BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 24, Number 4, December 1996

COMMUTING ADDITIVE MAPPINGS IN SEMIPRIME RINGS

BY

TSIU-KWEN LEE(李秋坤) AND TSONG-CHERNG LEE(李聰成)

Abstract. Let R be a semiprime ring with extended centroid C and U the right Utumi quotient ring of R.

(I) Let ρ be a right ideal of R and $f : \rho \to U$ be an additive mapping such that $[f(x), x] = 0$ for all $x \in \rho$. Then there exist $\lambda \in C$ and $\zeta : \rho \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in \rho$ provided that one of the following holds

(a) R is a prime ring with $[\rho, \rho]\rho \neq 0$;

(b) the left annihilator of ρ in R is zero.

(II) Let R be a prime ring and L a noncentral Lie ideal of R. Suppose that $f : L \to U$ is an additive mapping such that $[f(x), x] \in C$ for all $x \in L$. Then there exist $\lambda \in C$ and $\zeta : L \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in L$ unless $\mathrm{char} R = 2$ and $\dim_C RC = 4$.

As a corollary to (II), the only ring homomorphisms of R centralizing on a noncentral Lie ideal are also characterized.

Recall that a mapping f of a ring R into itself is said to be commuting on a subset S of R if $[f(x), x] = 0$ for all $x \in S$. The study of such mappings was initiated by a paper of Posner. In [17] Posner proved that if a prime ring R has a nonzero derivation commuting on R, then R must be commutative. Over the last twenty years, many related results have been published (for instance, see [2]-[9] and [11]-[16]). In [4] Brešar obtained a characterization of additive mappings commuting on prime rings and then he extended the result to the semiprime case. More precisely, in [6] Brešar proved the result:

Received by the editors June 29, 1995.

1991 AMS Subject Classification: 16W25, 16N60, 16D15.

Key words and phrases: Semiprime rings, Right Utumi quotient rings, Centralizing mappings, Derivations, Biderivations, Lie ideals.

Let R be a semiprime ring with extended centroid C, and let $f : R \to R$ be a commuting additive mapping. Then there exist $\lambda \in C$ and an additive mapping $\zeta : R \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in R$. The purpose of this paper is to give the one-sided version of Brešar's and to give its Lie ideal case.

Throughout this paper let R be always a semiprime ring with extended centroid C, and let U be the right Utumi quotient ring (i.e., the maximal right quotient ring) of R. For a subset S of R, denote by $\ell_R(S) = \{x \in R \mid xS = 0\}$, the left annihilator of S in R. Also, $r_R(S)$ is defined similarly. We note that $\ell_R(S) = r_R(S)$ if S is an ideal of R. For any subsets A, B of U, $[A, B]$ denotes the additive subgroup of U generated by all elements of the form $[a, b] = ab - ba$ with $a \in A$ and $b \in B$. The first main result of this paper is then to prove the following theorem.

Theorem 1. *Let R be a semiprime ring and let ρ be a right ideal of R. Suppose that $f : \rho \to U$ is a commuting additive mapping, i.e., $[f(x), x] = 0$ for all $x \in \rho$. Then there exist $\lambda \in C$ and an idempotent $e \in C$ such that*

$$e[\rho, \rho]\rho = 0 \quad \textit{and} \quad [f(x) - \lambda x, (1 - e)\rho] = 0 \quad \textit{for all} \quad x \in \rho.$$

To prove Theorem 1 we need a result about biderivation. Let ρ be a right ideal of R. An additive mapping $d : \rho \to U$ is said to be a derivation of ρ into U if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \rho$. A biadditive mapping $B : \rho \times \rho \to U$ is called a biderivation if for every $x \in \rho$ the mapping $y \mapsto B(x, y)$ is a derivation of ρ into U, and for every $y \in \rho$ the mapping $x \mapsto B(x, y)$ is a derivation of ρ into U. The following lemma has the same proof as that of [5, Lemma 3.1].

Lemma 1. *Let ρ be a right ideal of R and $B : \rho \times \rho \to U$ be a biderivation. Then*

$$B(x, y)z[u, v] = [x, y]zB(u, v) \quad \textit{for all} \quad x, y, z, u, v \in \rho.$$

Proof of Theorem 1.

