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INDEPENDENCE PROPERTY OF POLYNOMIALS IN PRIME RINGS

BY

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Abstract. In this paper we consider the independence property of polynomials in prime rings with assumptions on one-sided ideals. The following result is proved.

Let R be a prime ring with extended centroid C, λ a left ideal of R and let $g_i(X_1,\ldots,X_t)$, $i=1,\ldots,k$, be polynomials in $C\{X\}$, the free C-algebra in noncommuting indeterminates in $X=\{X_1,X_2,\ldots\}$. Assume that $a_1,\ldots,a_k$ are C-independent elements in RC.

(I) Suppose that $\sum_{i=1}^k a_i g_i(X_1,\ldots,X_t)$ is a GPI of λ. Then each $X_{t+1}g_i(X_1,\ldots,X_t)$ is a PI of λ for $i=1,\ldots,k$.

(II) Suppose that $\sum_{i=1}^k a_i g_i(X_1,\ldots,X_t)$ is central-valued on λ but is not a GPI of λ. Then each $g_i(X_1,\ldots,X_t)$ is central-valued on RC unless $R\cong M_2(GF(2))$ and $k\geq 2$.

In [13] Regev proved an analogue of a theorem of Amitsur for central polynomials. More precisely, Regev proved the theorem: Let Φ be an infinite field, $f(X_1,\ldots,X_t)$ and $g(Y_1,\ldots,Y_m)$ two polynomials over Φ in two disjoint indeterminates sets $\{X_1,\ldots,X_t\}$ and $\{Y_1,\ldots,Y_m\}$. Assume that $f(X_1,\ldots,X_t)g(Y_1,\ldots,Y_m)$ is central but is not an identity for $M_k(\Phi)$, the $k\times k$ matrix ring over Φ. Then both f and g are central polynomials for $M_k(\Phi)$. In [8] Kovacs gave the theorem a brief proof by using [7, Theorem 8] together with a famous theorem of Amitsur [1, Theorem 4]. The arguments given by Regev and Kavacs do depend on the infinity of the field Φ. In fact, the result is independent of the infinity of Φ as pointed out by Chuang. In [3]

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Chuang proved the following natural generalization without the assumption that Φ is infinite.

Theorem (Chuang). *Let Φ be a field, $n \geq 2$, and let I_n be the T-ideal of polynomial identities of $M_n(\Phi)$. For $i = 1, \ldots, k$, let $f_i(X_1, \ldots, X_t)$ and $g_i(Y_1, \ldots, Y_m)$ be polynomials with coefficients in Φ and in noncommuting indeterminates in the disjoint sets $\{X_1, \ldots, X_t\}$ and $\{Y_1, \ldots, Y_m\}$ respectively. Assume that the polynomial $\sum_{i=1}^{k} f_i(X_1, \ldots, X_t)g_i(Y_1, \ldots, Y_m)$ is central on $M_n(\Phi)$. Then, except only when $k \geq 2$, $n = 2$ and $\Phi = GF(2)$, the Galois field with two elements, the following hold :*
(1) If $f_i(X_1, \ldots, X_t)$, $i = 1, \ldots, k$, are Φ-independent modulo I_n, then all $g_i(Y_1, \ldots, Y_m)$, $i = 1, \ldots, k$, must be central on $M_n(\Phi)$.
(2) If both the sets $\{f_i(X_1, \ldots, X_t) | i = 1, \ldots, k\}$ and $\{g_i(Y_1, \ldots, Y_m) | i = 1, \ldots, k\}$ are Φ-independent modulo I_n, then all $f_i(X_1, \ldots, X_t)$ and $g_i(Y_1, \ldots, Y_m)$, $i = 1, \ldots, k$, must be central on $M_n(\Phi)$.

On the other hand, recall that a ring R is called prime if every nonzero left ideal of R has no nonzero left annihilators. In [4] Chuang and Lee extended this to a polynomial form. They proved the result: Let R be a prime algebra over a commutative ring K with unity, λ a left ideal of R and $g(X_1, \ldots, X_t)$ be a polynomial over K in noncommuting indeterminates $X_1, \ldots, X_t$. If $a \in R$ is such that $ag(x_1, \ldots, x_t) = 0$ for all $x_i \in \lambda$, then either $a = 0$ or $\lambda g(x_1, \ldots, x_t) = 0$ for all $x_i \in \lambda$.

