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AN APPLICATION OF RAMSEY THEORY TO THE CROSSING NUMBER OF POSETS

BY

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Abstract. The crossing number of a poset P is denoted by $\chi(P)$. Let P_n be the poset B_n(rank 1 ∪ rnak 2), i.e., P_n is the subposet of Boolean lattice B_n restricted to ranks 1 and 2. In this paper we use Ramsey theory to show that $\chi(P_n) = 3$ for large n.

1. Introduction. The crossing number of a finite poset was defined in [5], and was used to show the existence of a 4-dimensional noncircle order [5], a $(2n+2)$-dimensional non-n-gon order [5], a 5-dimensional nonangle order [7], and a 4-dimensional nonregular n-gon order [4]. It was shown [6] that the crossing number is a comparability graph invariant. Some properties of crossing numbers were derived in [3]. Let us give the definition.

For a function f, we use $G(f)$ to denote the graph of f, i.e., $G(f) = \{(t, f(t)) : t \in \text{domain of } f\}$. Now let $P = (X, \leq)$ be a poset. We use $x \in P$ to denote $x \in X$. For each $x \in P$, we associate a continuous, real-valued function f_x defined on the interval [0,1]. The set of functions $\xi = \{f_x : x \in P\}$ is called a function diagram for P, if

(1) for $x, y \in P$ with $x \neq y$, $G(f_x) \cap G(f_y)$ is a finite set, and $f_x(0) \neq f_y(0)$, $f_x(1) \neq f_y(1)$,

(2) each time the graphs of two different functions in ξ intersect, they cross each other, and

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(3) $x < y$ in $P \Leftrightarrow f_x(t) < f_y(t)$ for all $t \in [0,1]$.

The crossing number for the function diagram $\xi = \{f_x : x \in P\}$ is defined by $\chi(\xi) = \max\{|G(f_x) \cap G(f_y)| : x, y \in P, x \neq y\}$. The crossing number for the poset P is defined by $\chi(P) = \min\{\chi(\xi) : \xi$ is a function diagram for $P\}$. We may assume further that the graphs of any three different functions in a function diagram have an empty intersection. It is trivial from the definition that if Q is a subposet of a poset P, then $\chi(Q) \leq \chi(P)$.

We need some notation. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Let B_n be the poset $(\mathcal{P}[n]), \subset)$, where $\mathcal{P}([n])$ is the power set of $[n]$ and the relation $\subset$ is the set inclusion. The poset B_n is usually called a Boolean lattice. Let rank i denote the set $\{A \subset [n] : |A| = i\}$. If $P = (X, \leq)$ is a poset and $A \subset X$, we use $P(A)$ to denote the poset which is the restriction of P to A. If Q is a subposet of P, we write $Q \subset P$. Two distinct elements x, y in P are denoted $x||y$ if they are incomparable in P.

The following theorem was proved in [3].

Theorem. *Let n be an integer ≥ 4. For $2 \leq i \leq \frac{n}{2}$, let $Q_i = B_n(\text{rank}\,1 \cup \text{rank}\,i \cup \text{rank}\,n-1)$, and $S_i = B_n(\text{rank}\,1 \cup \text{rank}\,2 \cup \cdots \cup \text{rank}\,i \cup \text{rank}\,n-i+1 \cup \text{rank}\,n-i+2 \cup \cdots \cup \text{rank}\,n-1)$. If $Q_i \subset P \subset S_i$, then $\chi(P) = 2i-1$.*

From the above Theorem, we have $\chi(B_n(\text{ rank } 1 \cup \text{ rank } 2)) \leq 3$. We will show that this inequality is sharp. This result has been claimed in [3]. In this paper we give the proof.

2. The theorem and proof. We require the following Ramsey result [2]. For positive integers $k, \ell_1, \ell_2, \cdots$, and ℓ_r with $k \leq \ell_1$, $k \leq \ell_2, \cdots, k \leq \ell_r$, there exists an integer n such that if all the k-element subsets of $[n]$ are divided into r classes, say class 1, class 2, ..., and class r, then there exists $B \subset [n]$ with $|B| = \ell_i$ for some $i, 1 \leq i \leq r$, such that every k-element subset of B is in the class i. The least n which satisfies the above property is called a Ramsey number and is denoted by $R_k(\ell_1, \ell_2, \cdots, \ell_r)$.

