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A WAVELET-LIKE UNCONDITIONAL BASIS

BY

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Abstract. We show that the orthogonal wavelet basis suitably normalized under Wiener condition is an unconditional basis in $L^p(\mathbb{R})$. This basis is a bounded Besselian basis in the space for $1 < p \leq 2$, and bounded Hilbertian basis in for $2 \leq p < \infty$. Under the Wiener condition, we show that the pre-wavelet basis with stable shift is a frame, we also construct the dual frame. We show that this frame provides an unconditional basis in $L^p(\mathbb{R})(1 < p < \infty)$ using Calderon-Zygmund operator.

1. Introduction. The purpose of this paper is to construct unconditional bases for $L^p(\mathbb{R})$ $(1 < p < \infty)$ using 'nice' wavelet bases of $L^2(\mathbb{R})$. The research presented here is motivated by Meyer's result in [12].

The structure of a Banach space with a basis is simple since it is isometric to a sequence space. However, in general, not a lot more can be asserted. A very useful class of bases with more interesting properties is the class of unconditional bases [11], [16].

Definition 1.1. A basis (f_n) is unconditional in a Banach space if any convergent series $\sum_n a_n f_n$ converges unconditionally; that is, the series $\sum_n a_{\pi(n)} f_{\pi(n)}$ converges to the same limit for all permutations π of $\mathbb{N}$ (or $\mathbb{Z}$).

For instance, the natural bases of $\ell^p (p \geq 1)$ are unconditional bases. It

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is well known [16] that the space $L^1([0,1)]$ has no unconditional basis. Any orthonormal base in a separable Hilbert spaces is unconditional. From [16, Theorem 16.1] we know that unconditional bases can be characterized as follows.

Theorem 1.2. *Let* (f_n) *be a basis in a Banach space. Then the following properties are equivalent:*

(i) (f_n) *is unconditional.*

(ii) *There exists a positive constant* C *such that, for all* n*, for all* $\epsilon_i = \pm 1$ *and for all scalars* $a_1, \ldots, a_n$*,*

$$\left\| \sum_{i=1}^{n} \epsilon_i a_i f_i \right\| \leq C \left\| \sum_{i=1}^{n} a_i f_i \right\|.$$

This equivalence will be used to prove the nature of unconditional bases of $L^p(\mathbb{R})$ in the next two sections.

Recently, Meyer [12] showed that if ψ and its derivative ψ' satisfy the decay condition

$$|\psi(x)|, |\psi'(x)| \leq C(1+|x|)^{-1-\epsilon},$$

and if the functions $\psi_{jk}(\cdot) = 2^{j/2}\psi(2^j \cdot +k)$, $j,k \in \mathbb{Z}$, form an orthonormal basis for $L^2(\mathbb{R})$, then ψ_{jk} form an unconditional basis of $L^p(\mathbb{R})$ for $1 < p < \infty$. This theorem requires the Calderon-Zygmund operator and the Marcinkiewicz interpolation theorem. In order to understand Calderon-Zygmund operators we need to review a classic theorem of harmonic analysis, the Calderon-Zygmund decomposition [12, vol, II] or [8] and the space $L^p_{\text{weak}}(\mathbb{R})$.

If f is a measurable function on $\mathbb{R}$ and $0 < p < \infty$, we define

$$[f]_p = (\sup_{\alpha>0} \alpha^p \omega_f(\alpha))^{1/p},$$

where $\omega_f(\alpha) = m\{x : |f(x)| > \alpha\}$ (the distribution function of f) and we define $L^p_{\text{weak}}(\mathbb{R})$ to be the set of all f such that $[f]_p < \infty$. $[\cdot]_p$ is not a norm:

it is easy to verify that $[cf]_f = |c|[f]_p$, but the triangle inequality fails. However, $L^p_{\text{weak}}(\mathbb{R})$ is a topological vector space. The relationship between $L^P(\mathbb{R})$ and $L^p_{\text{weak}}(\mathbb{R})$ is the following:

$$L^p(\mathbb{R}) \subset L^p_{\text{weak}}(\mathbb{R}) \quad \textit{and} \quad [f]_p \leq \|f\|_p.$$

The following theorem of Calderon-Zygmund decomposition is taken from [8, p. 178].

Theorem 1.3. *Let $f \in L^1(\mathbb{R})$ and let $\alpha > 0$. Then there exists a sequence of pairwise disjoint intervals Q_k such that*

$$\alpha \leq m(Q_k)^{-1} \int_{Q_k} |f| \leq 2\alpha$$

for all $k \in \mathbb{N}$ and $|f(x)| \leq \alpha$ as $x \in E = \mathbb{R} \setminus \cup_{k \in \mathbb{N}} Q_k$. Set

$$g(x) = f(x)\chi_E + \sum_j \left(M(Q_k)^{-1} \int_{Q_k} |f| \right) \chi_{Q_k}(x),$$

$$h_j(x) = \left[f(x) - m(Q_k)^{-1} \int_{Q_k} |f| \right] \chi_{Q_k}, \quad h = \sum_j h_j.$$

Then $f = g + h$. Moreover

(i) $|g| \leq \alpha$

(ii) $\int_{Q_k} h_j = 0$ *and* $h_j(x) = 0$ *as* $x \notin Q_k$.

(iii) $\|g\|_1 + \sum_j \|h\|_j \leq 3\|f\|_1$.

