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ON THE UPPER LIMITS OF SUBSEQUENCES ON THE NUMBERS OF RUNS

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Abstract. Let $X, X_1, X_2, \ldots$ be independent, identically distributed random variables with $P(X=1) = p = 1 - P(X=0)$ for some $0 < p < 1$. For $n = 1, 2, \ldots$, the random variables

$$N_n = \inf\{j \geq 0 : X_{n+j} = 0\}$$

are called the number of runs. Newman (cf. Feller (1950, p.210) or Chow and Teicher (1988, p.61)) proved that

$$\limsup_{n\to\infty} \frac{N_n}{\log_{1/p} n} = 1 \qquad \text{a.s.}$$

Pattern after Chow, Teicher, Wei and Yu (1981), we have the following result. Let $(K_n, n \geq 1)$ be a subsequence of positive integers, and $(K'_n = K'_n(C), n \geq 1)$ is a thinner subsequence of $(K_n, n \geq 1)$ such that

$$K'_{n+1}(C) = \inf\{K_m : K_m > K'_n + C \log_{1/p} K'_n\}$$

for some $C > 0$. Then

$$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} = 1 \qquad \text{a.s.}$$

iff for every $0 < \beta < 1$

$$\sum_{n=1}^{\infty} K'_n(C)^{-\beta} = \infty.$$

1. Introduction. In 1981, Chow, Teicher, Wei and Yu proved the following result on the iterated logarithm law with subsequences: Let $Y_1, Y_2, \ldots$

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be independent, identically distributed random variables, $EY_1 = 0$, $EY_1^2 = 1$ and $W_n = \sum_{j=1}^n Y_j$. Let $(i_n, n \geq 1)$ be a subsequence of positive integers, and $(i'_n, n \geq 1)$ be a thinner subsequence defined by

$$i'_{n+1} = \inf\left\{i_m : i_m > i'_n \exp \frac{C}{\log n}\right\}$$

for some $C > 0$ and all $n \geq n_0 > 2$. Then

$$\limsup_{n\to\infty} \frac{W_{i_n}}{\sqrt{2i_n \log_2 i_n}} = 1 \quad \text{a.s.}$$

iff for every $0 < \beta < 1$

$$\sum_{n=1}^{\infty} (\log i'_n)^{-\beta} = \infty.$$

Motivated by their work, we will establish the following Theorem 1. Let $X, X_1, X_2, \ldots$ be independent, identically distributed random variables with $P(X = 1) = p = 1 - P(X = 0)$ for some $0 < p < 1$. For $n = 1, 2, \ldots$, the random variables

$$N_n = \inf\{j \geq 0 : X_{n+j} = 0\}$$

are called the number of runs. Newman (cf. Feller (1950, p.210) or Chow and Teicher (1988, p.61)) proved that

$$\limsup_{n\to\infty} \frac{N_n}{\log_{1/p} n} = 1 \qquad \text{a.s.}$$

For a given subsequence $(K_n, n \geq 1)$ of positive integers, we are interested in the necessary and sufficient conditions for

$$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} = 1 \qquad \text{a.s.}$$

To do that, we need the concept of a thinner subsequence of $(K_n, n \geq 1)$ which was introduced by Qualls (1974). Let $(K_n, n \geq 1)$ be a subsequence of positive integers and $C > 0$. Define $(K'_n = K'_n(C), n \geq 1)$ by $K'_1 = K_1$, $K'_2 = K_2$ and for $n \geq 2$

$$K'_{n+1}(C) = \inf\{K_m : K_m > K'_n + C \log_{1/p} K'_n\}. \tag{1}$$

The subsequence $(K'_n(C), n \geq 1)$ is called the thinner subsequence of $(K_n, n \geq 1)$ for a given C.

Theorem 1. *Let $(K_n, n \geq 1)$ be a subsequence of positive integers. Then*

$$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} = 1 \quad \text{a.s.} \tag{2}$$

iff for some (and then for all) $C > 0$, the thinner subsequence $(K'_n = K'_n(C))$ satisfies

$$\sum_{n=1}^{\infty} {K'_n}^{-\beta} = \infty, \tag{3}$$

for every $0 < \beta < 1$.

The proof of Theorem 1 will be given in the next section.