Linearizing $[f(x), x] = 0$ gives $[f(x), y] = [x, f(y)]$ for all $x, y \in \rho$. Define the mapping $B : \rho \times \rho \to U$ by $B(x, y) = [f(x), y]$ for all $x, y \in \rho$. Then B is a biderivation. By Lemma 1 we get

$$(1) \qquad B(x,y)z[u,v] = [x,y]zB(u,v)$$

for all $x, y, z, u, v \in \rho$. Since ρ is a right ideal of R, we have

$$(2) \qquad B(x,y)zt[u,v]w = [x,y]ztB(u,v)w$$

for all $x, y, z, u, v, w \in \rho$, all $t \in R$. By Beidar's result [1, Theorem 2] R and U satisfy the same generalized polynomial identities with coefficients in U. Therefore (2) holds for all $t \in U$. Note that U is a semiprime ring and the right Utumi quotient ring of U coincides with itself. By [6, Theorem 3.1] there exist idempotents $\epsilon_1, \epsilon_2, \epsilon_3 \in C$ and an invertible element $\beta \in C$ such that $\epsilon_i\epsilon_j = 0$ for $i \neq j, \epsilon_1 + \epsilon_2 + \epsilon_3 = 1$, and

$$(3) \qquad \epsilon_1 B(x,y)z = \epsilon_1\beta[x,y]z, \quad \epsilon_2[x,y]z = 0, \quad \epsilon_3 B(x,y)z = 0$$

for all $x, y, z \in \rho$. Since $B(x, y) = [f(x), y]$, by (3) we have $[\epsilon_1 f(x) - \epsilon_1\beta x, \rho]\rho = 0$ for all $x \in \rho$. Let $g : \rho \to U$ be defined by $g(x) = \epsilon_1 f(x) - \epsilon_1\beta x$ for all $x \in \rho$. Note that g is also a commuting additive mapping. Set $D : \rho \times \rho \to U$ to be defined by

$$D(x,y) = [g(x), y] = [x, g(y)] \quad \text{for all} \quad x, y \in \rho.$$

Then D is also a biderivation satisfying $D(x, y)\rho = 0$ for all $x, y \in \rho$. Applying Lemma 1 again we get $[\rho, \rho]\rho D(u, v) = 0$ for all $u, v \in \rho$. That is,

$$(4) \qquad [\rho,\rho]\rho[\epsilon_1 f(x) - \epsilon_1\beta x, \rho] = 0 \quad \text{for all} \quad x \in \rho.$$

On the other hand, applying Lemma 1 to $\epsilon_3 B(x, y)$ and taking into account $\epsilon_3 B(x, y)z = 0$ in (3) we get

$$(5) \qquad [\rho,\rho]\rho[\epsilon_3 f(x), \rho] = 0 \quad \text{for all} \quad x \in \rho.$$

It follows from (3) that $\epsilon_2[\rho,\rho]\rho = 0$. In particular, we have

$$(6) \qquad [\rho,\rho]\rho[\epsilon_2 f(x),\rho] = 0 \quad \text{for all} \quad x \in \rho.$$

Now combining (4), (5) and (6) and using the fact that $\epsilon_1+\epsilon_2+\epsilon_3 = 1$ we get that $[\rho,\rho]\rho[f(x) - \epsilon_1\beta x,\rho] = 0$ for all $x \in \rho$. By Beidar's theorem [1] again, $(U[\rho,\rho]\rho U)[f(x)-\epsilon_1\beta x,\rho] = 0$ for all $x \in \rho$. Since $U[\rho,\rho]\rho U$ is an ideal of the semiprime ring U, there exists an idempotent $e \in C$ such that $(1-e)x = x$ for all $x \in U[\rho,\rho]\rho U$ and $(1-e)y = 0$ for all $y \in \ell_U(U[\rho,\rho]\rho U) = r_U(U[\rho,\rho]\rho U)$. That is, $e[\rho,\rho]\rho = 0$ and $[f(x) - \lambda x, (1-e)\rho] = 0$ for all $x \in \rho$ where $\lambda = \epsilon_1\beta \in C$, since $[f(x) - \epsilon_1\beta x,\rho] \subseteq r_U(U[\rho,\rho]\rho U)$ for all $x \in \rho$. This finishes the proof of Theorem 1.