Theorem. *If $P_n = B_n(\text{ rank } 1 \cup \text{ rank } 2)$, then $\chi(P_n) = 3$ for large n.*

Proof. Let n be the Ramsey number $R_3(n_1,4,4,6,6)$ where $n_1 = 2R_2(2,2,8,8)$. We will show that $\chi(P_n) \geq 3$. Let $\xi = \{f_x : x \in P_n\}$ be an arbitrary function diagram for P_n. Assume that the graphs of any three distinct functions in ξ have an empty intersection. Let $G(C)$ denote the graph of a function f_C where $C \in P_n$. Without loss of generality, we assume that $f_{\{1\}}(0) > f_{\{2\}}(0) > \cdots > f_{\{n\}}(0)$.

We need to show that $\chi(\xi) \geq 3$. Suppose, on the contrary, that $\chi(\xi) \leq 2$. We can extend each function f_C in ξ to be a function on the interval [0,2] by joining the point $(1, f_C(1))$ to the point $(2, f_C(0))$ with a line segment. This generates a new function diagram ξ' for P_n with $\chi(\xi') \leq 2$ and $f_{\{1\}}(2) > f_{\{2\}}(2) > \cdots > f_{\{n\}}(2)$. So we may assume that in ξ we have $f_{\{1\}}(1) > f_{\{2\}}(1) > \cdots > f_{\{n\}}(1)$.

Let i,j,k be integers in $[n]$ with $i < j < k$. Since $\{i\}||\{j\}$ in P_n, $G(\{i\})$ and $G(\{j\})$ intersect. The graphs $G(\{i\})$ and $G(\{j\})$ are as in Fig.1. Since $\{i,k\} > \{i\}$, $\{i,k\} > \{k\}$ and $\{i,k\}||\{j\}$ in P_n, $G(\{i,k\})$ lies above $G(\{i\})$ and $G(\{k\})$, and intersects $G(\{j\})$. Thus some part of $G(\{j\})$ lies above both $G(\{i\})$ and $G(\{k\})$. Similarly since $\{i,j\} > \{i\}$, $\{i,j\} > \{j\}$ and $\{i,j\}||\{k\}$ in P_n, $G(\{i,j\})$ lies above $G(\{i\})$ and $G(\{j\})$, and intersects $G(\{k\})$. Thus some part of $G(\{k\})$ lies above both $G(\{i\})$ and $G(\{j\})$. Therefore the triple $\{G(\{i\}), G(\{j\}), G(\{k\})\}$ is of one of the five types in Figs 2.a, 2.b, 2.c, 2.d and 2.e.

For $i,j,k \in [n]$ with $i < j < k$, consider the type of $\{G(\{i\}), G(\{j\}), G(\{k\})\}$. Apply Ramsey theory. From the definition of $n = R_3(n_1,4,4,6,6)$, there exists $B \subset [n]$ such that one of the following five conditions holds.

(1) $|B| = n_1$, and for any $i,j,k \in B$ with $i < j < k$, $\{G(\{i\}), G(\{j\}), G(\{k\})\}$ is of the type of Fig. 2.a.

(2) $|B| = 4$, and for any $i,j,k \in B$ with $i < j < k$, $\{G(\{i\}), G(\{j\}), G(\{k\})\}$ is of the type of Fig. 2.b.

(3) $|B| = 4$, and for any $i,j,k \in B$ with $i < j < k$, $\{G(\{i\}), G(\{j\}), G(\{k\})\}$ is of the type of Fig. 2.c.

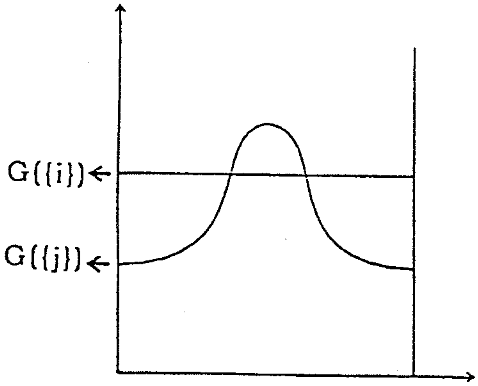

Fig. 1

(4) $|B| = 6$, and for any $i, j, k \in B$ with $i < j < k$, $\{G(\{i\}), G(\{j\}), G(\{k\})\}$ is of the type of Fig. 2.d.

