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STABILITY OF A SYSTEM OF FUNCTIONAL PARABOLIC EQUATIONS

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Abstract. This paper establishes stability theorems for a system of functional parabolic equations with vector components in terms of the eigenvalues of the coefficient matrices.

1. Introduction. Qualitative properties of functional partial differential equations have been the subjects of investigations in many studies. In this paper, we are concerned with the stability of a system of partial differential equations with delay. Our investigation is novel in the sense that a solution of our system is a vector function with vector components and the concept of stability is measured by means of L^2 and L^∞ norms. We shall derive stability criteria in terms of the eigenvalues of the coefficient matrices of the system. Several recent studies which are related to our study can be found in [1-4]. More specifically, both [1] and [2] are concerned with differential equations with Volterra integrals and delays. These equations arise from population dynamics. Such an equation is used to model the evolution of population density of a species which lives in a bounded domain undergoing diffusion and subject to memory effects. In this paper, we will be concerned with a system of diffusion equations which may be used to describe the evolution of N species living in a bounded domain undergoing diffusion and subject to memory effects as well as mutual interactions. In

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[4], weakly coupled reaction diffusion system similar to ours are also studied and stability criteria are established. However, memory effects are not considered. Our results will likely find applications in the stability of solutions of diffusion systems such as the Volterra-Lotka system and the Brusselator (see [4]).

We first recall a few preparatory definitions which will be useful in our later discussions. In what follows, Ω is a bounded domain in R^q such that

$$\Omega \subset \{(x_1, \ldots, x_q) \in R^q \,|\, |x_i| < \Gamma, 1 \leq i \leq q\}.$$

For convenience, the number $\Gamma/\sqrt{q}$ will be denoted by ω. The domain Ω will be assumed to have smooth boundary $\partial\Omega$, and outward unit normal ν. As usual, the Laplacian will be denoted by Δ and the gradient by ∇.

For $1 \leq p \leq \infty$, $L^q(\Omega)$ denotes the usual Banach space of real measurable functions u defined on the domain Ω equipped with the norm

$$\|u\|_p = \left\{ \int_\Omega |u|^p \right\}^{1/p}, \; 1 \leq p < \infty$$

or

$$\|u\|_p = \text{esssup}_{x\in\Omega} |u(x)|, \; p = \infty.$$

Let m be a positive integer and $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is the set of all $u \in L^p(\Omega)$ such that the distributional derivative $D^\alpha u \in L^p(\Omega)$ for all $0 \leq |\alpha| \leq m$, where $\alpha = (\alpha_1, \ldots, \alpha_q)$ is a tuple of non-negative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_q$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_q^{\alpha_q}}.$$

The space $W^{m,p}(\Omega)$ will be equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_p$$

or the equivalent norm

$$\|u\|_{W^{m,p}} = \left\{ \sum_{|\alpha|\le m} \int_\Omega |D^\alpha u|^p \right\}^{1/p}$$

when $1 \le p < \infty$.

Let $n_1, n_2, \ldots, n_N$ be N positive integers. Let $A_{ij}(t)$ and $B_{ij}(t)$ be continuous $n_i \times n_j$ matrix functions defined on $[0,\infty)$. Let

$$D_1(t) = \mathrm{diag}[d_{11}(t), \ldots, d_{1n_1(t)}],$$
$$D_2(t) = \mathrm{diag}[d_{21}(t) \ldots, d_{2n_2(t)}],$$
$$\cdots$$
$$D_N(t) = \mathrm{diag}[d_{N1}(t), \ldots, d_{Nn_N}(t)],$$

where $d_{ij}(t)$, $1 \le i \le n_i$, $1 \le i \le N$, are positive continuous functions uniformly bounded away from zero of $[0,\infty)$. Several quantities related to these matrices will be used in the sequel. These are (provided they are defined)

$$a_{ij} = \sup_{t\ge 0} \|A_{ij}(t)\|,$$
$$b_{ij} = \sup_{t\ge 0} \|B_{ij}(t)\|,$$

$$A^*_{ii}(t) = \frac{A^T_{ii}(t) + A_{ii}(t)}{2} - \frac{1}{\omega^2} D_i(t),$$

$$-\gamma_i(t) = \max\{\lambda(t) | \lambda(t) \text{ is an eigenvalue of } A^*_{ii}(t)\},$$

$$\gamma_i = \inf_{t\ge 0} \gamma_i(t),$$

where v^T denotes the transpose of the vector v. Finally, the Dirac delta function is denoted by δ_{ij}.