The objective of this paper is then to generalize the definition of primeness to a polynomial form with finite sum and to consider Chuang's theorem in the context of prime rings. More precisely, we obtain the following result.

Theorem 1. *Let R be a prime ring with extended centroid C, λ a left ideal of R and let $g_i(X_1, \ldots, X_t)$, $i = 1, \ldots, k$, be polynomials in $C\{X\}$, the free C-algebra in noncommuting indeterminates in $X = \{X_1, X_2, \ldots\}$. Assume that $a_1, \ldots, a_k$ are C-independent elements in RC.*
(I) Suppose that $\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)$ is a GPI of λ. Then each $X_{t+1} g_i(X_1, \ldots, X_t)$ is a PI of λ for $i = 1, \ldots, k$.

(II) *Suppose that* $\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)$ *is central-valued on* λ *but is not a* GPI *of* λ. *Then each* $g_i(X_1, \ldots, X_t)$ *is central-valued on* RC *unless* $R \cong M_2(GF(2))$ *and* $k \geq 2$.

In what follows, R always denotes a prime ring with extended centroid C. Let $C\{Z\}$ be the free C-algebra in noncommuting indeterminates in $Z = \{X_1, X_2, \ldots; Y_1, Y_2, \ldots\}$. For the simplicity of notation, if $T \subseteq RC$ and $f(X_1, \ldots, X_t) \in C\{Z\}$, we denote by $f(T)$ the additive subgroup of RC generated by all elements of the form $f(a_1, \ldots, a_t)$ with $a_1, \ldots, a_t \in T$. Now we start the proof of Theorem 1 with some observations.

Lemma 1. *Let* R *be a prime ring with extended centroid* C *and let* I *be a nonzero ideal of* RC. *Suppose that* $a_1, \ldots, a_n$ *are* C*-independent elements in* RC. *Then there exists an element* $h \in I$ *such that* $ha_1, \ldots, ha_n$ *are* C*-independent.*

Proof. Note that if R is not a PI-ring, then by [11, Lemma 3] we are done. So we assume that R is a PI-ring. Then by Posner's theorem RC is a finite-dimensional central simple C-algebra and hence $I = RC$. In this case, we can choose $h = 1$. This completes the proof.

Lemma 2. *Theorem* 1 *(I) holds when* $RC \cong M_n(C)$ *for some* $n \geq 1$.

Proof. We may suppose that $\lambda \neq 0$ and $n \geq 2$. According to [11, Lemma 2] λ and λC satisfy the same GPIs. Therefore replacing λ with λC we may assume from the start that λ is a left ideal of $RC \cong M_n(C)$. So $\lambda = RCe$ for some idempotent $e \in \lambda$. Denote by $\{e_{ij} | 1 \leq i, j \leq n\}$ a complete set of matrix units in RC, i.e., $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$ for all $1 \leq i, j, k, \ell \leq n$ and $\sum_{i=1}^{n} e_{ii} = 1$. Choose an invertible element $u \in RC$ such that $ueu^{-1} = e_{11} + \cdots + e_{mm}$, where $m = \text{rank}(e)$. Since $ua_1u^{-1}, \ldots, ua_ku^{-1}$ are still C-independent, we may assume further that $\lambda = RCe$ with $e = e_{11} + \cdots + e_{mm}$. Note that $k \leq n^2$ since $\dim_C RC = n^2$. If $k < n^2$, then we can choose $n^2 - k$ elements $a_{k+1}, \ldots, a_{n^2}$ in RC such that $\{a_1, \ldots, a_{n^2}\}$ forms a basis for RC over C. In this case, set $f_i(X_1, \ldots, X_t) = 0$ for $i > k$. Hence we may always assume that $k = n^2$. Since the two bases $\{a_1, \ldots, a_{n^2}\}$ and $\{e_{ij} | 1 \leq i, j \leq n\}$ can be transformed each other via an invertible $n^2 \times n^2$ matrix with entries in C, therefore we may assume that $\{a_1, \ldots, a_{n^2}\} = \{e_{ij} | 1 \leq i, j \leq n\}$. Rearrange these $f_i(X_1, \ldots, X_t)$, $i = 1, \ldots, n^2$, as $g_{ij}(X_1, \ldots, X_t)$, $1 \leq i, j \leq n$. Then we have that $\sum_{1 \leq i,j \leq n} e_{ij} g_{ij}(X_1, \ldots, X_t)$ is a GPI of λ. Clearly, for each $i = 1, \ldots, n$, we have that $\sum_{j=1}^{n} e_{ij} g_{ij}(X_1, \ldots, X_t)$ is a GPI of λ. Let $x_i \in \lambda$, $i = 1, \ldots, t$ and let $1 < k \leq n$. Then $(1 + e_{k1}) x_i (1 - e_{k1}) \in \lambda$ for $i = 1, \ldots, t$ and hence