(5) $|B| = 6$, and for any $i, j, k \in B$ with $i < j < k$, $\{G(\{i\}), G(\{j\}), G(\{k\})\}$ is of the type of Fig. 2.e.

For simplicity of notation we let $B = \{1, 2, 3, \cdots, |B|\}$ in each condition. The above five conditions of $\{G(\{i\}) : i \in B\}$ are shown in Figs. 3.a, 3.b, 3.c, 3.d and 3.e, respectively.

We divide these five conditions into three cases: (1) Fig. 3.b or 3.c, (2) Fig. 3.a and (3) Fig. 3.d or 3.e. We will show that each case leads to a contradiction.

Case 1. The condition of Fig. 3.b or 3.c.

By symmetry, it suffices to consider Fig. 3.b. Now $G(\{1,3\})$ is above $G(\{1\}), G(\{3\})$, and has some part below $G(\{2\})$ and some part below $G(\{4\})$, and $G(\{2,4\})$ is above $G(\{2\})$ and $G(\{4\})$, and has some part below $G(\{1\})$ and some part below $G(\{3\})$. We can easily see that $|G(\{1,3\}) \cap G(\{2,4\})| \geq 3$, a contradiction.

Case 2. The condition of Fig. 3.a.

We give the following notations. Suppose $1 \leq i < j < k \leq n_1$. Let $C_j(i,k)$ and $C'_j(i,k)$ be the left part and the right part, respectively, of

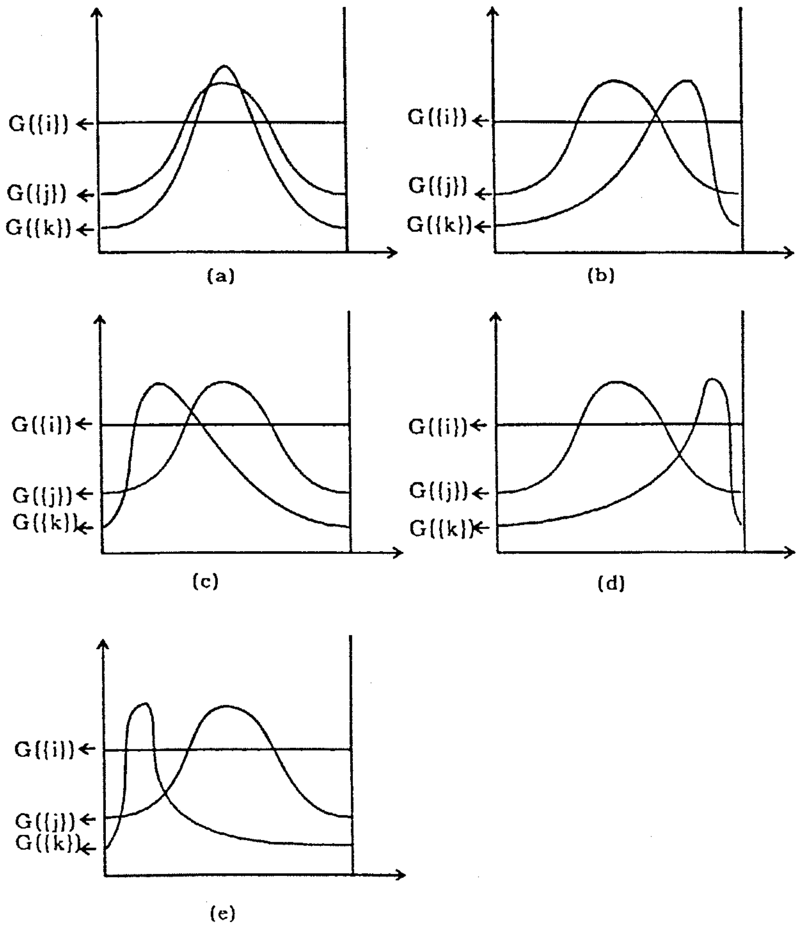

Fig. 2

$G(\{j\})$ which lie above both $G(\{i\})$ and $G(\{k\})$. Let $C_i(j)$ and $C_i'(j)$ be the left part and the right part, respectively, of $G(\{i\})$ which lie above $G(\{j\})$. Let $C_k(j,j)$ be the part of $G(\{k\})$ which lies above $G(\{j\})$. We give an illustration of these in Fig. 4.