2. The system. Consider the following system of equations

$$\frac{\partial u_i(x,t)}{\partial t} = D_i(t)\Delta u_i(x,t) + \sum_{j=1}^{N} [A_{ij}(t) u_j(x,t) - B_{ij}(t) u_j(x, t-\tau)], 1 \le i \le N, \tag{1}$$

where $(x,t) \in \Omega \times (0,\infty)$,

$$u_i(x,t) = \phi_i(x,t),\ (x,t) \in \Omega \times [-\tau,0],\ 1 \le i \le N, \tag{2}$$

$$\frac{\partial u_i(x,t)}{\partial \nu} = 0,\ (x,t) \in \partial\Omega \times [-\tau,\infty),\ 1 \le i \le N, \tag{3}$$

where $\tau \ge 0$ is the delay and $\phi_i : \Omega \times [-\tau, 0] \to R^{n_i}$, $1 \le i \le N$, are smooth functions.

A solution of the system (1-3) is a function of the form

$$u(x,t,\phi) = \mathrm{col}(u_1(x,t,\phi),\ldots,u_N(x,t,\phi)),$$

which satisfies (1-3) for $(x,t) \times [-\tau,\infty)$, where $\phi = \mathrm{col}(\phi_1,\ldots,\phi_N)$ and

$$u_i(x,t,\phi) = \mathrm{col}(u_{i1}(x,t,\phi),\ldots,u_{in_i}(x,t,\phi)),\ 1 \le i \le N.$$

In the sequel, ϕ will be depressed from the expression of a solution if no confusion will arise. We will be interested in the stability of the solutions of the system (1-3) measured by L^2 and L^∞ norms. The precise definitions are as follows.

We will agree that if $v = \mathrm{col}(v_1,\ldots,v_n)$ is a vector function, then

$$\|v\| = \{v^T v\}^{1/2},$$

$$\nabla v = \mathrm{col}(\nabla v_1,\ldots,\nabla v_n),$$

and

$$\|v\|_{L^p} = \|\{v^T v\}^{1/2}\|_p,\ 1 \le p \le \infty,$$

where v^T denotes the transpose of the vector v.

Definition 1. A solution $u(x,t,\phi) = \mathrm{col}(u_1(x,t,\phi),\ldots,u_N(x,t,\phi))$ of the system (1-3) is said to be $\aleph$-bounded if $\|u_i(x,t,\phi)\|_{L^\infty}$ and $\|\nabla u_i(x,t,\phi)\|_{L^\infty}$, $1 \le i \le N$, are bounded on $[-\tau,\infty)$.

Definition 2. The trivial solution of the system (1-3) is said to be $\aleph^{1,2}$-stable if given any $\epsilon > 0$, there is a $\delta > 0$ and $\mu > 0$ such that, if

$$M_i \equiv \sup_{-\tau\leq t\leq 0} \|\phi_i(x,t)\|^2_{L^2} < \delta,\ 1 \leq i \leq N$$

is satisfied, then for any solution $\mathrm{col}(u_i(x,t,\phi),\dots u_N(x,t,\phi))$ of (1), we have

$$\|u_i(x,t,\phi)\|_{L^2} \leq \epsilon$$

and

$$\|\nabla u_i(x,t,\phi)\|_{L^2} \leq \mu\epsilon$$

for $1 \leq i \leq N$ and $t > 0$. If in addition, we also have

$$\lim_{t\to\infty} \|u_i(x,t,\phi)\|_{L^2} = \lim_{t\to\infty} \|\nabla u_i(x,t,\phi)\|_{L^2} = 0,$$

then we say that the trivial solution of (1) is asymptotically $\aleph^{1,2}$-stable.