Remark. Without the idea of "thinner subsequence", one would have difficulties to formulate the necessary condition for the upper limit.

Corollary 1. *If $K_n = n^\gamma$, for some $\gamma > 1$ and $n = 1, 2, \ldots$, then*

$$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} < 1 \qquad \text{a.s.} \tag{4}$$

Proof. Obviously, for any $C > 0$, $K'_n = K'_n(C) \geq K_n$, $n = 1, 2, \ldots$. Choose $0 < \beta < 1$ such that $\gamma\beta > 1$. Then

$$\sum_{n=1}^{\infty} {K'_n}^{-\beta} \leq \sum_{n=1}^{\infty} K_n^{-\beta} = \sum_{n=1}^{\infty} n^{-\gamma\beta} < \infty.$$

By Theorem 1, (4) holds.

Corollary 2. *Let $K_n = [n \log^\gamma(n+2)]$, for some $\gamma > 1$ and all $n \geq 1$, where $[a]$ is the integral part of a. Then*

$$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} = 1 \qquad \text{a.s.} \tag{5}$$

Proof. There exists a positive integer n_0 such that for all $n \geq n_0$,

$$\begin{aligned} K_{n+1} &= [(n+1)\log^\gamma(n+3)] \\ &\geq n[\log^\gamma(n+3)] + [\log^\gamma(n+3)] - 1 \\ &\geq n[\log^\gamma(n+3)] + \log(n\log^\gamma(n+3)) \\ &\geq K_n + \frac{1}{2}\log K_n. \end{aligned}$$

Hence $(K'_n = K_n, n \geq 1)$ is a thinner subsequence defined by (2), with $C = \frac{1}{2}\log\frac{1}{p}$. Since $\sum_{n=1}^\infty K_n^{-\beta} = \infty$ for every $0 < \beta < 1$, by Theorem 1, we have (5).

2. Proof of Theorem 1. Before giving the proof of Theorem 1, we need the following lemmas.

Lemma 1. *For any* $0 < \beta < 1$, *if*

$$\sum_{n=1}^\infty K'_n(C_0)^{-\beta} = \infty \quad \textit{for some } C_0 > 0, \tag{6}$$

then

$$\sum_{n=1}^\infty K'_n C^{-\beta} = \infty \quad \textit{for all } C > 0. \tag{7}$$

Proof. By the definition of $K'_n(C)$, we know that

$$K'_n(C) \leq K'_n(C') \qquad \text{for } C' > C$$

Hence (7) holds for all $0 < C < C_0$. To prove that (7) hold for all $C > C_0$, we need to prove that

$$\sum_{n=1}^\infty K'_n(2C_0)^{-\beta} = \infty. \tag{8}$$

Since

$$\begin{aligned} K'_{2n+2}(C) &\geq K'_{2n+1}(C) + C\log_{1/p} K'_{2n+1}(C) \\ &\geq K'_{2n}(C) + C\log_{1/p} K'_{2n}(C) + C\log_{1/p} K'_{2n+1}(C) \\ &\geq K'_{2n}(C) + 2C\log_{1/p} K'_{2n}(C). \end{aligned}$$

If $K'_{2n}(C) \geq K'_n(2C)$, then $K'_{2n+2}(C) \geq K'_n(2C) + 2C\log_{1/p} K'_n(2C)$, and hence $K'_{2n+2}(C) \geq K'_{n+1}(2C)$. Since $K'_1(2C_0) = K_1 < K_2 = K'_2(C_0)$, $K'_{2n}(C_0) \geq K'_n(2C_0)$ for all $n \geq 1$, by induction. Hence

$$\begin{aligned}\infty &= \sum_{j=2}^{\infty} K'_j(C_0)^{-\beta} \\ &= \sum_{n=1}^{\infty} K'_{2n}(C_0)^{-\beta} + \sum_{n=1}^{\infty} K'_{2n+1}(C_0)^{-\beta} \\ &\leq 2\sum_{n=1}^{\infty} K'_n(2C_0)^{-\beta},\end{aligned}$$

yielding (8).

