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POINT BIFURCATIONS AND BUBBLES FOR SOME ONE-PARAMETER FAMILIES OF QUADRATIC POLYNOMIALS

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Abstract. We show that, for some suitably chosen constant a, the one-parameter family $g_c(x) = a - c(1 + x^2)$ with c as the parameter has a 'point bifurcation' of periodic points of some period ≥ 3 and the bifurcation diagram of g_c has 'bubbles' for $c \in [0, a]$. We also show that the topological entropy of g_c, as a function of c, is symmetric with respect to the vertical line $c = \frac{a}{2}$ and hence is not monotonic although g_c is a family of unimodal maps with negative Schwarzian derivatives.

Let f be a continuous map from the real line R into itself. For every positive integer n, define the n^{th} iterate f^n of f inductively by letting $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n > 1$. For $x_0 \in R$, we call x_0 a periodic point of f if $f^m(x_0) = x_0$ for some positive integer m and call the smallest such positive integer m the least period of x_0 with respect to f. If x_0 is a periodic point of f, then we also call the set $\{f^n(x_0) | n \geq 0\}$ the periodic orbit of x_0 with respect to f. Let f_c be a one-parameter family of continuous maps from the real line into itself. Assume that there exist two positive numbers δ and ε such that f_c has a periodic point p of some period n for $c = c_0$, but no periodic point of same period in $(p - \varepsilon, p + \varepsilon)$ for every c in $(c_0 - \delta, c_0) \cup (c_0, c_0 + \delta)$. Then we say that f_c has a point bifurcation of

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period n points at $c = c_0$. Although trivial examples of point bifurcations of periodic points can be easily constructed, for nontrivial examples (see, for example, [4]), the point bifurcations are usually very difficult to detect from the practical point of view. We may not notice them even when we encounter one such example. In this note, we show that point bifurcations of periodic points can occur *nontrivially* in families of simple well-behaved maps, the quadratic polynomials, for example, in the family $g_c(x) = a - c(1 + x^2)$, where a is some suitably chosen constant and c is the parameter. This also answers the question posed at the end of [4]. Since families of quadratic polynomials are often taken to model some situations in many disciplines, the interpretation of such bifurcations can be very interesting.

Theorem 1. *For every integer $n \geq 3$, let c_n be a value in the interval $(1, 2)$ such that the one-parameter family $f_c(x) = 1 - cx^2$ has a tangent bifurcation of periodic points of least period n at $c = c_n$. Then the one-parameter family $g_c(x) = 2\sqrt{c_n} - c(1+x^2)$ has a point bifurcation of periodic points of least period n at $c = \sqrt{c_n}$.*

Theorem 2. *Let $n \geq 2$ be a fixed integer. For $\sqrt{3} < b < a < 2\sqrt{2}$, let $c_1 = \frac{a}{2} - \frac{\sqrt{a^2-b^2}}{2}$ and $c_2 = \frac{a}{2} + \frac{\sqrt{a^2-b^2}}{2}$. Assume that y is a periodic point of the map $1 - \frac{b^2}{4}x^2$ with least period n. Then, for each $i = 1, 2$, the point $x_i = (a - c_i)y$ is also a periodic point of the map $a - c_i(1 + x^2)$ with least period n. Furthermore, if y is attracting, then so are x_1 and x_2.*

Theorem 3. *Let a be a fixed number in $(0, 2\sqrt{2})$ and, for $0 \leq c \leq a$, let $g_c(x) = a - c(1+x^2)$. Then the topological entropy of g_c, as a function of c, is monotone increasing on $[0, \frac{a}{2}]$ and monotone decreasing on $[\frac{a}{2}, a]$ and is symmetric with respect to the vertical line $c = \frac{a}{2}$.*

For the proof of theorems, we shall use the notion of conjugacy. Let $h(x)$ be a map from the real line R into itself. We shall say that $h(x)$ is a homeomorphism on R if $h(x)$ is one-to-one, onto, and both $h(x)$ and $h^{-1}(x)$ are continuous maps of R. Assume that both $f(x)$ and $g(x)$ are continuous maps from R into itself. We shall say that $g(x)$ is topologically conjugate to $f(x)$ through $h(x)$ if $h(x)$ is a homeomorphism on R such that $(g \circ h)(x) = (h \circ f)(x)$ for all $x \in R$. The following result is easy to prove [5]:

Lemma 1. *Let $f(x)$ and $g(x)$ be continuous maps from R into itself and let $h(x)$ be a homeomorphism on R. Assume that $g(x)$ is topologically conjugate to $f(x)$ through $h(x)$. Then $f(x)$ is topologically conjugate to $g(x)$ through $h^{-1}(x)$ and the following also hold:*

(1) *$g^n \circ h = h \circ f^n$ for all positive integers n.*

(2) *If y is a periodic point of f with least period n, then $h(y)$ is a periodic point of g with least period n. Furthermore, if y is attracting, then so is $h(y)$.*

Consequently, for every positive integer n, there is a one-to-one correspondence between the periodic orbits of f with least period n and those of g.

The following result can be proved by direct computation or see, for example, [5]:

Lemma 2. *Let $F_{a,c}(x) = 1 - c(a-c)x^2$, $G_{a,c}(x) = a - c(1+x^2)$, and let $H_{a,c}(x) = (a-c)x$. Then $(G_{a,c} \circ H_{a,c})(x) = (a-c) - c[(a-c)^2x^2] = (H_{a,c} \circ F_{a,c})(x)$. In particular, if $a \neq c$, then $G_{a,c}(x)$ is topologically conjugate to $F_{a,c}(x)$ through the linear map $H_{a,c}(x)$.*

Proof of Theorem 1. By Lemma 2, $G_{a,c}(x)$ is topologically conjugate to $F_{a,c}(x)$ through $H_{a,c}(x) = (a-c)x$. By direct computation, $F_{a,c}(x) = 1 - [\frac{a^2}{4} - (c - \frac{a}{2})^2]x^2$. So, if, for some positive integer n, $f_c(x) = 1 - cx^2$ has a tangent bifurcation of periodic points of least period n at $c = c_n$, then it is clear that the one-parameter family (with c as the parameter) $F_{2\sqrt{c_n},c}(x) = 1 - [c_n - (c - \sqrt{c_n})^2]x^2$ has a point bifurcation of periodic points of least period n at $c = \sqrt{c_n}$. By Lemma 1, the one-parameter family (with c as the parameter) $g_c(x) = 2\sqrt{c_n} - c(1+x^2)$ has a point bifurcation of periodic points of least period n at $c = \sqrt{c_n}$. This completes the proof of Theorem 1.

Proof of Theorem 2. By direct computation, we see that $c_1(a - c_1) =$

$\frac{b^2}{4} = c_2(a-c_2)$. For each $i = 1, 2$, it follows from Lemma 2 that the map $a - c_i(1+x^2)$ is topologically conjugate to the map $1 - c_i(a-c_i)x^2 = 1 - \frac{b^2}{4}x^2$ through the linear map $h_i(x) = (a-c_i)x$. Let y be a periodic point of the map $1 - \frac{b^2}{4}x^2$ with least period n and, for each $i = 1, 2$, let $x_i = (a-c_i)y$. Then it follows from Lemma 1 that each x_i, $i = 1, 2$ is a periodic point of the map $a - c_i(1+x^2)$ with least period n. Furthermore, if y is attracting, so are x_1 and x_2. This completes the proof of Theorem 2.

Proof of Theorem 3. By Lemma 2, $G_{a,c}(x)$ is topologically conjugate to $F_{a,c}(x) = 1 - c(a-c)x^2 = 1 - [\frac{a^2}{4} - (c - \frac{a}{2})^2]x^2$. Since topological entropy is a topological invariant [1], it is easy to see that the topological entropy of $G_{a,\frac{a}{2}-c}(x)$ is equal to that of $G_{a,\frac{a}{2}+c}$ for all $0 \le c \le \frac{a}{2}$. This shows that the topological entropy of $g_c(= G_{a,c})$ is symmetric with respect to the vertical line $c = \frac{a}{2}$. Since the family $f_c(x) = 1 - cx^2$ is topologically conjugate to the family $h_c(x) = cx(1-x)$ and the topologically entropy of h_c is known [12] to be monotone increasing on $[0, 4]$, this, combined with the conjugacy of $G_{a,c}$ with $F_{a,c}$, implies that the topological entropy of $g_c(= G_{a,c})$, as a function of c, is monotone increasing on $[0, \frac{a}{2}]$. This completes the proof of Theorem 3.