3. A lemma. Let ϵ be a positive number. Let $u = \mathrm{col}(u_1,\dots,u_N)$ be a solution of the system (1-3). Since $u_i^T(\partial u_i/\partial t) = (\partial u_i/\partial t)^T u_i$, we obtain from (1) that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(u_i^T(x,t)u_i(x,t)\right)\\
=&2u_i^T(x,t)D_i(t)\Delta u_i(x,t) + 2\sum_{j=1}^{N}\left[u_i^T(x,t)A_{ij}(t)u_j(x,t)\right.\\
&\qquad\qquad\qquad\qquad\left. + u_i^T(x,t)B_{ij}(t)u_j(x,t-\tau)\right]\\
=&2\sum_{m=1}^{n_i} d_{im}(t)u_{im}(x,t)\Delta u_{im}(x,t) + u_i^T(x,t)\left[A_{ii}^T(t)+A_{ii}(t)\right]u_i(x,t)\\
&+2\sum_{j=1,j\neq i}^{N} u_i^T(x,t)A_{ij}(t)u_j(x,t) + 2\sum_{j=1}^{N} u_i^T(x,t)B_{ij}(t)u_j(x,t-\tau).
\end{aligned}$$

For convenience, let us write

$$\left\{\int_\Omega \|u_i(x,t)\|^2 dx\right\}^{1/2} = p_i(t),\ t \geq -\tau. \tag{4}$$

After integrating the above equality over Ω, we obtain

$$\begin{aligned}\frac{dp_i^2(t)}{dt} =& 2\sum_{m=1}^{n_i} d_{im}(t)\int_\Omega u_{im}(x,t)\Delta u_{im}(x,t)dx\\ &+\int_\Omega u_i^T(x,t)\left[A_{ii}^T(t)+A_{ii}(t)\right]u_i(x,t)dx\\ &+2\sum_{j=1,j\neq i}^{N}\int_\Omega u_i^T(x,t)A_{ij}(t)u_j(x,t)dx\\ &+2\sum_{j=1}^{N}\int_\Omega u_i^T(x,t)B_{ij}(t)u_j(x,t-\tau)dx.\end{aligned}\tag{5}$$

By means of the divergence theorem and the boundary condition (2), we have

$$\int_\Omega(\nabla u_{im}\nabla u_{im}+u_{im}\Delta u_{im})dx=\int_{\partial\Omega}u_{im}\frac{\partial u_{im}}{\partial\nu}dx=0,$$

so that

$$\int_\Omega u_{im}\Delta u_{im}dx=-\int_\Omega(\nabla u_{im})^2dx.$$

By means of Poincare's inequality, we then obtain

$$\int_\Omega u_{im}^2dx\le\omega^2\int_\Omega(\nabla u_{im})^2dx,\ \omega=\frac{\Gamma}{\sqrt{q}}.$$

Thus,

$$\begin{aligned}&d_{im}(t)\int_\Omega u_{im}(x,t)\Delta u_{im}(x,t)dx=-d_{im}(t)\int_\Omega(\nabla u_{im}(x,t))^2dx\\ &=-\epsilon\int_\Omega(\nabla u_{im}(x,t))^2dx-(d_{im}(t)-\epsilon)\int_\Omega(\nabla u_{im}(x,t))^2dx\\ &\le-\epsilon\int_\Omega(\nabla u_{im}(x,t))^2dx-(d_{im}(t)-\epsilon)\omega^{-2}\int_\Omega u_{im}^2(x,t)dx.\end{aligned}\tag{6}$$

Next, note that $A_{ii}^T(t)+A_{ii}(t)=2A_{ii}^*(t)+2\omega^{-2}D_i(t)$, thus assuming $\gamma_i>0$, we have

$$\begin{aligned}&\int_\Omega u_i^T(x,t)\left[A_{ii}^T(t)+A_{ii}(t)\right]u_i(x,t)dx\\ &=2\int_\Omega u_i^T(x,t)A_{ii}^*(t)u_i(x,t)dx+\sum_{m=1}^{n_i}\int_\Omega 2d_{im}(t)\omega^{-2}u_{im}^2(x,t)dx\\ &\le-2\gamma_ip_i^2(t)+\sum_{m=1}^{n_i}\int_\Omega 2d_{im}(t)\omega^{-2}u_{im}^2(x,t)dx.\end{aligned}\tag{7}$$