$$\begin{aligned}
0 &= \sum_{j=1}^{n} e_{ij} g_{ij}((1 + e_{k1}) x_1 (1 - e_{k1}), \ldots, (1 + e_{k1}) x_t (1 - e_{k1})) \\
&= \sum_{j=1}^{n} e_{ij} (1 + e_{k1}) g_{ij}(x_1, \ldots, x_t)(1 - e_{k1}) \\
&= \sum_{j=1}^{n} e_{ij} g_{ij}(x_1, \ldots, x_t)(1 - e_{k1}) + e_{i1} g_{ik}(x_1, \ldots, x_t)(1 - e_{k1}) \\
&= e_{i1} g_{ik}(x_1, \ldots, x_t)(1 - e_{k1}),
\end{aligned}$$

since $\sum_{j=1}^{n} e_{ij} g_{ij}(x_1, \ldots, x_t) = 0$ by assumption. But $1 - e_{k1}$ is invertible, we have that $e_{i1} g_{ik}(X_1, \ldots, X_t)$ is a GPI of λ for $k > 1$. By [4, Lemma 3] $\lambda g_{ik}(\lambda) = 0$ for $k > 1$. In particular, $e g_{ik}(x_1, \ldots, x_t) = 0$, that is, $g_{ik}(e x_1, \ldots, e x_t) = 0$. Now

$$\begin{aligned}
0 = \sum_{j=1}^{n} e_{ij} g_{ij}(e x_1, \ldots, e x_t) &= e_{i1} g_{i1}(e x_1, \ldots, e x_t) \\
&= e_{i1} e g_{i1}(x_1, \ldots, x_t) = e_{i1} g_{i1}(x_1, \ldots, x_t).
\end{aligned}$$

Applying [4, Lemma 3] again yields that $\lambda g_{i1}(\lambda) = 0$. So we have proved that $\lambda g_{ij}(\lambda) = 0$ for $1 \leq i, j \leq n$. This completes the proof.

Proof of Theorem 1.

We first prove (I). According to the C-independence of $a_1, \ldots, a_k$, each $g_i(X_1, \ldots, X_t)$ has no constant term. Also, we may assume that these polynomials $g_i(X_1, \ldots, X_t)$ are blended in $X_1, \ldots, X_t$. We define the height of a polynomial in $C\{Z\}$ as given in [6, p.15]. Set $h = \sum_{i=1}^{k} \mathrm{ht}(g_i)$. Proceed the proof by induction on h. For the case $h = 0$, each $g_i(X_1, \ldots, X_t)$ is multilinear. Let $x_1, \ldots, x_t, y \in \lambda$. Then

$$[y, g_i(x_1, \ldots, x_t)] = \sum_{s=1}^{t} g_i(x_1, \ldots, [y, x_s], \ldots, x_t).$$

So we have

$$\begin{aligned} 0 &= \sum_{s=1}^{t} \sum_{i=1}^{k} a_i g_i(x_1, \ldots, [y, x_s], \ldots, x_t) \\ &= \sum_{i=1}^{k} a_i [y, g_i(x_1, \ldots, x_t)] \\ &= \sum_{i=1}^{k} a_i y g_i(x_1, \ldots, x_t) - \sum_{i=1}^{k} a_i g_i(x_1, \ldots, x_t) y \\ &= \sum_{i=1}^{k} a_i y g_i(x_1, \ldots, x_t). \end{aligned}$$

Let $z \in R$. Then $zy \in \lambda$. The above implies that $\sum_{i=1}^{k} a_i z y g_i(x_1, \ldots, x_t) = 0$. By [12, Theorem 2(a)], $y g_i(x_1, \ldots, x_t) = 0$ for $i = 1, \ldots, k$. That is, $\lambda g_i(\lambda) = 0$ for $i = 1, \ldots, k$.