Let i, j, k, ℓ be integers with $1 \leq i < j < k < \ell \leq n_1$. We consider the graphs $G(\{i,k\})$ and $G(\{j,\ell\})$. Since $G(\{i,k\})$ is above $G(\{i\})$, $G(\{k\})$, and has some part below $G(\{j\})$, we see that $G(\{i,k\})$ has some part below $C_j(i,k)$ or $C_j'(i,k)$. Similarly $G(\{j,\ell\})$ has some part below $C_k(j,\ell)$ or $C_k'(j,\ell)$. We thus give the following terminologies.

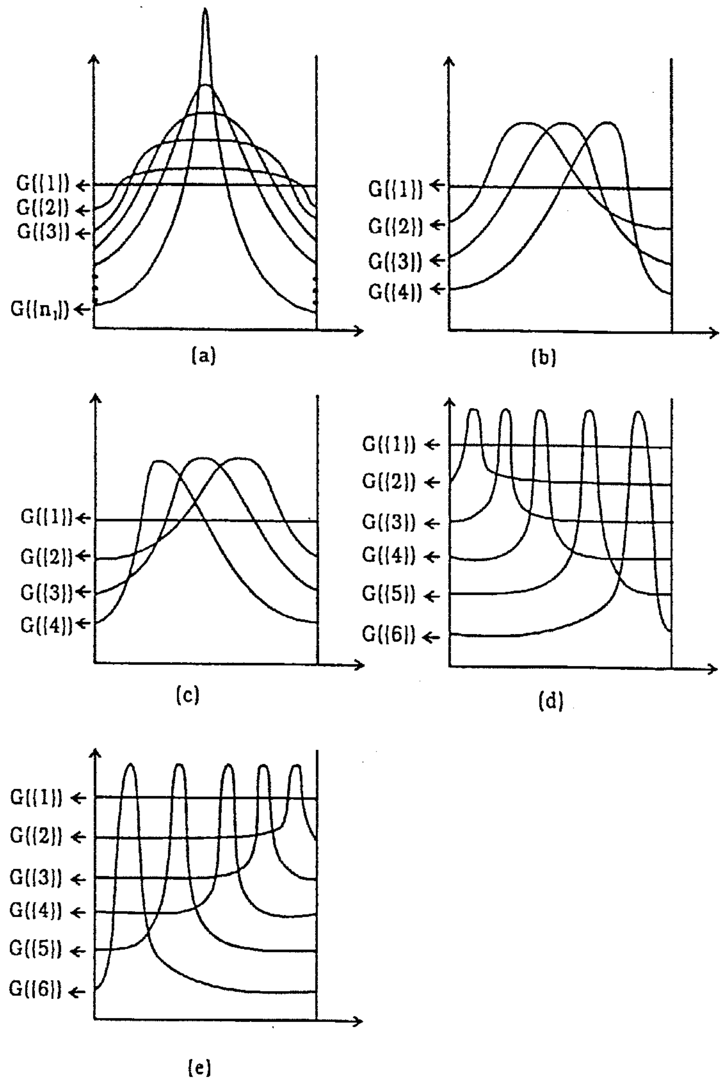

Fig. 3

(1) If $G(\{i,k\})$ has some part below $C_j(i,k)$, and $G(\{j,\ell\})$ has some part below $G_k(j,\ell)$, then we say that the pair $\{G(\{i,k\}),G(\{j,\ell\})\}$ is of type I.

(2) If $G(\{i,k\})$ has some part below $C'_j(i,k)$, and $G(\{j,\ell\})$ has some part below $G'_k(j,\ell)$, then we say that the pair $\{G(\{i,k\}),G(\{j,\ell\})\}$ is of type II.