Substituting these inequalities into (5) and simplifying, we have

$$
\begin{aligned}
\frac{dp_i^2(t)}{dt} + 2(\gamma_i - \epsilon\omega^{-2})p_i^2(t) &\le -2\epsilon \sum_{m=1}^{n_i} \int_\Omega (\nabla u_{im}(x,t))^2 dx \\
&+ 2 \sum_{j=1, j\neq i}^{N} \int_\Omega \|u_i(x,t)\| \, \|A_{ij}(t)\| \, \|u_j(x,t)\| dx \\
&+ 2 \sum_{j=1}^{N} \int_\Omega \|u_i(x,t)\| \, \|B_{ij}(t)\| \, \|u_j(x,t-\tau)\| dx,
\end{aligned} \tag{8}
$$

where $\|A_{ij}\|$ and $\|B_{ij}\|$ are induced matrix norms of the matrix A_{ij} and B_{ij} respectively.

Multiply both sides of (8) by the integrating factor

$$\exp\{2(\gamma_i - \epsilon\omega^{-2})t\}$$

and then integrate from $t = 0$ to t, we obtain

$$
\begin{aligned}
& p_i^2(t) \le p_i^2(0) \exp\{-2(\gamma_i - \epsilon\omega^{-2})t\} \\
& -2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} \exp\{-2(\gamma_i - \epsilon\omega^{-2})(t-s)\} (\nabla u_{im}(x,s))^2 dx ds \\
& +2 \sum_{j=1, j\neq i}^{N} \int_0^t \int_\Omega \exp\{-2(\gamma_i - \epsilon\omega^{-2})(t-s)\} \|u_i(x,t)\| \, \|A_{ij}(t)\| \, \|u_j(x,t)\| dx ds \\
& +2 \sum_{j=1}^{N} \int_0^t \int_\Omega \exp\{-2(\gamma_i - \epsilon\omega^{-2})(t-s)\} \|u_i(x,t)\| \, \|B_{ij}(t)\| \, \|u_j(x,t-\tau)\| dx ds.
\end{aligned} \tag{9}
$$

We summarize the above as follows: Suppose ϵ and $\gamma_1, \ldots, \gamma_N$ are positive. If $u = \mathrm{col}(u_1, \ldots, u_N)$ is a solution of the system (1-3), then (9) holds where $p_i(t)$ is defined by (4).

4. Stability criteria. Let $c_{ij} = 2b_{ii}/\gamma_i$ when $i = j$ and $c_{ij} = 2(a_{ij} + b_{ij})/\gamma_i$ when $i \neq j$. Let C be the block matrix (c_{ij}) and

$$\rho = \sup\{\lambda | \lambda \text{ is an eigenvalue of } (C^T + C)/2\}.$$

Theorem 1. *Suppose $\gamma_i > 0$ for $1 \le i \le N$ and suppose $\rho < 1$. Then the trivial solution of (1) is asymptotically $\aleph^{1,2}$-stable.*

Proof. Since the eigenvalues of a matrix depends continuously on the components of the matrix, in view of the assumption that $\rho < 1$, there are positive numbers ϵ and σ such that $\gamma_i - \epsilon\omega^{-2} - \sigma > 0$ and the largest eigenvalue μ of the matrix $(H^T + H)/2$ satisfies $\mu < 1$. Here $H = (h_{ij})$ is defined by

$$h_{ii} = \frac{2b_{ii}e^{\sigma\tau}}{\gamma_i - \epsilon\omega^{-2} - \sigma},$$

and

$$h_{ij} = \frac{2(a_{ij} + b_{ij}e^{\sigma\tau})}{\gamma_i - \epsilon\omega^{-2} - \sigma},\ j \ne i.$$