(iv) $m(\cup_{k \in \mathbb{N}} Q_k) \leq \alpha^{-1} \|f\|_1$.

Next we define a wide class of Calderon-Zygmund operators on $\mathbb{R}$ (see [8, p. 221-223]). Assume that $\mathcal{D}(\mathbb{R})$ denotes the class of all C^∞-functions with compact supports in $\mathbb{R}$.

Definition 1.4. A bounded linear operator S in $L^2(\mathbb{R})$ is call a Calderon-Zygmund operator if there exists a function $K(x, y)$ defined for $x, y \in L^2(\mathbb{R})$, $x \neq y$ such that for all $f \in \mathcal{D}(\mathbb{R})$ and $x \notin \operatorname{supp} f$,

$$(Sf)(x) = \int_{\mathbb{R}} K(x, y) f(y) dy$$

for which the integral kernel satisfies

$$|K(x,y)| \le \frac{C}{|x-y|}, \tag{1.5}$$

$$|K(x,y)-K(x,y')| \le C\frac{|y-y'|^\alpha}{|x-y|^{1+\alpha}}, \quad |y-y'| \le \frac{1}{2}|x-y|, \tag{1.6}$$

and

$$|K(x,y)-K(x',y)| \le C\frac{|x-x'|^\alpha}{|x-y|^{1+\alpha}}, \quad |x-x'| \le \frac{1}{2}|x-y|. \tag{1.7}$$

Here, $\alpha > 0$ and $0 < C < \infty$ are constants.

For now on we keep α fixed and set

$$\|S\|_\alpha = \|S\| + \inf C.$$

Remark. If K is continuously differentiable and

$$\left|\frac{\partial}{\partial x}K(x,y)\right| + \left|\frac{\partial}{\partial y}K(x,y)\right| \le \frac{C}{|x-y|^2}, \tag{1.8}$$

then (1.6) and (1.7) are satisfied for $\alpha = 1$.

The following theorem is an application of Theorem 1.3 (see [8, p. 224]).

Theorem 1.9. *A Calderon-Zygmund operator S is a bounded operator from $L^1(\mathbb{R})$ to $L^1_{\text{weak}}(\mathbb{R})$ and $Sf_1 \le C_1\|S\|_\alpha\|f\|_1$.*

Once we know that S maps $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and $L^1(\mathbb{R})$ to $L^1_{\text{weak}}(\mathbb{R})$, we can extend S to othere $L^p(\mathbb{R})$ spaces by the Marcinkiewicz interpolation theorem [5, p. 195].

Theorem 1.10. *Suppose that S is a linear operator from $L^{p_i}(\mathbb{R})$ to $L^{q_i}_{\text{weak}}(\mathbb{R})$; where $1 \le p_i \le q_i < \infty$ for $i = 0, 1$, and $q_0 \ne q_1$. Let $0 < t < 1$ and define*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If $[Sf]_{q_i} \leq C_i\|f\|_{p_i}$ *for* $i = 0, 1$, *then* $\|Sf\|_q \leq B_p\|f\|_p$ *where* B_p *depends only on* p_i, q_i, C_i *and* p.

Combining Theorems 1.9 and 1.10, we have the following main property of Calderon-Zygmund operators (see, [8, p. 224]).

Theorem 1.11. *If* S *is a Calderon-Zygmund operator on* $L^2(\mathbb{R})$, *then* S *extends to a bounded operator from* $L^p(\mathbb{R})$ *to* $L^p(\mathbb{R})$ *for all* p *with* $1 < p < \infty$ *and* $\|Sf\|_p \leq \|S\|_\alpha\|f\|_p$.

Note that if we choose $\epsilon_{jk} = \pm 1$, and define $K(x,y) = \sum_{j,k} \epsilon_{jk}\psi_{jk}(x)\overline{\psi_{jk}(y)}$. Then $K(x,y)$ satisfies the inequalities (1.6) and (1.7). Thus, Theorem 1.11 implies that the operator S defined by

$$S(f) = \sum_{j,k} \epsilon_{jk}(f, \psi_{jk})\psi_{jk}$$

is a bounded operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$. This is an outline of the proof of Meyer's theorem.

This paper is organized as follows. In Section 2, we prove that if ψ satisfies the Wiener condition and $\{\psi_{jk}\}_{j,k}$ forms an orthonormal basis, then the result of Meyer extends without using the Calderon-Zygmund operators. Section 3 is devoted to the connection between pre-wavelets and Meyer's result.

2. Besselian and Hilbertian Bases from orthonormal wavelets. In this section, the Besselian and Hilbertian bases are studied in spaces $L^p(\mathbb{R})$. We prove the classical result that the space $L^p(\mathbb{R})$ is linearly isometric to $L^p([0,1])$, for all $1 \leq p < \infty$. Let ψ be a function that ψ satisfies the Wiener condition [17]; that is, ψ is a continuous function on $\mathbb{R}$ and satisfies the following inequality:

$$\|\psi\|_\omega := \sum_{k\in\mathbb{Z}} \max_{0\leq x\leq 1} |\psi(x+k)| < \infty.$$

Define a sequence of functions by

$$\psi^p_{jk}(\cdot) := 2^{j/p}\psi(2^j \cdot + k) \quad j,k \in \mathbb{Z} \quad \text{and} \quad 1 < p < \infty.$$