Assume next that $h > 1$. There is no loss of generality in assuming that $\mathrm{ht}(g_1) > 0$ and that $\deg(g_1) = \deg_{x_1}(g_1) > 1$. Denote by $\tilde{g}_i$ the linearlization of g_i at X_1, i.e.,

$$\begin{aligned} &\tilde{g}_i(Y_1, X_1, \ldots, X_t) \\ =&g_i(X_1 + Y_1, X_2, \ldots, X_t) - g_i(X_1, X_2, \ldots, X_t) - g_i(Y_1, X_2, \ldots, X_t). \end{aligned}$$

Then $\sum_{i=1}^{k} a_i \tilde{g}_i(Y_1, X_1, \ldots, X_t)$ is a GPI of λ with height less than h. By inductive hypothesis, in particular, $Y_2 \tilde{g}_1(Y_1, X_1, \ldots, X_t)$ is a PI of λ. By [10, Proposition] $\lambda C = He$, where H stands for the socle of RC and e is an idempotent in H. By Lemma 1 we can choose an element $v \in H$ such that $va_1, \ldots, va_t$ are still C-independent. Note that λC and λ satisfy the same GPIs with coefficients in RC by [11, Lemma 2]. So replacing λ and $a_1, \ldots, a_k$ by λC and $va_1, \ldots, va_k$ respectively, we may assume that $\lambda = He$ and that $a_i \in H$ for $i = 1, \ldots, k$. By Litoff's theorem [5, p.90], there exists an idempotent $u \in H$ such that $e, a_1, \ldots, a_k \in uHu$. Also, R satisfies a nontrivial GPI, since λ satisfies a nontrivial PI. So Martindale's theorem implies that $uHu = M_n(D)$, where D is a finite-dimensional central division algebra over C. Choose a maximal subfield L of D. Then

$$(uHu)\lambda \otimes_C L = (uHu)e \otimes_C L = uHe \otimes_C L \subseteq uHu \otimes_C L \cong M_q(L)$$

for some $q \geq 1$. Note that $(uHu)\lambda \otimes_C L$ is a left ideal of $(uHu) \otimes_C L$. Also, $(uHu)\lambda \otimes_C L$ still satisfies $\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)$. Indeed, if C is a finite field, then $D = C = L$ by Wedderburn's theorem on finite division rings. If C is an infinite field, this case can be proved by a standard arguments; see, for instance, [6, Lemma 1, p.89] for the PI case and [9, Proposition] for the GPI case. Applying Lemma 2 to the present case yields that $(uHe \otimes_C L)g_i(uHe \otimes_C L) = 0$ and hence $eg_i(He) = 0$ since $e \in uHe$. That is, $\lambda g_i(\lambda) = 0$ for $i = 1, \ldots, k$. This proves (I).

For (II), by [11, Lemma 2] $\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)$ is central-valued on λC but is not a GPI of λC. Therefore $\lambda C = RC$. Suppose that $g_i(X_1, \ldots, X_t)$ is not a PI of RC for some i. We may assume that $g_i(X_1, \ldots, X_t)$ is not a PI of RC for $i = 1, \ldots, k$. Then $[Y_1, \sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)]$ is a nontrivial GPI of RC. Martindale's theorem [12, Theorem 3] implies that RC is a strongly primitive ring. Denote by H the socle of RC. Since $\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)$ is central-valued on RC but is not a GPI of RC and C is a field, $1 \in RC$ follows. By [2, Theorem 2], H and RC satisfy the same GPIs. So $1 \in H$. Recall that H itself is s simple ring with minimal right ideals. Therefore $H = RC = M_n(D)$ for some $n \geq 1$, where D is a finite-dimensional central division algebra over C. Take a maximal subfield L of D. As before, $\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)$ is central-valued on $H \otimes_C L$. Note that $H \otimes_C L \cong M_{n\ell}(L)$, where $[D : C] = \ell^2$. By [3, Lemma 1], each $g_i(X_1, \ldots, X_t)$ is central-valued on $H \otimes_C L$ and hence on $H = RC$ unless $RC \cong M_2(GF(2))$. Note that $RC \cong M_2(GF(2))$ if and only if $R \cong M_2(GF(2))$. Finally, we settle the case when $R \cong M_2(GF(2))$ and $k = 1$. Denote by A the set $\{a_1 g_1(x_1, \ldots, x_t) | x_1, \ldots, x_t \in R\}$. Then clearly $A = \{0, 1\}$ since $C = GF(2)$.