Let $u = \mathrm{col}(u_i, \ldots, u_N)$ be a solution of (1-3), then in view of (9), Hölder's inequality and the definitions of a_{ij} and b_{ij}, we have

$$\begin{aligned} p_i^2(t) \le\ & p_i^2(0)e^{-2(\gamma_i - \epsilon\omega^{-2})t} \\ & - 2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2(\gamma_i - \epsilon\omega^{-2})(t-s)} (\nabla u_{im}(x,s))^2 dxds \\ & + 2 \sum_{j=1, j\ne i}^{N} \int_0^t e^{-2(\gamma_i - \epsilon\omega^{-2})(t-s)} a_{ij} p_i(s) p_j(s) ds \\ & + 2 \sum_{j=1}^{N} \int_0^t e^{-2(\gamma_i - \epsilon\omega^{-2})(t-s)} b_{ij} p_i(s) p_j(s-\tau) ds. \end{aligned}$$

For convenience, let us denote $\gamma_i - \epsilon\omega^{-2} - \sigma$ by ξ_i (which is positive by assumption), we have

$$\begin{aligned} e^{2\sigma t}p_i^2(t) \le\ & p_i^2(0)e^{-2\xi_i t} - 2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2\xi_i(t-s)} e^{2\sigma s} (\nabla u_{im}(x,s))^2 dxds \\ & + 2 \sum_{j=1, j\ne i}^{N} \int_0^t e^{-2\xi_i(t-s)} a_{ij} (e^{\sigma s} p_i(s))(e^{\sigma s} p_j(s)) ds \\ & + 2 \sum_{j=1}^{N} \int_0^t e^{-2\xi_i(t-s)} b_{ij} e^{\tau\sigma} (e^{\sigma s} p_i(s)) (e^{\sigma(s-\tau)} p_j(s-\tau)) ds \end{aligned} \tag{10}$$

$$\leq p_i^2(0) - 2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2\xi_i(t-s)} e^{2\sigma s} (\nabla u_{im}(x,s))^2 dxds$$
$$+ 2 \sum_{j=1, j\neq i}^{N} \int_0^t e^{-2\xi_i(t-s)} a_{ij} \Big\{ \sup_{-\tau\leq\theta\leq s} \big(e^{\sigma\theta} p_i(\theta)\big)\Big\}\Big\{ \sup_{-\tau\leq\theta\leq s} \big(e^{\sigma\theta} p_j(\theta)\big)\Big\} ds$$
$$+ 2 \sum_{j=1}^{N} \int_0^t e^{-2\xi_i(t-s)} b_{ij} e^{\tau\sigma} \Big\{ \sup_{-\tau\leq\theta\leq s} \big(e^{\sigma\theta} p_i(\theta)\big)\Big\}\Big\{ \sup_{-\tau\leq\theta\leq s} \big(e^{\sigma\theta} p_j(\theta)\big)\Big\} ds.$$

Define

$$q_i(t) = \sup_{-\tau\leq\theta\leq t} \big(e^{\sigma\theta} p_i(\theta)\big).$$

Then $q_i(t)$ is non-decreasing in t. Hence the last wo terms in (10) is less than or equal to

$$2 \sum_{j=1, j\neq i}^{N} \int_0^t e^{-2\xi_i(t-s)} a_{ij} q_i(t) q_j(t) ds = \sum_{j=1, j\neq i}^{N} \xi_i^{-1} e^{-2\xi_i t} a_{ij} q_i(t) q_j(t)$$

and to

$$2 \sum_{j=1}^{N} \int_0^t e^{-2\xi_i(t-s)} b_{ij} e^{\tau\sigma} q_i(t) q_j(t) ds = \sum_{j=1}^{N} \xi_i^{-1} e^{-2\xi_i t} b_{ij} e^{\tau\sigma} q_i(t) q_j(t)$$

respectively.