It follows that $\|\psi^p_{jk}\|_p = \|\psi\|_p$; in particular, $\{\psi^p_{jk} : j,k \in \mathbb{Z}\}$ is a bounded sequence. Our goal will be to show that $\{\psi^p_{jk}\}_{j,k}$ is a unconditional basis in $L^p(\mathbb{R})$, $1 < p < \infty$. Moreover, in the spaces $L^p(\mathbb{R})$, where $1 < p \leq 2$, ψ^p_{jk} form a bounded and Besselian basis. Consequently, in the spaces $L^p(\mathbb{R})$, where $2 \leq p < \infty$, $\{\psi^p_{jk}\}_{j,k}$ is a bounded and Hilbertian basis. The basis $\{\psi^p_{jk}\}_{j,k}$ is simultaneously Besselian and Hilbertian if and only if $p = 2$. Recall that a basis

$$\{\psi_n : n \in \mathbb{N}(\text{or } \mathbb{Z})\}$$

is called a *Riesz Basis* in Hilbert space if there exist positive constants C_1 and C_2 such that, for all scalars $a_1, \ldots, a_n$,

$$C_1\sqrt{\sum_{i=1}^{n}|a_i|^2} \leq \left\|\sum_{i=1}^{n} a_i\psi_i\right\| \leq C_2\sqrt{\sum_{i=1}^{n}|a_i|^2}.$$

From this motivation, we have the following definition (see [16, p. 338]).

Definition 2.1. A basis (f_n) of a Banach space is said to be
(i) *Besselian* if there exists a positive constant C such that, for all scalars $a_1, \ldots, a_n$, and for all n,

$$C\sqrt{\sum_{i=1}^{n}|a_i|^2} \leq \left\|\sum_{i=1}^{n} a_i f_i\right\|.$$

(ii) *Hilbertian* if there exists a positive constant C such that, for all scalars $a_1, \ldots, a_n$, and for all n,

$$\left\|\sum_{i=1}^{n} a_i f_i\right\| \leq C\sqrt{\sum_{i=1}^{n}|a_i|^2}.$$

It is clear that a Riesz basis for a Hilbert space is equivalent[1] to an orthonormal basis. Thus, it must be both bounded and unconditional. On the other hand, any bounded unconditional basis is equivalent to a Riesz basis in any seprable Hilbert space (see [16, p. 640]). The following theorem gives some characterizations of Besselian and Hilbertian bases.

Theorem 2.2 [16, Theorem 11.1]. *Let (f_n) be a basis of a Banach space E. Then the following statements are equivalent:*

(i) *(f_n) is Besselian (Hilbertian).*

(ii) *There exists a continuous linear mapping u of E into ℓ^2 (ℓ^2 into E) such that*

$$u(f_n) = e_n \quad (u(e_n) = f_n)$$

where (e_n) is the unit vector basis of ℓ^2.

Next we want to show that if ψ satisfies the Wiener condition and $\{\psi_{jk}\}$ form an orthonormal basis for $L^2(\mathbb{R})$, then $\{\psi^p_{jk}\}$ is an unconditional basis of $L^p(\mathbb{R})$, for all $1 < p < \infty$. The idea of the proof presented here comes from a proof of it by Burkholder [2]. It is well known that the Haar basis is an unconditional basis of $L^p(\mathbb{R})$ $(1 < p < \infty)$. This result is due to Paley [14], but Burkholder gave an elegant and elementary proof. The trick of Burkholder's proof is also used to prove our main result, Theorem 2.5 below.

Define a function $v : \mathbb{R}^2 \to \mathbb{R}$ by

$$v(x,y) = \left|\frac{x+y}{2}\right|^p - (p^*-1)^p\left|\frac{x-y}{2}\right|^p \tag{2.3}$$

where $1 < p < \infty$ and $p^* := \max\{p, p/(p-1)\}$. In order to prove this result we construct a function $g : \mathbb{R}^2 \to \mathbb{R}$ with continuous first partial derivatives as follows:

Lemma 2.4 [2]. *There exists a function g with continuous first partial derivatives such that*

(i) *For given x and y in $\mathbb{R}$, $g(\cdot, y)$ and $g(x, \cdot)$ are concave.*

(ii) *$v \le g$ on $\mathbb{R}^2$.*

(iii) *$g(x,y) \le 0$ if $xy = 0$.*

Outline of Proof:

We define

$$A_1 = \{(x,y) : x > 0 \quad \text{and} \quad (1-2/p^*)a < y < x\},$$
$$A_2 = \{(x,y) : x > 0 \quad \text{and} \quad -x < y < (1-2/p^*)x\}.$$

Set $\alpha_p = p[p^*/(p^*-1)]^{1-p}$ and let g be the continuous function on $\mathbb{R}^2$ satisfying

$$g(x,y) = g(y,x) = g(-x,-y),$$

and the following further condition: if $1 < p \leq 2$, then

$$g(x,y) = v(x,y) \qquad \text{if} \quad (x,y) \in A_1$$
$$g(x,y) = \alpha_p x^p \left[1 - \frac{p^*(x-y)}{2x}\right] \text{ if } \quad (x,y) \in A_2.$$

If $p > 2$, then A_1 and A_2 are interchanged.