Lemma 2. *Let*

(9) $$\sum_{n=1}^{\infty} K'_n(C)^{-\beta} = \infty \text{ for some } C > 0 \text{ and all } 0 < \beta < 1.$$

Then

(10) $$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} \geq 1 \qquad \text{a.s.}$$

Proof. Let $0 < \beta < 1$. Put $\alpha_n = [\beta \log_{1/p} K'_n]$,

(11) $$P\{N_{K'_n} \geq \alpha_n\} = p^{\alpha_n} \geq K_n'^{-\beta},$$

$$\{N_{K'_n} \geq \alpha_n\} = \{X_{K'_n} = 1, \ldots, X_{(K'_n+\alpha_n-1)} = 1\}.$$

Now we choose $C = \beta$. $K'_{n+1}(\beta) - K'_n(\beta) > \beta \log_{1/p} K'_n(\beta)$, $K'_{n+1}(\beta) - K'_n(\beta)$
$\geq \alpha_n$. Since $(X_n, n \geq 1)$ are independent and $\{N_{K'_n} \geq \alpha_n\} \in \sigma(X_{K'_n}, X_{K'_n+1}, \ldots, X_{K'_n+\alpha_n-1})$, $(\{N_{K'_n} \geq \alpha_n\}, n \geq 1)$ are independent.

By (9), (11) and the Borel-Cantelli theorem,

$$P\{N_{K'_n} \geq \alpha_n, \text{ i.o.}\} = 1.$$

Hence

$$P\{N_{K'_n} > \beta \log_{1/p} K'_n - 1, \text{ i.o.}\} = 1.$$

Consequently,

$$\limsup_{n\to\infty} \frac{N_{K_n'}}{\log_{1/p} K_n'} \geq \beta \qquad \text{a.s.}$$

for every $0 < \beta < 1$. Therefore (10) holds.

Lemma 3. *Let*

$$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} = 1 \qquad \text{a.s.}$$

Then for some $C > 0$,

$$\sum_{n=1}^{\infty} K_n'(C)^{-\beta} = \infty \textit{ for every } 0 < \beta < 1.$$

Proof. For a given $0 < \beta < 1$, choose $\beta < \gamma < 1$. Put

$$A_n = \{N_{K_n} > \gamma \log_{1/p} K_n\}. \tag{12}$$

By (2),

$$P(A_n,\ \text{i.o.}) = 1. \tag{13}$$

Let $C = \gamma - \beta$, $K_{n'} = K_n'(C)$, $n'' = (n+1)' - 1$, and

$$B_n = \bigcup_{n'}^{n''} A_j. \tag{14}$$

Then by (13),

$$\sum_{n=1}^{\infty} P\{B_n\} = \infty.$$

For $j = 1, 2, 3, \ldots$, and $m = j+1, j+2, \ldots$, if $N_j = k > m - j$, then $X_j = 1$, $X_{j+1} = 1, \ldots, X_m = 1, \ldots, X_{j+k-1} = 1, X_{j+k} = 0$. Therefore $N_m = k - (m-j) = (j+k-1) - (m-1)$, and $N_j - N_m = m - j$. Of course, if $N_j \leq m - j$, then $N_j - N_m \leq m - j$. Hence for some $m > j$,

$$N_j - N_m \leq m - j. \tag{15}$$

For every $n' \leq j \leq n''$, on A_j,

$$\begin{aligned} N_{K_{n''}} &= N_{K_j} + N_{K_{n''}} - N_{K_j} \\ &> (\gamma \log_{1/p} K_j) - (K_{n''} - K_j) \quad \text{(by (12) and (15))} \\ &> (\gamma \log_{1/p} K_j) - (C \log_{1/p} K_{n'}) \\ &\geq (\gamma \log_{1/p} K_{n'}) - (C \log_{1/p} K_{n'}) \\ &= \beta \log_{1/p} K_{n'}. \end{aligned}$$

By (14), $B_n \subset \{N_{K_{n''}} > \beta \log_{1/p} K_n'\}$. Hence

$$P\{B_n\} \leq {K_n'}^{-\beta},$$

and (3) holds.

Proof of Theorem 1. Assume that (3) holds. By Newman's result, we have

$$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} \leq 1 \qquad \text{a.s.}$$

and by Lemma 2,

$$\limsup_{n\to\infty} \frac{N_{K_n}}{\log_{1/p} K_n} \geq 1 \qquad \text{a.s.}$$

Hence (2) holds. Now assume that (2) holds . Then by Lemma 3, (3) holds.

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