Note that

$$\sum_{j=1, j\neq i}^{N} \xi_i^{-1} e^{-2\xi_i t} a_{ij} q_i(t) q_j(t) + \sum_{j=1}^{N} \xi_i^{-1} e^{-2\xi_i t} b_{ij} e^{\tau\sigma} q_i(t) q_j(t)$$
$$\begin{aligned} &\leq \sum_{j=1, j\neq i}^{N} \frac{a_{ij} q_i(t) q_j(t)}{\gamma_i - \epsilon\omega^{-2} - \sigma} + \sum_{j=1}^{N} \frac{b_{ij} e^{\tau\sigma} q_i(t) q_j(t)}{\gamma_i - \epsilon\omega^{-2} - \sigma} \\ &= \sum_{j=1, j\neq i}^{N} \Big\{ \frac{a_{ij} + b_{ij} e^{\tau\sigma}}{\gamma_i - \epsilon\omega^{-2} - \sigma} \Big\} q_i(t) q_j(t) + \frac{b_{ii} e^{\tau\sigma}}{\gamma_i - \epsilon\omega^{-2} - \sigma} \\ &= \sum_{j=1}^{N} \frac{1}{2} q_i(t) h_{ij} q_j(t), \end{aligned} \tag{11}$$

thus from (10), we have

$$e^{2\sigma t}p_i^2(t) \le p_i^2(0) + \sum_{j=1}^{N} \frac{1}{2} q_i(t) h_{ij} q_j(t) \tag{12}$$
$$-2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2\xi_i(t-s)} e^{2\sigma s} (\nabla u_{im}(x,s))^2 dxds.$$

It is not difficult to see from (12) that if $p_i(t) \le M_i$ for $-\tau \le t \le 0$, then

$$\sum_{j=1}^{N} q_i^2(t) \le \frac{3}{1-\mu} \sum_{j=1}^{N} M_i^2. \tag{13}$$

Indeed,

$$q_i(t) = \sup_{-\tau \le s \le t} (e^{\sigma s} p_i(s)) \le \sup_{-\tau \le s \le 0} (e^{\sigma s} p_i(s)) + \sup_{0 \le s \le t} (e^{\sigma s} p_i(s))$$
$$\le M_i + \sup_{0 \le s \le t} (e^{\sigma s} p_i(s)),$$

thus,

$$q_i^2(t) \le 2M_i^2 + 2\left\{ \sup_{0 \le s \le t} \left(e^{\sigma s} p_i(s)\right) \right\}^2 \tag{14}$$
$$\le 2M_i^2 + 2 \sup_{0 \le s \le t} \left(e^{2\sigma s} p_i^2(s)\right) \le 3M_i^2 + \sum_{j=1}^{N} q_i(t) h_{ij} q_j(t).$$

In view of the fact that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} q_i(t) h_{ij} q_j(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} q_i(t) \frac{h_{ij}^T + h_{ij}}{2} q_j(t) \le \mu \sum_{i=1}^{N} q_i^2(t), \tag{15}$$

we see from (14) that

$$\sum_{i=1}^{N} q_i^2(t) \le 3 \sum_{i=1}^{N} M_i^2 + \mu \sum_{i=1}^{N} q_i^2(t),$$

which implies (13), as desired.

Next, we find a lower bound for the integral term in (12). Note that for any $0 < a < t$, by the integral mean value theorem, there is η in $[t-a, t]$ such that

$$
\begin{aligned}
&\int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2\xi_i(t-s)} e^{2\sigma s} (\nabla u_{im}(x,s))^2 dxds \\
&\geq \int_{t-a}^t \sum_{m=1}^{n_i} e^{-2\xi_i(t-s)} \int_\Omega e^{2\sigma s} (\nabla u_{im}(x,s))^2 dxds \\
&\geq a \sum_{m=1}^{n_i} e^{-2\xi_i(t-\eta)} \int_\Omega e^{2\sigma\eta} (\nabla u_{im}(x,\eta))^2 dx \\
&\geq a \sum_{m=1}^{n_i} e^{-2\xi_i a} \int_\Omega e^{2\sigma\eta} (\nabla u_{im}(x,\eta))^2 dx.
\end{aligned}
\tag{16}
$$