Theorem 2.5. *Let ψ satisfy the Wiener condition and assume that $\{\psi_{jk}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. Then $\{\psi^p_{jk}\}$ is a bounded unconditional basis of $L^p(\mathbb{R})$ for all $1 < p < \infty$. Moreover, we have, for all $\epsilon_{jk} = \pm 1$ and all scalars a_{jk},*

$$\left\| \sum_{(j,k)\in F} \epsilon_{jk} a_{jk} \psi^p_{jk} \right\|_p \leq (p^*-1) \left\| \sum_{(j,k)\in F} a_{jk} \psi^p_{jk} \right\|_p, \tag{2.6}$$

for any finite set F in $\mathbb{Z} \times \mathbb{Z}$.

Proof. Assume that $n = \sharp F_n$ and use induction on n. Let $Z_n : \mathbb{R} \to \mathbb{R}^2$ be the function given by $Z_n = (X_n, Y_n)$, where

$$X_n = \sum_{(j,k)\in F_n} (\epsilon_{jk}+1) a_{jk} \psi^p_{jk},$$

and

$$Y_n = \sum_{(j,k)\in F_n} (\epsilon_{jk}-1) a_{jk} \psi^p_{jk}.$$

Let the function v be defined as in (2.3). Then we have

$$\int_{\mathbb{R}} v \circ Z_n = \Big\| \sum_{(j,k)\in F_n} \epsilon_{jk} a_{jk} \psi^p_{jk} \Big\|_p^p - (p^* - 1)^p \Big\| \sum_{(j,k)\in F_n} a_{jk} \psi^p_{jk} \Big\|_p^p .$$

Clearly (2.6) is equivalent to

$$\int_{\mathbb{R}} v \circ Z_n \leq 0.$$

By Lemma 2.4, there exists a function g on $\mathbb{R}^2$ such that 2.4(i)-(iii) hold. Moreover, g satisfies

$$\int_{\mathbb{R}} g \circ Z_n < \infty \quad \text{for all } n$$

To establish this, fix n and write

$$\begin{aligned} \int_{\mathbb{R}} \Big| \sum_{(j,k)\in F_n} a_{jk} \psi^p_{jk}(t) \Big|^p dt &\leq \int_0^1 \Big(\sum_{(j,k)\in F_n} |a_{jk}| \sum_{l\in\mathbb{Z}} |\psi^p_{jk}(t+l)| \Big)^p dt \\ &\leq \Big(\sum_{(j,k)\in F_n} |a_{jk}| \sum_{l\in\mathbb{Z}} \max_{0\leq t\leq 1} |\psi^p_{jk}(t+l)| \Big)^p \\ &= \Big(\sum_{(j,k)\in F_n} 2^{j+j/p} |a_{jk}| \Big)^p \|\psi\|^p_\omega < \infty. \end{aligned}$$

Similarly, we can prove that for each n, X_n and Y_n are integrable on $\mathbb{R}$. So is $v \circ X_n$.

Since $X_1 Y_1 = 0$, we know that $\int_{\mathbb{R}} g \circ Z_1 \leq 0$. In fact, the first partial derivatives g_x and g_y exist so, by the concavity condition,

$$g(x+h, y+h) \leq g(x,y) + g_x(x,y)h + g_y(x,y)k \quad \text{if } hk = 0 \tag{2.7}$$

if $hk = 0$. It is now a short step to

$$\int_{\mathbb{R}} g \circ Z_n \leq \int_{\mathbb{R}} g \circ Z_{n-1} \leq \cdots \leq \int_{\mathbb{R}} g \circ Z_1 \leq 0, \tag{2.8}$$

which, since $v \leq g$, gives (2.6). Let $n \geq 2$. By (2.7) and

$$(X_n - Y_n)(Y_n - Y_{n-1}) = 0,$$

we have

$$g \circ Z_n \leq g \circ Z_{n-1} + g_x \circ Z_{n-1}(X_n - X_{n-1}) + g_y \circ Z_{n-1}(Y_n - Y_{n-1}).$$

Consequently, (2.8) will follows if we can show that

$$\begin{aligned}
&\int_{\mathbb{R}} g_x(Z_{n-1}(t))(X_n(t) - X_{n-1}(t))dt \\
(2.9)\qquad &= \int_{\mathbb{R}} g_y(Z_{n-1}(t))(Y_n(t) - Y_{n-1}(t))dt = 0.
\end{aligned}$$

Each of the integrands in (2.9) is equal to the product of $\psi^p_{j_n k_n}$ with a function of ψ^p_{jk}, $(j,k) \in F_{n-1}$. Since the sequence $\{\psi_{jk}\}$ is orthonormal in $L^2(\mathbb{R})$, by induction, (2.9) follows.

Notice that similarly, we can also prove that ψ_{jk} form a unconditional basis of $L^p(\mathbb{R})$, for all $1 < p < \infty$. Next we want to show that $L^p(\mathbb{R})$ is linearly isometric to $L^p([0,1])$. In general cases, it is sufficient, up to an isometry, to work with $L^p([0,1])$ or ℓ^p instead of general L^p-spaces (see [9, chapter 5]).