So we have $ua_1u^{-1} = a_1$ for all invertible elements $u \in R$. This implies that $a_1 \in C$. Note that $a_1 \neq 0$. Therefore $g_1(X_1, \ldots, X_t)$ is central-valued on R. This finishes the proof of the theorem.

Remarks. 1. In Theorem 1 (I) we cannot conclude that each $g_i(X_1, \ldots, X_t)$ is a PI of λ for $i = 1, \ldots, k$. Indeed, let $R = M_n(F)$, $n > 1$, where F is a field, and let $\lambda = Re$, where e is an idempotent of R of rank k, $1 \leq k < n$. Denote by $S_{2k}(X_1, \ldots, X_{2k})$ the standard polynomial of degree $2k$. Then by the Amitsur-Levitzki theorem $X_{2k+1}S_{2k}(X_1, \ldots, X_{2k})$ is a PI of λ but clearly $S_{2k}(X_1, \ldots, X_{2k})$ is not a PI of λ.
2. In Theorem 1 (II) the exceptional case does occur by Chuang's example [3, p.239]. Indeed, let $R = M_2(GF(2))$. Choose $h(X) = X^2(X+1)^2 = (X^2+X+1)^2+1$ and let $f_1(X) = Xh(X)$, $f_2(X) = f_1(X)^2$, and $a = e_{11}+e_{12}+e_{21}$. Then $af_1(X)+a^2f_2(x)$ is central-valued on R but a and a^2 are $GF(2)$-independent.

As an immediate consequence of Theorem 1 we can consider Chuang's theorem in the context of prime rings. To give its precise statement we need one more terminology. Let λ be a left ideal of prime ring R. The polynomials $f_i(X_1, \ldots, X_t) \in C\{X\}$, $i = 1, \ldots, k$, are called properly C-independent modulo the identities of λ if they satisfy the following condition: If $\delta_1, \ldots, \delta_k \in C$ satisfy that $X_{t+1}\sum_{i=1}^{k} \delta_i f_i(X_1, \ldots, X_t)$ is a PI of λ then $\delta_i = 0$ for all $i = 1, \ldots, k$,.

Theorem 2. *Let R be a prime ring with extended centroid C, λ a left ideal of R and let $g_i(X_1, \ldots, X_t) \in C\{X\}$, $i = 1, \ldots, k$, be properly C-independent modulo the identities of λ.*
(I) Let $a_1, \ldots, a_k \in RC$ be such that $\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)$ is a GPI *of λ. Then $a_i = 0$ for $i = 1, \ldots, k$.*
(II) Let $a_1, \ldots, a_k \in RC$ be such that $\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_t)$ is central-valued on λ but is not a GPI *of λ. Then $a_i \in C$, $i = 1, \ldots, k$, unless $R \cong M_2(GF(2))$ and $k \geq 2$.*

Proof. We first prove (I). Suppose on the contrary that $a_i \neq 0$ for some

i. If $a_1, \ldots, a_k$ are C-independent, then we are done by Theorem 1 (I). So we may assume that $a_1, \ldots, a_k$ are C-dependent. Without loss of generality, we may assume that for some $1 \leq k' < k$, $a_1, \ldots, a_{k'}$ is a maximal C-independent subset of $\{a_1, \ldots, a_k\}$. Write $a_j = \sum_{s=1}^{k'} \beta_{js} a_s$ for $k' < j \leq k$, where $\beta_{js} \in C$. Then

$$\sum_{i=1}^{k'} a_i \Big[g_i(X_1, \ldots, X_t) + \sum_{j=k'+1}^{k} \beta_{ji} g_j(X_1, \ldots, X_t) \Big]$$

is a GPI of λ. By Theorem 1 (I), $Y_1[g_1(X_1, \ldots, X_t) + \sum_{j=k'+1}^{k} \beta_{j1} g_j(X_1, \ldots, X_t)]$ is a PI of λ, which is absurd since the $g_i(X_1, \ldots, X_t)$, $i = 1, \ldots, k$, are properly C-independent modulo the identities of λ. This proves (I).