(3) If $G(\{i,k\})$ has some part below $C_j(i,k)$, and $G(\{j,\ell\})$ has some part

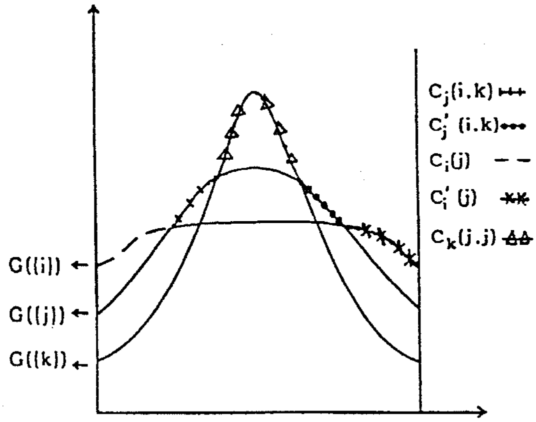

Fig. 4

below $G'_k(j,\ell)$, then we say that the pair $\{G(\{i,k\}), G(\{j,\ell\})\}$ is of type III.

(4) If $G(\{i,k\})$ has some part below $C'_j(i,k)$, and $G(\{j,\ell\})$ has some part below $G_k(j,\ell)$, then we say that the pair $\{G(\{i,k\}), G(\{j,\ell\})\}$ is of type IV.

We also give some notation. Let $G(A)$ be the graph of f_A for some $A \in P_n$, and let $C = C_i(j), C'_i(j), C_j(i,k)$ or $C_\ell(k,k)$. We use $\frac{G(A)}{C}$ to denote that $G(A)$ is above C, and $\frac{C}{G(A)}$ to denote that some part of $G(A)$ is below C.

We have two Remarks.

Remark 1. Let $1 \le i < j < k < \ell \le n_1$. If $\{G(\{i,k\}), G(\{j,\ell\})\}$ is of type I or type II, then $|G(\{i,k\}) \cap G(\{j,\ell\})| \ge 3$.

Check. Due to the symmetry, we only need to consider the type I case, i.e., $G(\{i,k\})$ has some part below $C_j(\{i,k\})$, and $G(\{j,\ell\})$ has some part below $C_k(\{j,\ell\})$. Since $\{i,k\} > \{i\}, \{i,k\} > \{k\}, \{i,k\} \| \{\ell\}$ in P_n, we see that $G(\{i,k\})$ is above $C_i(j), C_k(j,\ell)$ and $C'_i(j)$, and has some part below $C_\ell(k,k)$. Since $\{j,\ell\} > \{j\}, \{j,\ell\} > \{\ell\}$ in P_n, $G(\{j,\ell\})$ is above $C_j(i,k)$ and $C_\ell(k,k)$.

Thus we have

X	$\square$	X	$\square$	X
$C_i(j)$	$C_j(i,k)$	$C_k(j,\ell)$	$C_\ell(k,k)$	$C'_i(j)$
	X	$\square$	X	

where $X = G(\{i,k\}), \square = G(\{j,\ell\})$.

Furthermore since $\{j,\ell\} > \{j\}$, $\{j,\ell\} \| \{i\}$ in P_n, $G(\{j,\ell\})$ has some part below $C_i(j)$ or $C_i'(j)$. Then we can see that $|G(\{i,k\}) \cap G(\{j,\ell\})| \geq 3$. This completes the check of Remark 1.

Remark 2. Let $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \leq n_1$. If the pair $\{G(\{i_1,i_3\}), G(\{i_2,i_4\})\}$ and the pair $\{G(\{i_2,i_5\}), G(\{i_3,i_6\})\}$ are both of type III (or both of type IV), then $|G(\{i_1,i_3\}) \cap G(\{i_2,i_5\})| \geq 3$.

Check. Due to the symmetry, we only need to consider the type III case. The fact that the pair $G(\{i_1,i_3\}), G(\{i_2,i_4\})\}$ is of type III implies that $G(\{i_1,i_3\})$ has some part below $C_{i_2}(i_1,i_3)$. And the fact that the pair $\{G(\{i_2,i_5\}), G(\{i_3,i_6\})\}$ is of type III implies that $G(\{i_2,i_5\})$ has some part below $C_{i_3}(i_2,i_5)$. Thus $\{G(\{i_1,i_3\}), G(\{i_2,i_5\})\}$ is of type I. Hence, by Remark 1, $|G(\{i_1,i_3\}) \cap G(\{i_2,i_5\})| \geq 3$. This completes the check of Remark 2.