Theorem 2.10. *The space $L^p(\mathbb{R})$ is linearly isometric to $L^p([0,1])$, for all $1 \le p < \infty$.*

Proof. The set [0,1] and $\mathbb{R}$ can be written as countable unions of disjoint half-open intervals by

$$[0,1] = \cup_{n=1}^{\infty} I_n \quad \text{and} \quad \mathbb{R} = \dot{\cup}_{n=1}^{\infty} J_n,$$

where $I_n = [\frac{1}{n+1}, \frac{1}{n})$, and $J_{2n} = [n-1, n)$ and $J_{2n-1} = [-n-1, -n)$, respectively. If $f \in L^p([a,b])$ and $g(t) = f[a + \frac{t-c}{d-c}(b-a)]$, then $g \in L^p([c,d])$. Use the fact that there exists a linearly isometric mapping Λ_n, $n = 1, 2, \ldots$ from $L^p(J_n)$ onto $L^p(I_n)$. Define a map Λ from $L^p(\mathbb{R})$ onto $L^p([0,1])$ by

$$\Lambda f = \sum_{n=1}^{\infty} (f\chi_{J_n}).$$

We find that Λ is linear and norm-preserving since

$$\|\Lambda f\|_{L^p([0,1])} = \|\sum_n \Lambda_n (f\chi_{J_n})\|_{L^p(I_n)} = \sum_n \|f\chi_{J_n}\|_{L^p(J_n)} = \|f\|_{L^p(\mathbb{R})}$$

for all $f \in L^p(\mathbb{R})$.

The next result is easily proved with the aid of Theorem 1.2.

Theorem 2.11. *Let X and Y be two Banach spaces and let Λ be an isometry from X onto Y. If $\{x_n\}$ is an unconditional basis, then $\{\Lambda x_n\}$ is also an unconditional basis.*

Combining Theorems 2.5, 2.10, and 2.11 leads to the following result.

Theorem 2.12. *The set $\{\Lambda\psi^p_{jk}\}$ is a bounded unconditional basis of $L^p([0,1])$, for all $p \in (1,\infty)$.*

The following is a well known result [16, Proposition 14.1] about bounded unconditional bases in $L^p([0,1])$, for $1 < p < \infty$.

Theorem 2.13.

(i) *If $1 < p < 2$, every bounded unconditional basis is Besselian in $L^p([0,1])$.*

(ii) *If $2 \leq p < \infty$, every bounded unconditional basis is Hilbertian in $L^p([0,1])$.*

(iii) *A bounded unconditional basis is simultaneously Besselian and Hilbertian in $L^p([0,1])$ if and only if $p = 2$.*

The proof of Theorem 2.13 relies on the Khinchin inequality (see, [16, p. 425]). This states that there exist two positive constants A_p and B_p such that for any n and for any scalars $a_1, \ldots, a_n$

$$A_p\left(\sum_{i=1}^{n}|a_i|^2|\right)^{1/2} \leq \left(\int_{[0,1]}\left|\sum_{i=1}^{n} a_i r_i(t)\right|^p dt\right)^{1/p} \leq B_p\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}$$

Here the Rademacher functions r_i on $[0,1]$ are defined by

$$r_i(t) = \text{sign } \sin(2^i\pi t) \quad (i = 1, 2 \ldots).$$

Combining Theorems 2.12 and 2.13, we have the following main results.

Theorem 2.14. *Under the hypotheses of Theorem 2.5, in the spaces $L^p(\mathbb{R})$, where $1 < p \leq 2$, $\{\psi^p_{jk}\}$ is a bounded and Besselian basis. Consequently, in the spaces $L^p(\mathbb{R})$, where $2 \leq p < \infty$, $\{\psi^p_{jk}\}$ is a bounded and Hilbertian basis. The basis $\{\psi^p_{jk}\}$ is simultaneously Besselian and Hilbertian if and only if $p = 2$.*

Proof. By Theorem 2.5 and 2.12, there exists a linearly isometric map Λ from $L^p(\mathbb{R})$ onto $L^p([0,1])$, for all $1 < p < \infty$, such that $\{\Lambda(\psi^P_{jk})\}$ is a bounded unconditional basis in $L^p(\mathbb{R})$ since $\{\psi^p_{jk}\}$ is. Theorem 2.12 implies that $\{\Lambda(\psi^p_{jk})\}$ is Besselian for $p \in (1,2]$ and Hilbertian for $p \in [2,\infty)$. Moreover, $\{\psi^p_{jk}\}$ is a besselian and Hilbertian basis if and only if $p = 2$. If $1 < p \leq 2$ and $\{\psi^p_{jk}\}$ is a Besselian basis, then by Theorem 2.2, there exists a continuous linear mapping u of $L^p([0,1])$ into $\ell^2(\mathbb{Z}\times\mathbb{Z})$ such that $u(\Lambda\psi^p_{jk}) = e_{jk}$, where e_{jk} form the unit vector basis of $\ell^2(\mathbb{Z}\times\mathbb{Z})$; that is, the composition operator $u \circ \Lambda$ of u and Λ is also a continuous linear mapping from $L^p(\mathbb{R})$ onto $\ell^2(\mathbb{Z}\times\mathbb{Z})$ by $\psi^p_{jk} \mapsto e_{jk}$. Consequently, the basis $\{\psi^p_{jk}\}$ is Besselian by Theorem 2.2 again. Similarly, we can prove that the basis $\{\psi^p_{jk}\}$ is Hilbertian for all $2 \leq p < \infty$.