In view of (13), (15) and (16), we obtain from (12) that

$$
\begin{aligned}
\sum_{i=1}^{N} e^{2\sigma t} p_i^2(t) + 2\epsilon a \sum_{i=1}^{N} \sum_{m=1}^{n_i} e^{-2\xi_i a} \int_\Omega e^{2\sigma\eta} (\nabla u_{im}(x,\eta))^2 dx \\
\leq \frac{2+\mu}{2-2\mu} \sum_{j=1}^{N} M_i^2.
\end{aligned}
\tag{17}
$$

Since the right hand side of (17) is independent of t, we see that

$$
\begin{aligned}
&\sum_{i=1}^{N} \sup_{t-a\leq s\leq t} \left(e^{2\sigma s} p_i^2(s)\right) \\
&\qquad + 2\epsilon a \sum_{i=1}^{N} \sum_{m=1}^{n_i} e^{-2\xi_i a} \int_\Omega \sup_{t-a\leq s\leq t} \left(e^{2\sigma s} (\nabla u_{im}(x,s))^2\right) dx \\
&\leq \frac{2+\mu}{2-2\mu} \sum_{j=1}^{N} M_i^2.
\end{aligned}
\tag{18}
$$

But then

$$
\begin{aligned}
&\sum_{i=1}^{N} \left\{ p_i^2(t) + 2\epsilon a e^{-2\xi_i a} \sum_{m=1}^{n_i} \int_\Omega (\nabla u_{im}(x,t))^2 dx \right\} \\
&= \sum_{i=1}^{N} e^{-2\sigma t} \left\{ e^{2\sigma t} p_i^2(t) + 2\epsilon a e^{-2\xi_i a} \sum_{m=1}^{n_i} \int_\Omega e^{2\sigma t} (\nabla u_{im}(x,t))^2 dx \right\} \\
&\qquad \leq e^{-2\sigma t} \{\text{right hand side of (18)}\} \\
&\qquad \leq \frac{2+\mu}{2-2\mu} \sum_{i=1}^{N} M_i^2 e^{-2\sigma t}.
\end{aligned}
\tag{19}
$$

Finally, we see that

$$p_i^2(t) \le \sum_{i=1}^{N} p_i^2(t) \le \frac{2+\mu}{2-2\mu} \sum_{i=1}^{N} M_i^2 e^{-2\sigma t}$$

and

$$\sum_{m=1}^{n_i} \int_\Omega (\nabla u_{im}(x,t))^2 dx \le \sum_{i=1}^{N} \sum_{m=1}^{n_i} \int_\Omega (\nabla u_{im}(x,t))^2 dx$$
$$\le \frac{1}{2\epsilon a} e^{2a\{\max(\gamma_1,\ldots,\gamma_N)-\epsilon\omega^{-2}-\sigma\}} \frac{2+\mu}{2-2\mu} \sum_{i=1}^{N} M_i^2 e^{-2\sigma t}$$

Thus if

$$M_i = \sup_{-\tau \le t \le 0} \|\phi_i(x,t)\|_{L^2}^2 < \infty, \ 1 \le i \le N,$$

then $\lim_{t\to\infty} \|u_i(x,t,\phi)\|_{L^2} = \lim_{t\to\infty} \|\nabla u_i(x,t,\phi)\|_{L^2} = 0$ as desired.

Corollary 1. *Under the assumptions of Theorem 1, any $\aleph$-bounded solution of* (1-3) *converges to zero uniformly in $x \in \bar{\Omega}$ as $t \to \infty$.*

Proof. Let $u(x,t,\phi) = \mathrm{col}(u_1(x,t,\phi),\ldots,u_N(x,t,\phi))$ be a $\aleph$-bounded solution of (1-3). By means of the following inequality [6],

$$\|y\|_{L^p} \le \|y\|_{L_\infty}^{(p-2)/p} \|y\|_{L^2}^{2/p}, \ p \ge 2,$$

we have

$$\|u_i(x,t,\phi)\|_{L^p} \le \|u_i(x,t,\phi)\|_{L^\infty}^{(p-2)/p} \|u_i(x,t,\phi)\|_{L^2}^{2/p}, \ 1 \le i \le N.$$

and

$$\|\nabla u_i(x,t,\phi)\|_{L^p} \le \|\nabla u_i(x,t,\phi)\|_{L^\infty}^{(p-2)/p} \|\nabla u_i(x,t,\phi)\|_{L^2}^{2/p}, \ 1 \le i \le N.$$