Recall that $n_1 = 2R_2(2,2,8,8)$ (in the beginning of the proof). Let $\ell_1 = R_2(2,2,8,8)$. Thus $n_1 = 2\ell_1$. For every pair i,j with $1 \leq i < j \leq \ell_1$, we note $1 \leq i < j < i+\ell_1 < j+\ell_1 \leq n_1$, and consider the type of $\{G(\{i,i+\ell_1\}), G(\{j,j+\ell_1\})\}$. Apply Ramsey theory. By the definition of ℓ_1, the following cases may happen.

Case 2.a. For some $1 \leq i < j \leq \ell_1$, $\{G(\{i,i+\ell_1\}), G(\{j,j+\ell_1\})\}$ is of type I or type II.

Case 2.b. There exists $D \subset [\ell_1], |D| = 8$ such that for every $i,j \in D, i < j$, we have $\{G(\{i,i+\ell_1\}), G(\{j,j+\ell_1\})\}$ all of type III or all of type IV.

We consider these cases.

Case 2.a. By Remark 1, we have $|G(\{i,i+\ell_1\}) \cap G(\{j,j+\ell_1\})| \geq 3$, a contradiction to $\chi(\xi) \leq 2$.

Case 2.b. By symmetry, we only need to consider the type III case. For simplicity of notation, we let $D = [8]$. Thus for $1 \leq i < j \leq 8$, $\{G(\{i,i+\ell_1\}), G(\{j,j+\ell_1\})\}$ is of type III. Now for every i,j, with $1 \leq i < j \leq 4$, we note $1 \leq i < j < i+4 < j+4 \leq n_1$, and consider the type of $G(\{i,i+4\}), G(\{j,j+4\})\}$. We distinguish two subcases.

Case 2.b.1. For some $1 \leq i < j \leq 4$, $\{G(\{i,i+4\}), G(\{j,j+4\})\}$ is of type I, type II or type III.

Case 2.b.2. For every $1 \leq i < j \leq 4$, $\{G(\{i, i+4\}), G(\{j, j+4\})\}$ is of type IV.

We consider these cases.

Case 2.b.1. For type I case and type II case, we have, by Remark 1, $|G(\{i, i+4\}) \cap G(\{j, j+4\})| \geq 3$, a contradiction. Consider type III case, i.e., $\{G(\{i, i+4\}), G(\{j, j+4\})\}$ is of type III. Since $1 < j < i+4 < 8$, $\{G(\{j, j+\ell_1\}), G(\{i+4, i+4+\ell_1\})\}$ is also of type III. Now $1 \leq i < j < i+4 < j+4 < j+\ell_1 < i+4+\ell_1 \leq n_1$, we have, by Remark 2, $|\{G(\{i, i+4\}) \cap G(\{j, j+\ell_1\})| \geq 3$, a contradiction.

Case 2.b.2. We consider the type of $\{G(\{1,3\}), G(\{2,4\})\}$. If it is of type I or II, then by Remark 1, $\chi(\xi) \geq 3$, a contradiction. If it is of type III, then, combined with the fact that $\{G(\{2, 2+\ell_1\}), G(\{3, 3+\ell_1\})\}$ is of type III, this implies, by Remark 2, that $\chi(\xi) \geq 3$, since $1 < 2 < 3 < 4 < 2+\ell_1 < 3+\ell_1 \leq n_1$, a contradiction. If it is of type IV, then, combined with the fact that $\{G(\{2, 2+4\}), G(\{3, 3+4\})\}$ is of type IV, this implies, again by Remark 2, that $\chi(\xi) \geq 3$, a contradiction. This completes Case 2.

Case 3. Condition of Fig. 3.d. or 3.e.

By symmetry, it suffices to consider Fig. 3.d.