Remarks.

(1) Since it is well-known [3] that the one-parameter family $f_c(x) = 1 - cx^2$ has a tangent bifurcation of period 3 points at $c = \frac{7}{4}$, we immediately obtain that the one-parameter family $g_c(x) = \sqrt{7} - c(1+x^2)$ has a point bifurcation of period 3 points at $c = \frac{\sqrt{7}}{2}$.

(2) It is well-known [2] that the one-parameter family $f_c(x) = 1 - cx^2$ can have a tangent bifurcation of periodic points of least period n for some integer $n \ge 3$ only when $c \in (c_\infty, 2)$, where $c_\infty \approx 1.40115$ is the accumulation point of the cascade of the first period-doubling bifurcations of $f_c(x)$. Actually [10], if, for every integer $n \ge 3$, k_n is the number of distinct parameter values c_n in $(0, 2)$ such that $f_c(x)$ has a tangent bifurcation of periodic points of least period n at $c = c_n$, then $k_3 = 1$, $k_4 = 1$, $k_5 = 3$, $k_6 = 4$, $k_7 = 9$, $k_8 = 14$, $k_9 = 28$, $k_{10} = 48$,

$k_{11} = 93$, $k_{12} = 165$, and $\lim_{n\to\infty} k_n = \infty$. Especially, when $n \geq 3$ is prime, we have $k_n = \frac{2^n-2}{2n}$. Consequently, for any fixed integer $n \geq 3$, there exist k_n values of a in $(2\sqrt{c_\infty}, 2\sqrt{2}) \approx (2.36740, 2\sqrt{2})$ such that, for each such a, the one-parameter family $g_c(x) = a - c(1 + x^2)$ has a point bifurcation of periodic points of least period n at $c = \frac{a}{2}$.

(3) It is well-known that the period 2 points of the family $f_c(x) = 1 - cx^2$ bifurcate from one branch of the fixed points at $c = \frac{3}{4}$ and exist for all $c > \frac{3}{4}$. Also, it is well-known that all periodic points of the family f_c with $c \geq 2$ are unstable. So, let $n \geq 2$ be a fixed integer and let $a \in (\sqrt{3}, 2\sqrt{2})$ be a fixed number. For $b \in (\sqrt{3}, a)$, let $c_1(b) = \frac{a}{2} - \frac{\sqrt{a^2-b^2}}{2}$ and $c_2(b) = \frac{a}{2} + \frac{\sqrt{a^2-b^2}}{2}$. Then $c_1(b)$ and $c_2(b)$ are symmetric with respect to the point $\frac{a}{2}$. By Theorem 2, if $y(b)$ is a branch of period n points of the family $1 - \frac{b^2}{4}x^2$, then, for each $i = 1, 2$, the points $(a - c_i(b))y(b)$ form a branch of period n points of the family $a - c_i(b)(1+x^2)$. Furthermore, if $y(b)$ is attracting, then so is $(a-c_i(b))y(b)$ for each $i = 1, 2$. On the other hand, if $b > 0$ is increased to a, then $c_1(b)$ is increased to $\frac{a}{2}$ and vice versa. Also, if $b > 0$ is decreased from a, then $c_2(b)$ is increased from $\frac{a}{2}$ and vice versa. Consequently, if the family $1 - \frac{b^2}{4}x^2$ (with b as the parameter) has a bifurcation of period n points at $b = b_0 \in (\sqrt{3}, a)$ (note that, at the bifurcation point $b = b_0$, two branches of period n points are created, one is usually called upper branch and the other lower branch) and if these period n points exist for all $b_0 \leq b \leq a$, let $y(b)$ denote any branch of these period n points, then as the parameter c in the family $a - c(1+x^2)$ is varied increasingly (by letting $c = c_1(b)$ when $c \leq \frac{a}{2}$ and letting $c = c_2(b)$ when $c \geq \frac{a}{2}$) from $\frac{a}{2} - \frac{\sqrt{a^2-b_0^2}}{2}$ through $\frac{a}{2}$ to $\frac{a}{2} + \frac{\sqrt{a^2-b_0^2}}{2}$, we see that the parameter $b > 0$ in the family $1 - \frac{b^2}{4}x^2$ is varied increasingly from b_0 to a and then decreasingly from a back to b_0 and so the branch $(a - c)y$ of period n points of the family $a - c(1 + x^2)$ is varied from $(a - c_1(b_0))y(b_0)$ to $\frac{a}{2} \cdot y(a)$ and then back to $(a - c_2(b_0))y(b_0)$. Therefore, the bifurcation diagram of the one-parameter family $g_c(x) = a - c(1 + x^2)$ will have