For the proof of (II) we proceed the proof by induction on k. For $k = 1$, we have that $a_1 g_1(X_1, \ldots, X_t)$ is central-valued on λ but is not a GPI of λ. By Theorem 1 (II), $g_1(X_1, \ldots, X_t)$ is central-valued on λ. But $g_1(X_1, \ldots, X_t)$ is not a PI of λ, $a_1 \in C$ follows. Now suppose that R (and hence RC) is not isomorphic to $M_2(GF(2))$. Suppose that $a_1, \ldots, a_k$ are C-independent. Then by Theorem 1 (II) each $g_i(X_1, \ldots, X_t)$ is central-valued on RC. Note that in this case $\lambda C = RC$. Hence $\sum_{i=1}^{k} a_1 g_(X_1, \ldots, X_t)$ is central-valued on RC. Let $x_1, \ldots, x_t, y \in RC$. Then

$$0 = \Big[a_1 y, \sum_{i=1}^{k} a_i g_i(x_1, \ldots, x_t) \Big] = \sum_{i=1}^{k} [a_1 y, a_i] g_i(x_1, \ldots, x_t).$$

By (I), $[a_1 y, a_i] = 0$ for $i = 1, \ldots, k$. That is, $[a_1 R, a_i] = 0]$ which implies $a_i \in C$. Hence we are done. So we may assume that $a_1, \ldots, a_k$ are C-dependent. As before, we may assume that there exist k', $1 \leq k' < k$, such that $\{a_1, \ldots, a_{k'}\}$ is a maximal C-independent subset of $\{a_1, \ldots, a_k\}$. Write $a_j = \sum_{s=1}^{k'} \beta_{js} a_s$ for $j > k'$, where $\beta_{js} \in C$. Then

$$\sum_{i=1}^{k'} a_i \Big[g_i(X_1, \ldots, X_t) + \sum_{j=k'+1}^{k} \beta_{ji} g_j(X_1, \ldots, X_t) \Big]$$

is central-valued on λ but is not a GPI of λ. By inductive hypothesis, $a_i \in C$ for $i = 1, \ldots, k'$ since the $g_i(X_1, \ldots, X_t) + \sum_{j=k'+1}^{k} \beta_{ji} g_j(X_1, \ldots, X_t)$, $i =$

$1,\ldots,k'$, are still properly C-independent modulo the identities of λ. So $k'=1$. That is, $a_i=\beta_i a_1$ for $i=2,\ldots,n$, where $\beta_i\in C$. So $a_i[g_1(X_1,\ldots,X_t)+\sum_{i=2}^k \beta_i g_i(X_1,\ldots,X_t)]$ is central-valued on λ but is not GPI of λ. Now this is just the case of length one. So $a_1\in C$. This finishes the proof of Theorem 2.

As an immediate application of Theorem 2 Chuang's theorem can be obtained in the context of prime rings.

Theorem 3. *Let R be a prime ring with extended centroid C, λ a left ideal of R and let $f_i(X_1,\ldots,X_t)$ and $g_i(Y_1,\ldots,Y_m)$, $i=1,\ldots,k$, be polynomials in $C\{Z\}$. Suppose that the $g_i(Y_1,\ldots,Y_m)$, $i=1,\ldots,k$, are properly C-independent modulo the identities of λ.*
(I) *Suppose that $\sum_{i=1}^k f_i(X_1,\ldots,X_t)g_i(Y_1,\ldots,Y_m)$ is a* PI *of λ. Then each $f_i(X_1,\ldots,X_t)$ is* PI *of λ for $i=1,\ldots,k$.*
(II) *Suppose that $\sum_{i=1}^k f_i(X_1,\ldots,X_t)g_i(Y_1,\ldots,Y_m)$ is central-valued on λ but is not a* PI *of λ. Then each $f_i(X_1,\ldots,X_t)$ is central-valued on RC unless $R\cong M_2(GF(2))$ and $k\geq 2$.*

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