For the prime case we have a generalization of Brešar's result [4, Theorem 3.2].

Theorem 2. *Let R be a prime ring and let ρ be a right ideal of R. Suppose that $f : \rho \to U$ is a commuting additive mapping. Then either $[\rho,\rho]\rho = 0$ or there exist $\lambda \in C$ and an additive mapping $\zeta : \rho \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in \rho$.*

Proof. By Theorem 1, there exist $\lambda \in C$ and an idempotent $e \in C$ such that

$$e[\rho,\rho]\rho = 0 \quad \text{and} \quad [f(x) - \lambda x, (1-e)\rho] = 0 \quad \text{for all} \quad x \in \rho.$$

It is well-known that C is a field since R is a prime ring. Therefore the only idempotents in C are 0 and 1. If $e = 1$, then $[\rho,\rho]\rho = 0$ as desired. If $e = 0$, then $[f(x)-\lambda x,\rho] = 0$ for all $x \in \rho$. By Beidar's theorem, $[f(x)-\lambda x,\rho U] = 0$ for all $x \in \rho$. We may assume that $\rho \neq 0$. Now U itself is a prime ring by the primeness of R. Therefore ρU is a nonzero right ideal of U and hence $f(x) - \lambda x \in C$ for all $x \in \rho$. Let $\zeta : \rho \to C$ be defined by $\zeta(x) = f(x) - \lambda x$ for all $x \in \rho$. Then ζ is clearly an additive mapping and $f(x) = \lambda x + \zeta(x)$ for all $x \in \rho$. This completes the proof.

We remark that in Theorem 2 there indeed exists a commuting additive mapping $f : \rho \to U$ such that $[\rho, \rho]\rho = 0$ but f does not take the form $\lambda x + \zeta(x)$.

Example. Let $R = M_n(F)$ be the ring of all $n \times n$ matrices over a field F, where $n \geq 4$. As usual, denote by e_{ij} the matrix units, $1 \leq i, j \leq n$. Let $\rho = e_{11}R$ and let $f : \rho \to R$ be the linear mapping over F defined by $f(e_{11}) = e_{11}$, $f(e_{12}) = e_{12} + e_{34}$, $f(e_{13}) = e_{13} - e_{24}$ and $f(e_{1j}) = e_{1j}$ if $j \geq 4$. Then it is easy to check that $[f(x), x] = 0$ for all $x \in \rho$. Also, f cannot take the form $\lambda x + \zeta(x)$. Of course, in this example $[\rho, \rho]\rho = 0$.

Remark. In a recent paper [7], Brešar obtained the same result by assuming $f : \rho \to R$ [7, Theorem 5.2 (ii)]. He also gave a characterization of a prime ring with a nonzero right ideal ρ satisfying $[\rho, \rho]\rho = 0$. In [7, Lemma 5.1], it is shown that $[\rho, \rho]\rho = 0$ if and only if RC is a strongly primitive ring with minimal right ideal ρC and with associated division ring C.

The next result is a characterization of a commuting additive mapping $f : \rho \to U$ with $\ell_R(\rho) = 0$. This is a generalization of Brešar's result [6, Corollary 4.2].

Theorem 3. *Let R be a semiprime ring and ρ a right ideal of R such that $\ell_R(\rho) = 0$. Suppose that $f : \rho \to U$ is a commuting additive mapping. Then there exist $\lambda \in C$ and an additive mapping $\zeta : \rho \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in \rho$.*

To prove Theorem 3 we need the following easy lemma.