'bubbles' ([13, p.231 & p.257] or see Figure 1). It is visible on the bifurcation diagram when $y(b)$ is attracting and invisible otherwise. Since the bifurcation diagram of the family $1 - \frac{b^2}{4}x^2$ (to be exact, the family $1 - bx^2$) is well-known, Theorem 2 gives the existence of 'bubbles' for the family $g_c(x) = a - c(1 + x^2)$. Note that, under the linear map $h_c^{-1}(x) = \frac{1}{a-c}x$ with $c \neq a$, the bifurcation diagram of $g_c(x)$, for $c \in (0, a)$, is mapped onto that of $F_{a,c}(x) = 1 - c(a-c)x^2 = 1 - [\frac{a^2}{4} - (c - \frac{a}{2})^2]x^2$ whose bifurcation diagram is obviously symmetric with respect to the vertical line $c = \frac{a}{2}$ (Figure 2).

(4) The two-parameter family $Q_{a,c}(x) = x^3 - ax + c$ of cubic polynomials has been widely studied ([7,8,11]). A special case of it is the one-parameter family $q_c(x) = Q_{2,c}(x) = x^3 - 2x + c$. When $c = \frac{1}{\sqrt{3}}$, it can be easily verified that $q^3_{1/\sqrt{3}}(x) - x = (x^3 - 3x + \frac{1}{\sqrt{3}})(x^3 - \sqrt{3}x^2 + \frac{1}{\sqrt{3}})^2(x^{18} + 2\sqrt{3}x^{17} - 6x^{16} - 16\sqrt{3}x^{15} + 19x^{14} + 59\sqrt{3}x^{13} - 55x^{12} - \frac{403\sqrt{3}}{3}x^{11} + 133x^{10} + \frac{1793\sqrt{3}}{9}x^9 - \frac{664}{3}x^8 - \frac{560\sqrt{3}}{3}x^7 + \frac{707}{3}x^6 + \frac{973\sqrt{3}}{9}x^5 - \frac{400}{3}x^4 - \frac{301\sqrt{3}}{9}x^3 + \frac{272}{9}x^2 + \frac{70\sqrt{3}}{9}x + \frac{181}{27}$. That is, each zero of the polynomial $x^3 - \sqrt{3}x^2 + \frac{1}{\sqrt{3}}$ is a period 3 point of $q_{1/\sqrt{3}}(x)$ with multiplicity two. The computer experiments seem to suggest that the value $c = \frac{1}{\sqrt{3}}$ is the only value of c such that the map $q_c(x)$ has period 3 points and so the one-parameter family $q_c(x)$ seems to have a point bifurcation of period 3 points at $c = \frac{1}{\sqrt{3}}$. However, we are unable to prove it. On the other hand, it is easy to see that, for each fixed $a \geq 0$, the map $Q_{a,c}(x)$ has no periodic points other than fixed points for every sufficiently large c. So, when the fixed number $a \geq 0$, is not too large, say, when $Q_{a,0}(x)$ has only a finite number of periodic points, then as the parameter c in the one-parameter family $Q_{a,c}(x)$ is varied increasingly from 0, new periodic points may be born and then eventually disappear. Therefore, the bifurcation diagram of such one-parameter family $Q_{a,c}(x)$ (with c as the parameter) may have 'bubbles'. Figure 3 is one such example. Similar argument applies to the two-parameter family $T_{a,c}(x) = x^4 - x^2 - ax + c$. Figure 4 is one such example. Note that, in Figure 4, the left shows stable