3. Unconditional bases from Biorthonormal wavelets. In order to generalize the Meyer's theorem we need some terminology. A *pre-wavelet* is a square-integrable function ψ such that $\psi(2^j\cdot+k)$ is orthogonal to $\psi(2^m\cdot+l)$ for all $k,l \in \mathbb{Z}$ and all integers $j \neq m$. This terminology "pre-wavelet" first was used by Battle [1]. A linear space B of functions on $\mathbb{R}^n$ is said to be shift invariant if for any $f \in B$ and $k \in \mathbb{Z}^n$, $T_k f \in B$ where the translation operator T is defined by $T_k f(\cdot) = f(\cdot + k)$. A family $\{\psi_i\}_{i\in I}$ (I may be countable or uncountable) in a Hilbert space is said to be stable when there exist two positive constants C_1 and C_2 so that the inequalities

$$C_1 \sum_i |a_i|^2 \leq \|\sum_i a_i\psi_i\|^2 \leq C_2 \sum_i |a_i|^2$$

hold for all a in $\ell^2(I)$. In particular, if ψ is an element of $L^2(\mathbb{R}^n)$ whose integer shifts make a stable family, we shall say that ψ have stable shifts.

Assume that ψ and its derivative ψ' satisfy the decay condition $\mathcal{O}(|x|^\rho)$, $(\rho < -n)$. It is well know [4] that if ψ has stable shifts, then there exists a bi-orthogonal function $\tilde{\psi}$ whose integer translates are stable. We want to show that both function ψ and $\tilde{\psi}$ have the same rate of decay. This is related to a Wiener-Tauberian Theorem with a "weight". Let σ be a positive sequence satisfying the condition

$$\sigma(k+l) \leq \sigma(k)\sigma(l), \quad \text{for all} \quad k,l \in \mathbb{Z}^n. \tag{3.1}$$

Following Gelfand, *et al.* [6] for one-dimension or Mitjagin [13] and Lei [10] for multi-dimensions, we denote by $W[\sigma]$ the set of all formal power series $f = \sum_{k\in\mathbb{Z}^n} a_k X^k$ for which

$$\|f\| := \sum_{k\in\mathbb{Z}^n} |a_k|\sigma(k) < \infty.$$

Here X is an indeterminate vector, $X := (X_1, \dots, X_n)$. It plays the role of a dummy variable in defining the formal power series. From (3.1) it follows that if $W[\sigma]$ contains two series $f = \sum_k a_k X^k$ and $g = \sum_k b_k X^k$, it also contains their formal product

$$\sum_m c_m X^m = \sum_m \Big(\sum_l a_{m-l} b_l\Big) X^m,$$

and that

$$\begin{aligned}\|fg\| &= \sum_m \Big|\sum_l a_{m-l} b_l\Big|\sigma(m)\\ &\leq \sum_m \sum_l |a_{m-l}|\sigma(m-l)|b_l|\sigma(l)\\ &= \|f\|\ \|g\|\end{aligned}$$

Thus, $W[\sigma]$ is a Banach algebra having a unit under the formal operations on power series. Let M be a maximal ideal of $W[\sigma]$. Then the residue class algebra $W[\sigma]/M$ is isomorphic to the complex number field. If we denote by $f(M)$ the complex number corresponding to $f \in W[\sigma]$ under this canonical mapping, then $|f(M)| \leq \|f\|$. For each $p \in \{1, \dots, n\}$, the complex number $X_p(M)$ is non-zero since X_p is invertible. Let z be the element in $\mathbb{C}^n$ whose p-th component is $X_p(M)$. Then for each $k \in \mathbb{Z}^n$, $z^k = X^k(M)$ and

$$|z^k| = |X^k(M)| \leq \|X^k\| = \sigma(k).$$

This leads us to consider the set

$$S_\sigma := \{z \in \mathbb{C}^n \setminus \{0\} : |z^k| \le \sigma(k) \quad \text{for all} \quad k \in \mathbb{Z}^n\}.$$

It is clear that for any formal power series $\sum_k a_k X^k$ in $W[\sigma]$, the power series $\sum_k a_k z^k$ converges uniformly in z on S_σ. This implies that every element f in $W[\sigma]$ defines a continuous function on S_σ, still denoted by f.

In our application, σ is chosen to be

$$\sigma(x) = (1 + \|x\|)^\rho, \quad x \in \mathbb{R}^n, \tag{3.2}$$

where ρ is a positive number. Thus, $\sigma(x+y) \le \sigma(x)\sigma(y)$. Lei [10] proved that S_σ is exactly the torus T^n in $\mathbb{C}^n$. The following theorem follows immediately from [6, Sect.6, Theorem 1].

Theorem 3.3. *Let σ be as in* (3.2). *If f is a member of $W[\sigma]$ and if G is analytic on a neighborhood of the range of the function on T^n, then there exists a member g in $W[\sigma]$ such that $G(f(z)) = g(z)$ for all $z \in T^n$.*

This theorem is useful in estimating the decay rate of $\tilde{\phi}$ in terms of that of ϕ, where ϕ is a stable function, and $\tilde{\phi}$ is a bi-orthogonal stable function with respect to ϕ. We can show that $\tilde{\phi}$ has the same decay rate. Moreover, we will show that

$$\|\sigma\tilde{\phi}\|_w := \sum_k \max_{x\in[0,1]^n} |(\sigma\tilde{\phi})(x+k)| = \sum_k \max_{x\in[0,1]^n} |\sigma(x+k)||\tilde{\phi})(x+k)| < \infty.$$

Given a function ϕ on $\mathbb{R}^n$ and a sequence a on $\mathbb{Z}^n$, the semi-discrete convolution product $\phi *' a$ is, by definition, the sum $\sum_k a_k \phi(\cdot + k)$ (see, [7]).