Lemma 2. *Let R be semiprime ring and ρ a right ideal of R such that $\ell_R(\rho) = 0$. Let $a \in U$ be such that $[a, \rho] = 0$. Then $a \in C$.*

Proof. Let $x \in \rho$ and $r \in R$. By assumption, $[a, xr] = 0$ since $xr \in \rho$. Thus $x[a, r] = 0$. That is, $\rho[a, R] = 0$. By the definition of U there exists a dense right ideal I of R such that $aI \subseteq R$. Then $[a, I]I \subseteq R$ and $([a, I]I\rho)^2 = 0$. Therefore by the semiprimeness of R we have $[a, I]I\rho = 0$

and hence $[a,I]I = 0$ since $[a,I]I \subseteq \ell_R(\rho) = 0$. So $[a,I] = 0$. Let $y \in U$. Take a dense right ideal J of R such that $yJ \subseteq I$ and $J \subseteq I$. Then $[a,yJ] = 0$ and $[a,J] = 0$. Therefore $[a,y]J = 0$ and hence $[a,y] = 0$. That is, $[a,U] = 0$, i.e., $a \in C$ as desired.

Proof of Theorem 3.

By Theorem 1 there exist $\lambda \in C$ and an idempotent $e \in C$ such that

$$e[\rho,\rho]\rho = 0 \ \text{ and } \ [f(x) - \lambda x, (1-e)\rho] = 0 \ \text{ for all } \ x \in \rho.$$

Since $\ell_R(\rho) = 0$, by Lemma 2 we have $(1-e)(f(x)-\lambda x) \in C$ for all $x \in \rho$. Also, by $e \in C$ there exists an essential ideal I of R such that $Ie \subseteq R$. Then $Ie[\rho,\rho] \subseteq \ell_R(\rho) = 0$. Therefore $Ie[\rho,\rho] = 0$, which implies $e[\rho,\rho] = 0$. By Lemma 2 again, $[e\rho, U] = 0$ and so $[\rho, eU] = 0$ which implies $eU \subseteq C$. Now, $f(x) - \lambda x = (1-e)(f(x)-\lambda x) + e(f(x)-\lambda x) \in C$ for all $x \in \rho$. Set $\zeta : \rho \to C$ to be defined by $\zeta(x) = f(x) - \lambda x$ for all $x \in \rho$. Then ζ is an additive mapping such that $f(x) = \lambda x + \zeta(x)$ for all $x \in \rho$. This finishes the proof of Theorem 3.

Finally we handle the Lie ideal case. Let A be an additive subgroup of R. Recall that an additive mapping f of A into U is called centralizing if $[f(x),x] \in C$ for all $x \in A$.

Theorem 4. *Let R be a prime ring and let L be a noncentral Lie ideal of R. If $f : L \to U$ is a centralizing mapping, then there exist $\lambda \in C$ and an additive mapping $\zeta : L \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in L$ except when* $\text{char}R = 2$ *and* $\dim_C RC = 4$.

Proof. Suppose that $\text{char}R \neq 2$ or $\dim_C RC \neq 4$. Set $I = R[L,L]R$ and $J = U[L,L]U$. Then by [10, Theorem 4] we have $[L,L] \neq 0$. So I and J are nonzero. The key step to the proof is implicit in the proof of [5, Lemma 6.3]. Let $a, x \in L$. Note that $[f(u),v] + [f(v),u] \in C$ for all $u, v \in L$. Then

$$\begin{aligned}
C \ni & [f([x,a]),[x,a]] \\
&= [[f([x,a]),x],a]+[x,[f([x,a]),a]] \\
&= [[[x,a],f(x)],a]+[x,[[x,a],f(a)]] \\
&= [[x,[a,f(x)]],a]+[x,[[x,a],f(a)]] \\
&= [[x,[f(a),x]],a]+[x,[[x,a],f(a)]] \\
&= [[x,a],[f(a),x]]+[x,[[f(a),x],a]]+[x,[[x,a],f(a)]] \\
&= [[x,a],[f(a),x]]+[x,[[f(a),x],a]]+[x,[[x,f(a)],a]] \\
&= [[x,a],[f(a),x]].
\end{aligned}$$

Since it is well-known that $[I,R] \subseteq L$, by the above we have

$$(7) \qquad [[a,y],[f(a),y]] \in C \quad \text{for all} \quad y \in [I,R].$$

By Beidar's theorem [1] again we obtain that (7) holds for all $y \in [J,U]$. However, $[J,U]$ is a noncentral Lie ideal of the prime ring U. Applying [12, Theorem 4] to (7) yields that either $a \in C$ of $f(a) - \lambda_a a \in C$ where $\lambda_a \in C$ depends on a.