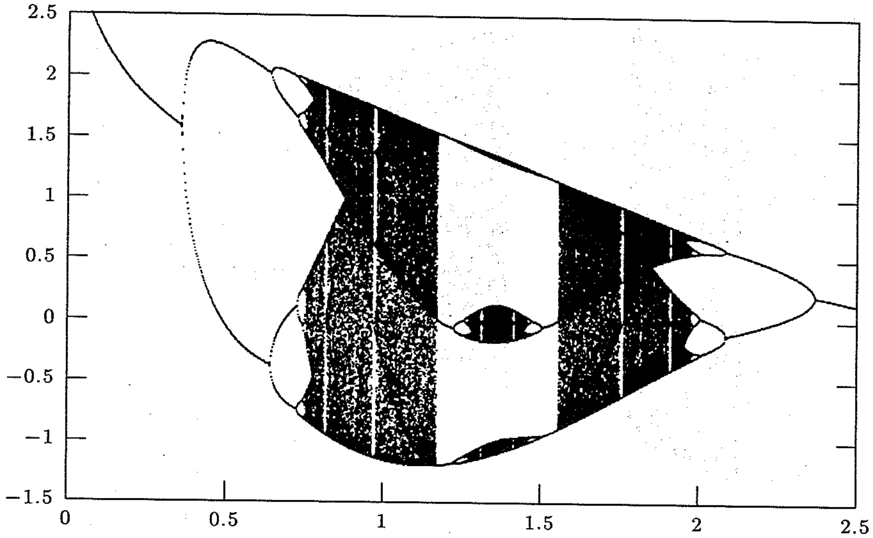

Figure 1. The bifurcation diagram with "bubbles" for the family $g_c(x) = 2.675 - c(1+x^2)$.

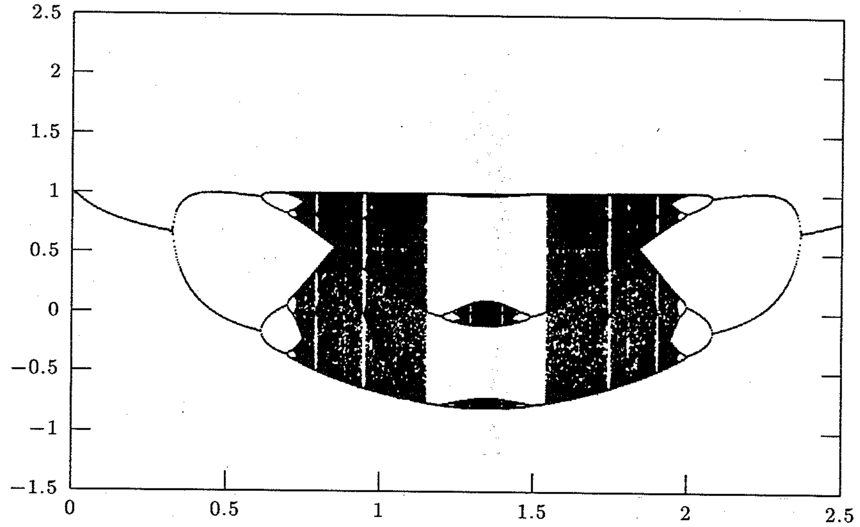

Figure 2. The bifurcation diagram with "bubbles" for the family $f_c(x) = 1 - c(2.675 - c)x^2$.

period 1 and period 2 attractors, the right shows 'bubbles', and in the middle there seems to have a sudden appearing chaotic attractor (see also [13, p.264]). On the other hand, if $p_c(x) = x^3 + ax^2 + bx + c$ and $d = \frac{-4a^3+18ab-18a}{27} - c$, then it is easy to see that $\frac{1}{2}[(c, p_c(x)) + (d, p_d(-\frac{2}{3}a - x))] = (\frac{-4a^3+18ab-18a}{54}, -\frac{a}{3})$. Thus, for fixed a and b, the one-parameter family $p_c(x) = x^3 + ax^2 + bx + c$ will have a bifurcation diagram (on the c–x plane) which is symmetric with respect to the

