lambda) F(\Omega_2, \Lambda_2) + \epsilon$. Continue this process m times, the subsequence $\{\Gamma_\ell^m\}$ of $\{\Gamma_\ell\}$ satisfies:

$$F(\Gamma_\ell^m, \Lambda_j) \leq \lambda F(\Omega_1, \Lambda_j) + (1 - \lambda) F(\Omega_2, \Lambda_j) + \epsilon$$

for all $1 \leq j \leq m$.

This shows the case for $n = 2$.

Now assume that it is true for n. Let $\xi_i \geq 0, i = 1, 2, \ldots, n+1$ and $\sum_{i=1}^{n+1} \xi_i = 1$ with $\xi_{n+1} \neq 1$. Let $\lambda = 1 - \xi_{n+1}$ and $\lambda_i = \frac{\xi_i}{\lambda}$ for $i = 1, 2, \ldots, n$. Then $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = \frac{\sum_{i=1}^{n} \xi_i}{\lambda} = 1$. Let $\Gamma \in \mathcal{A}$ be such that

$$F(\Gamma, \Lambda_j) \leq \sum_{i=1}^{n} \lambda_i F(\Omega_i, \Lambda_j) + \epsilon \quad \text{for} \quad 1 \leq j \leq m.$$

Choose $\Omega_0 \in \mathcal{A}$ so that

$$F(\Omega_0, \Lambda_j) \leq \lambda F(\Gamma, \Lambda_j) + (1-\lambda) F(\Gamma, \Lambda_j) + (1-\lambda)\epsilon, \quad 1 < j \leq m.$$

It follows that

$$F(\Omega_0, \Lambda_j) \leq \sum_{i=1}^{n+1} \xi_i F(\Omega_i, \Lambda_j) + \epsilon \quad \text{for} \quad 1 \leq j \leq m.$$

The claim is thus proved.

Since F is concave on $\mathcal{B}$, i.e., $-F$ is convex on $\mathcal{B}$, there exists $\Lambda_0 \in \mathcal{B}$ such that

$$F(\Omega_i, \Lambda_0) \geq \sum_{j=1}^{m} \eta_j F(\Omega_i, \Lambda_j) - \epsilon \tag{3}$$

for all $1 \leq i \leq n$.

Combining (1), (2) and (3), it follows that

$$F(\Omega_0, \Lambda_j) \leq F(\Omega_i, \Lambda_0) + 2\epsilon \quad (1 \leq i \leq n, \ 1 \leq j \leq m).$$

Since ϵ is arbitrary, the proof is complete.

Remark: *If $u : X_1 \times X_2 \longrightarrow R$ is almost periodic and $F_i : \mathcal{A}_i \longrightarrow X_i$ is a set function for $i = 1, 2$. Then the function $G : \mathcal{A}_1 \times \mathcal{A}_2 \longrightarrow R$ defined by $G(\Omega, \Lambda) = u(F_1(\Omega), F_2(\Lambda))$ is almost periodic.*

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