Since L is not central, we can take a fixed element $u \in L \setminus C$. Let $v \in L$. Suppose that $[u,v] \notin C$. In particular, $v \notin C$. It follows from the fact that $[f(u),v]+[f(v),u] \in C$ that $(\lambda_u - \lambda_v)[u,v] \in C$. Since $[u,v] \notin C$, $\lambda_u = \lambda_v$ follows. In other words, for any $v \in L$ we have either $[u,v] \in C$ or $f(v) - \lambda_u v \in C$. Hence the additive group L is the union of its two additive subgroups $\{v \in L \mid [u,v] \in C]$ and $\{v \in L \mid f(v) - \lambda_u v \in C\}$. This implies that either $[u,L] \subseteq C$ or $f(v) - \lambda_u v \in C$ for all $v \in L$. But the first case implies $u \in C$, a contradiction, we have $f(v) - \lambda_u v \in C$ for all $v \in L$. Set $\lambda = \lambda_u$ and $\zeta : L \to C$ is defined by $\zeta(v) = f(v) - \lambda v$ for all $v \in L$. This finishes the proof of Theorem 4.

We conclude this paper with two applications to Theorem 4. In [16] Mayne proved the following result.

Theorem M. *If R is a prime ring of characteristic not equal to two and T is an automorphism of R which is centralizing and nontrivial on a*

Lie ideal L of R, then L is contained in the center of R.

In the following theorem we extend Theorem M to its full generality.

Theorem 5. *Let R be a prime ring and let L be a noncentral Lie ideal R. If T is a homomorphism of R which is centralizing on L, then either $T(L) \subseteq Z(R)$, the center of R, or T is the identity mapping unless* $\text{char}R = 2$ *and* $\dim_C RC = 4$.

Proof. By assumption, $[T(x), x] \in Z(R)$ for all $x \in L$. It follows from Theorem 4 that there exist $\lambda \in C$ and an additive mapping $\zeta : L \to C$ such that $T(x) = \lambda x + \zeta(x)$ for all $x \in L$. For $u, v \in L$, we have $[u, v] \in L$ and hence $T([u, v]) = \lambda[u, v] + \zeta([u, v])$. On the other hand, $T([u, v]) = [T(u), T(v)] = [\lambda u, \lambda v] = \lambda^2[u, v]$. Therefore, $(\lambda^2 - \lambda)[u, v] \in C$. That is, $(\lambda^2 - \lambda)[L, L] \subseteq C$. Suppose that either $\text{char}R \neq 2$ or $\dim_C RC \neq 4$. Then $[L, L] \not\subseteq C$ by [10, Theorem 4]. Therefore, $\lambda^2 = \lambda$, i.e., $\lambda = 0$ or 1.

If $\lambda = 0$ then $T(L) \subseteq Z(R)$ as desired. So we may assume $\lambda = 1$. That is, $T(u) - u \in Z(R)$ for all $u \in L$. For $u \in L$ and $x \in R$ we have $[u, x] \in L$ and hence

$$[u, T(x)] = [T(u), T(x)] = T([u, x]) = [u, x] + \zeta([u, x]).$$

Therefore $[T(x) - x, u] \in Z(R)$ for all $u \in L$, which implies $T(x) - x \in Z(R)$ for all $x \in R$. For $x, y \in R$ we have

$$T(xy) - xy = T(x)T(y) - xy = (T(x) - x)T(y) + x(T(y) - y) \in Z(R).$$

Expanding $0 = [T(y), T(xy) - xy]$ and using the above we yield $[T(y), x](T(y) - y) = 0$. That is, $[T(y), R]\ (T(y) - y) = 0$ for all $y \in R$. Since the right annihilator of $[T(y), R]$ in R is zero if $T(y) \notin Z(R)$, this implies that for any $y \in R$ either $T(y) = y$ or $T(y) \in Z(R)$. Using the same argument given in the proof of Theorem 4 we have that either T is the identity mapping or $T(R) \subseteq Z(R)$. Now the proof is complete.