Theorem 3.4. *Let ρ be a positive number. If ϕ belongs to $C(\mathbb{R}^n)$ with $\|\sigma\phi\|_w < \infty$ and has stable shifts, then there exists a sequence b for which the formal power series $g := \sum_k b_k X^k$ is in $W[\sigma]$ and $\tilde{\phi} := \phi *' b$ has stable shifts with $\|\sigma\tilde{\phi}\|_w < \infty$.*

Proof. Let a be the sequence given by $a_k := \phi * \phi^*(k)$ and let f be the formal power series $\sum_k a_k X^k$. We want to show that f belongs to $W[\sigma]$. For any $k \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\sum_k \sigma(k)|a_k| &\le \sum_k \sigma(k)\int_{\mathbb{R}^n} |\phi(x)\phi(x+k)|dx \\ &\le \sum_k \int_{\mathbb{R}^n} |(\sigma\phi)(x)|\,|(\sigma\phi)(x+k)|dx \\ &\le \sum_k\sum_l \int_{[0,1]^n} |(\sigma\phi)(x+l)|\,|(\sigma\phi)(x+k+l)|dx \\ &\le \|\sigma\phi\|_w^2 < \infty.\end{aligned}$$

Thus, $f \in W[\sigma]$. Since ϕ has stable shifts, $f(z)$ is positive on T^n (see [7, Theorems 3.3]). By Theorem 3.3, there exists a sequence b for which the formal power series $g := \sum_k b_k X^k$ is in $W[\sigma]$ and

$$\frac{1}{f(z)} = g(z).$$

This implies that $\tilde{\phi} := \phi *' b$ (see, [7]) is bi-orthogonal with respect to ϕ. Indeed, by the Poisson summation formula,

$$\hat{\tilde{\phi}}(\xi) = \left(\sum_k b_k e^{2\pi i(k,\xi)}\right)\hat{\phi}(\xi) = \frac{\hat{\phi}(\xi)}{\sum_k |\hat{\phi}(\xi+k)|^2},$$

and hence

$$\sum_k (\tilde{\phi} * \phi^*)(k)e^{2\pi i(k,\xi)} = \frac{1}{\sum_k |\hat{\phi}(\xi+k)|^2}\sum_k |\hat{\phi}(\xi+k)|^2 = 1.$$

This implies that

$$\tilde{\phi} * \phi^*(k) = 0\,(k \ne 0), \quad \tilde{\phi} * \phi^*(k) = 1(k = 0).$$

Hence

$$\int_{\mathbb{R}^n} \tilde{\phi}(x+j)\overline{\phi(x+k)}dx = \tilde{\phi} * \phi^*(j-k) = \delta_{jk}.$$

Finally, in order to show $\|\sigma\tilde{\phi}\|_w < \infty$, we note that

$$\begin{aligned}\sum_k \max_{x\in[0,1]^n} |(\sigma\tilde{\phi})(x+k)| &\le \sum_k\sum_l \max_{x\in[0,1]^n} |(\sigma\phi)(x+k+l)|\,|\sigma(l)b_l| \\ &\le \|\sigma\phi\|_w \|g\| < \infty.\end{aligned}$$

That is $\|\sigma\tilde{\phi}\|_w < \infty$. Thus, ϕ and $\tilde{\phi}$ have the same decay rate.

Corollary 3.5 *Assume that* $\|\sigma\psi\|_w < \infty$ *and* $\|\sigma\psi'\|_w < \infty$. *If* ψ *has stable shifts, then there exists a sequence* $b \in \ell^1(\mathbb{Z})$ *such that* $\tilde{\psi} = \psi *' b$ *and* $\tilde{\psi}' = \psi' *' b$ *with* $\|\sigma\tilde{\psi}\|_w < \infty$ *and* $\|\sigma\tilde{\psi}'\|_w < \infty$.

Recall the definition of frame.

Definition 3.6. A family $\{\phi_i : i \in I\}$ is said to be a frame for a Hilbert space $\mathcal{H}$ if there exist positive constants A, B such that the inequalities

$$A\|f\|^2 \leq \sum_i |(f, \sigma_i)|^2 \leq B\|f\|^2 \tag{3.7}$$

hold for all f in $\mathcal{H}$. The numbers A and B are called bounds for the frame.

We want to give an equivalent condition for a frame. First, a lemma is needed.

Lemma 3.8 [15]. *Suppose that* X *and* Y *are Hilbert spaces, and* S *is a bounded linear operator from* X *to* Y. *Then* S *is surjective if and only if the adjoint operator* S^* *is bounded below; that is, there exists a constant* $m > 0$ *such that* $\|S^*g\| \geq m\|g\|$. *For every* $y \in Y$.

Theorem 3.9. *For a family* $\{\phi_i : k \in I\}$ *in* $\mathcal{H}$ *the following are equivalent:*

(i) $\{\phi_i\}$ *is a frame,*

(ii) *The map* $S : \{a_i\}_{i\in I} \mapsto \sum_i a_i\phi_i$ *is a bounded operator from* $\ell^2(I)$ *onto* $\mathcal{H}$.