As shown in Fig. 5, suppose that the x-coordinates of the points where $G(\{2\})$ intersects $G(\{3\})$ are t_1 and t_2 where $t_1 < t_2$, and that those of the points where $G(\{2\})$ intersects $G(\{5\})$ are t_3 and t_4 where $t_3 < t_4$.

We see that $G(\{3,5\})$ is above $G(\{3\})$ and $G(\{5\})$, and has some part below $G(\{2\})$. Thus $G(\{3,5\})$ has some part below $G(\{2\})$ between the lines $x = a$ and $x = b$, where a, b satisfy one of the following conditions: (1) $a = 0, b = t_1$ (2) $a = t_2, b = t_3$ (3) $a = t_4, b = 1$. We distinguish these conditions.

First we put conditions (1) and (3) together; hence, $G(\{3,5\})$ has some part below $G(\{2\})$ either between the lines $x = 0$ and $x = t_1$ or between $x = t_4$ and $x = 1$. Since $G(\{2\})$ is below $G(\{2,4\})$, we have that

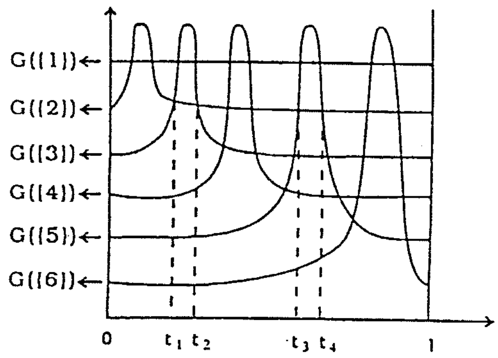

Fig. 5

(i) $G(\{3,5\})$ has some part below $G(\{2,4\})$ either between $x = 0$ and $x = t_1$ or between $x = t_4$ and $x = 1$.

Furthermore since $G(\{2,4\})$ is above $G(\{2\})$ and has some part below $G(\{3\})$, we see that $G(\{2,4\})$ has some part below $G(\{3\})$ between the lines $x = t_1$ and $x = t_2$. For a similar reason, $G(\{2,4\})$ has some part below $G(\{5\})$ between the lines $x = t_3$ and $x = t_4$. Then since $G(\{3\})$ and $G(\{5\})$ are below $G(\{3,5\})$, we have that

(ii) $G(\{2,4\})$ has some part below $G(\{3,5\})$ between $x = t_1$ and $x = t_2$, and also some part below $G(\{3,5\})$ between $x = t_3$ and $x = t_4$.

We see that $G(\{3,5\})$ has some part below $G(\{4\})$ between the lines $x = t_2$ and $x = t_3$. Since $G(\{4\})$ is below $G(\{2,4\})$ we have that

(iii) $G(\{3,5\})$ has some part below $G(\{2,4\})$ between $x = t_2$ and $x = t_3$.

From (i) (ii) (iii), we have $|G(\{3,5\}) \cap G(\{2,4\})| \geq 3$, a contradiction.

Next we consider condition (2), i.e., $G(\{3,5\})$ has some part below $G(\{2\})$ between the lines $x = t_2$ and $x = t_3$. Then we have that

(i) $G(\{3,5\})$ has some part below $G(\{2,6\})$ between $x = t_2$ and $x = t_3$.

We can see that $G(\{2,6\})$ has some part below $G(\{3\})$ between the lines $x = t_1$ and $x = t_2$, and some part below $G(\{5\})$ between the lines $x = t_3$ and $x = t_4$. Thus we have that

(ii) $G(\{2,6\})$ has some part below $G(\{3,5\})$ between $x = t_1$ and $x = t_2$ and also some part below $G(\{3,5\})$ between $x = t_3$ and $x = t_4$.

We see that $G(\{3,5\})$ has some part below $G(\{6\})$ between the lines $x = t_4$ and $x = 1$. Thus we have that

(iii) $G(\{3,5\})$ has some part below $G(\{2,6\})$ between $x = t_4$ and $x = 1$.

From (i) (ii) (iii), we have $|G(\{3,5\}) \cap G(\{2,6\})| \geq 3$, a contradiction. This completes case 3, and hence the proof of the theorem.

Remark. As pointed out by an anonymous referee, the techniques used in [1] to prove that P_5 is not a circle order, are of relevance for the area of research studied in this paper.

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