The final application is to present an easy argument to prove a known result, obtained by Lee and Lee [13] if char$R \neq 2$ and by Lanski [11] if char$R = 2$.

Theorem 6. *Let R be a prime ring, L a Lie ideal of R and d a nonzero derivation of R. If $[d(u), u] \in Z(R)$, the center of R, for all $u \in L$, then $L \subseteq Z(R)$ unless* char$R = 2$ *and* $\dim_C RC = 4$.

Proof. Suppose that either char$R \neq 2$ or $\dim_C RC \neq 4$. By assumption, $[d(u), u] \in Z(R)$ for all $u \in L$. Suppose on the contrary that $L \not\subseteq Z(R)$. It follows from Theorem 4 that there exist $\lambda \in C$ and an additive mapping $\zeta : L \to C$ such that $d(u) = \lambda u + \zeta(u)$ for all $u \in L$. For $u, v \in L$ we have $d([u, v]) = \lambda[u, v] + \zeta([u, v])$ and on the other hand, $d([u, v]) = [d(u), v] + [u, d(v)] = 2\lambda[u, v]$. Thus $\lambda[u, v] \in Z(R)$, that is, $\lambda[L, L] \subseteq Z(R)$. Note that $[L, L] \not\subseteq Z(R)$. Therefore, $\lambda = 0$ follows. So $d(L) \subseteq Z(R)$, which is a contradiction by [3, Lemma 6] if char$R \neq 2$ and by [9, Lemma 2] if char$R = 2$. This finishes the proof.

References

1. K. I. Beidar, *Rings with generalized identities III*, Vestnik Moskov. Univ. Ser. I Mat. Meh. **33(4)** (1978), 66-73. (Engl. Transl. Moscow Univ. Math. Bull. **33(4)** (1978), 53-58.)

2. H. E. Bell and W. S. Martindale III, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. **30** (1987), 92-101.

3. J. Bergen, I. N. Herstein and J. W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra **71** (1981), 259-267.

4. M. Brešar, *Centralizing mappings and derivations in prime rings*, J. Algebra **156** (1993), 385-394.

5. M. Brešar, W. S. Martindale III and C. R. Miers, *Centralizing maps in prime rings with involution*, J. Algebra **161** (1993), 342-357.

6. M. Brešar, *On certain pairs of functions of semiprime rings*, Proc. Amer. Math. Soc. **120** (1994), 709-713.

7. M. Brešar, *On generalized biderivations and related maps*, J. Algebra **172** (1995) 764-786.

8. C. L. Chuang and T. K. Lee, *On the one-sided version of hypercenter theorem*, Chinese J. Math. **23** (1995), 211-223.

9. W. F. Ke, *On derivations of prime rings of characteristic 2*, Chinese J. Math. **13** (1985), 273-290.

10. C. Lanski and S. Montgomery, *Lie structure of prime rings of characteristic 2*, Pacific J. Math. **42** (1972), 117-136.

11. C. Lanski, *Differential identities, Lie ideals, and Posner's theorems*, Pacific J. math. **134** (1988), 275-297.

12. C. Lanski, *Differential identities of prime rings, Kharchenko's theorem, and applications*, Contemporary Math. **124** (1992), 111-128.

13. P. H. Lee and T. K. Lee, *Lie ideals of prime rings with derivations*, Bull. Inst. Math. Acad. Sinica **11** (1983), 75-80.

14. J. H. Mayne, *Centralizing automorphisms of prime rings*, Canad. Math. Bull., **19** (1976), 113-115.

15. J. H. Mayne, *Centralizing mappings of prime rings*, Canad. Math. Bull. **27** (1984), 122-126.

16. J. H. Mayne, *Centralizing automorphisms of Lie ideals in prime rings*, Canad. Math. Bull, **35** (1992), 510-514.

17. E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc **8** (1957), 1093-1100.

Department of Mathematics, National Taiwan University, Taipei, Taiwan 10764, R.O.C.
E-mail: tklee@math.ntu.edu.tw

Department of Business Mathematics, Soochow University, Taipei, Taiwan 10001, R.O.C.
E-mail: tclee@mbc1.edu.tw