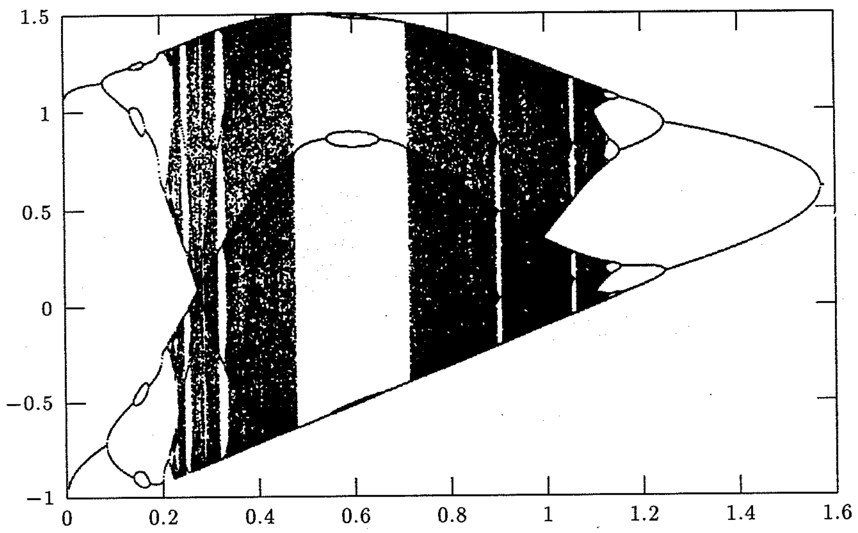

Figure 3. The bifurcation diagram with "bubbles" for the family $q_c(x) = x^3 - 2.018x + c$.

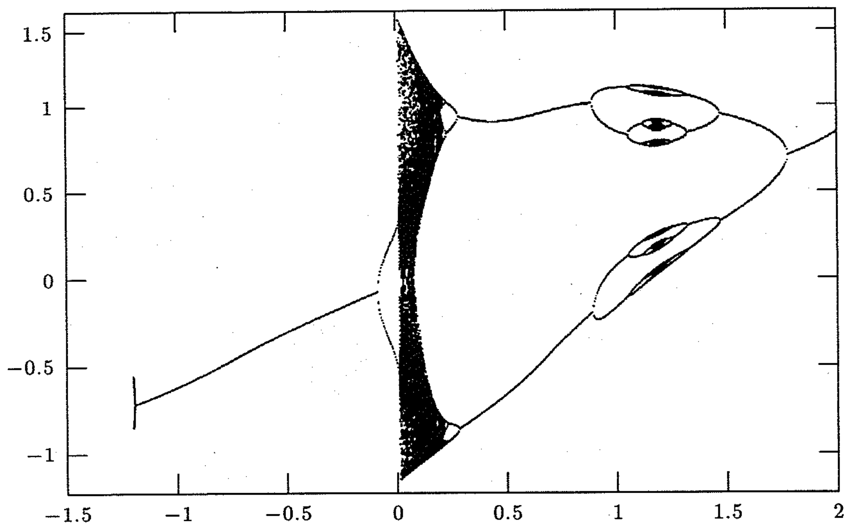

Figure 4. The bifurcation diagram for the family $t_c(x) = x^4 - x^2 - 1.084x + c$.

point $(\frac{-4a^3+18ab-18a}{54}, -\frac{a}{3})$ and so may have bubbles.

(5) For one-parameter families of continuous maps, the question, whether the topological entropy [1] varies in a monotone way with the parameter, is often asked. The widely studied families of quadratic polynomials $h_c(x) = cx(1-x)$, $f_c(x) = 1 - cx^2$, or $u_c(x) = c - x^2$ [6] (they are topologically conjugate to one another) are known to have this

monotone property [12]. However, the family $g_c(x) = a - c(1 + x^2)$ does not although it is also a family of unimodal maps with negative Schwarzian derivatives (see also [6, 9]).

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