Proof. Suppose that the operator S is bounded from $\ell^2(I)$ onto $\mathcal{H}$. Using Lemma 3.8 we conclude that the adjoint operator S^* is bounded below and $S^*f(i) = (f, \phi_i)$ for all $i \in I$ and $f \in \mathcal{H}$ since

$$(S\{a_i\}, f) = (\sum_i a_i\phi_i, f) = \sum_i a_i(\phi_i, f) = \sum_i a_i\overline{(f, \phi_i)} = (\{a_i\}, S^*f).$$

Thus $\{\phi_i\}$ is a frame. On the other hand, assume that $\{\phi_i\}$ satisfies inequalities (3.7) and let a be any element of $\ell^2(I)$. Put $f := \sum_i a_i\phi_i$. Then

$$\|f\|^4 = |(f,f)|^2 = \left|\sum_i a_i(f,\phi_i)\right|^2 \le \sum_i |a_i|^2 \sum_i |(f,\phi_i)|^2 \le A\|f\|^2 \sum_i |a_i|^2.$$

Dividing by $\|f\|^2$, we find $\|\sum_i a_i\phi_i\|^2 \le A\sum_i |a_i|^2$. This implies that the map $S : \{a_i\} \mapsto \sum_i a_i\phi_i$ is well-defined and bounded so that the adjoint operator S^* is bounded below by (3.7). Lemma 3.8 again implies that S is onto.

Corollary 3.10. *If a family $\{\phi_i : i \in I\}$ is a frame in a Hilbert space $\mathcal{H}$, then $\mathcal{H} = \{\sum_i a_i\phi_i : a \in \ell^2(I)\}$.*

Under certain conditions, we can apply Corollary 3.5 to prove that if the hypotheses of Corollary 3.5 hold and if ψ_{jk} form a pre-wavelet basis for $L^2(\mathbb{R})$, then ψ_{jk} also provide a unconditional basis for $L^p(\mathbb{R})$, for $1 < p < \infty$. By Theorem 1.2, we need to show that if

$$f = \sum_{jk} a_{jk}\psi_{jk} \in L^p(\mathbb{R}), \tag{3.11}$$

then

$$\sum_{j,k} \epsilon_{jk} a_{jk}\psi_{jk} \in L^p(\mathbb{R})$$

whenever $\epsilon_{jk} = \pm 1$.

Since ψ belongs to $L^p(\mathbb{R})$, for all $1 < p < \infty$, (3.11) implies that $a_{jk} = (f, \tilde{\psi}_{jk})$ (the inner product of f and $\tilde{\psi}_{jk}$) by the bi-orthogonality of the ψ. Assume that the linear operator U defined by

$$Uf = \sum_{j,k} \epsilon_{jk}(f, \tilde{\psi}_{jk})\psi_{jk} \tag{3.12}$$

is bounded on $L^2(\mathbb{R})$. The L^p-boundedness will follow by Theorem 1.10 if we can show that U is an integral operator with kernel satisfying (1.5) and (1.6). By Corollary 3.5, we can prove that if we choose $\epsilon_{jk} = \pm 1$, and define

$$H(x,y) = \sum_{j,k} \epsilon_{jk}\psi_{jk}(x)\overline{\tilde{\psi}_{jk}(y)},$$

then H satisfies (1.5) and (1.6) but the proof is similar to the orthonormal case (see [4, Lemma 9.1.5]).

Theorem 3.13. *Under the hypotheses given above, if U is an integral operator with integral kernel H, and if U is bounded on $L^2(\mathbb{R})$, then U extends to a bounded operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$, for all $p \in (1,\infty)$; that is, $\{\psi_{jk}\}$ is a unconditional basis of $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.*

A sufficient condition is given for the boundedness of the operator U. This is related to the Riesz bases. If ψ is a pre-wavelet and a stable function, we can conclude that ψ_{jk} forms a Riesz basis.

Theorem 3.14. *If ψ_{jk} form a pre-wavelet basis and if ψ has stable shift, then $\{\psi_{jk}\}$ is a Riesz basis (hence a frame) and $\{\tilde{\psi}_{jk}\}$ is the dual frame with $(\psi_{jk}, \tilde{\psi}_{il}) = \delta_{ji}\delta_{kl}$.*

Using Theorem 3.14, $\{\psi_{jk}, \tilde{\psi}_{jk}\}_{j,k}$ are dual frames and so one can prove that U in (3.12) is a bounded operator.

Corollary 3.15. *Adopt the assumptions of Theorem 3.14, U is a bounded operator on $L^2(\mathbb{R})$.*

proof. For any f in $L^2(\mathbb{R})$, we have

$$\begin{aligned}\|Uf\|_2^2 &= \Big\| \sum_{j,k} \epsilon_{jk}(f,\tilde{\psi})\psi_{jk} \Big\|_2^2 = \sum_j \| \sum_k \epsilon_{jk}(f,\tilde{\psi})\psi_{jk}\|_2^2 \\ &\leq C \sum_{j,k} |(f,\tilde{\psi})|^2 \leq C^2 \|f\|_2^2.\end{aligned}$$

whenever $\epsilon_{jk} = \pm 1$. Thus, U is bounded.

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