From Sobolev inequality, there is a positive number K such that

$$\|v\|_{L^\infty} \le K\{\|v\|_{L^p} + \|\nabla v\|_{L^p}\} = K\|v\|_{W^{1,p}},$$

thus

$$\|u_i(x,t,\phi)\|_{L^\infty} \le K\{\|u_i(x,t,\phi)\|_{L^p} + \|\nabla u_i(x,t,\phi)\|_{L^p}\}.$$

But the right hand side converges to zero since $u(x,t,\phi)$ is asymptotically $\aleph^{1,2}$-stable by Theorem 1, thus $u_i(x,t,\phi)$ converges to zero uniformly in $x \in \bar{\Omega}$ as $t \to \infty$.

5. Additional criteria. We will derive two more stability criteria for the system (1-3). The proofs are similar to that of Theorem 1. Hence they will be sketched only.

Theorem 2. *Let $G = (g_{ij})$ be the matrix defined by $g_{ii} = b_{ii}/\gamma_i$ and $g_{ij} = (a_{ij} + b_{ij})/\gamma_i$ for $j \neq i$. Suppose $\gamma_i > 0$ for $1 \leq i \leq N$, and suppose the spectral radius $\rho(G)$ of G is less than* 1, *then the trivial solution of* (1-3) *is asymptotically $\aleph^{1,2}$-stable.*

Proof. Since $\rho(G) < 1$, by continuity, there are positive numbers ϵ and σ such that $\xi_i \equiv \gamma_i - \epsilon\omega^{-2} - \sigma > 0$ and the spectral radius $\rho(S)$ of the matrix S is less than 1, where $S = (s_{ij})$ is defined by $s_{ii} = \xi_i^{-1} b_{ii} e^{\sigma\tau}$, and $s_{ij} = \xi_i^{-1}(a_{ij} + b_{ij}e^{\sigma\tau})$ for $i \neq j$. Let

$$M_i^2 \equiv \sup_{-\tau \leq t \leq 0} \|\phi_i(x,t)\|_{L^2}^2, \quad -\tau \leq t \leq 0,$$

and

$$\begin{aligned} W_i(t) = {} & M_i^2 \exp\{-2(\gamma_i - \sigma\omega^{-2})t\} \\ & -2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} \exp\{-2(\gamma_i - \epsilon\omega^{-2})(t-s)\} (\nabla u_{im}(x,s))^2 dx ds \\ & + 2\sum_{j=1}^N \int_0^t \exp\{-2(\gamma_i - \epsilon\omega^{-2})(t-s)\} a_{ij} p_i(s) p_j(s) ds \\ & + 2\sum_{j=1}^N \int_0^t \exp\{-2(\gamma_i - \epsilon\omega^{-2})(t-s)\} b_{ij} p_i(s) p_j(s) ds, \quad t \geq 0, \end{aligned}$$

where $p_i(t)$ is defined by (4). Then in view of (9), Hölder's inequality and the definitions of a_{ij} and b_{ij}, we see that $p_i^2(t) \leq W_i(t)$. Note that for $t \geq 0$,

$$\frac{dW_i(t)}{dt} \le -2(\gamma_i - \epsilon\omega^{-2})W_i(t) - 2\epsilon \sum_{m=1}^{n_i} \int_\Omega (\nabla u_{im}(x,t))^2 dx$$
$$+ 2 \sum_{j=1, j\neq i}^{N} a_{ij} W_i^{1/2}(t) W_j^{1/2}(t) + 2\sum_{j=1}^{N} b_{ij} W_i^{1/2}(t) W_j^{1/2}(t-\tau). \tag{20}$$

If we denote $W_i^{1/2}(t)$ by $Z_i(t)$, then

$$\frac{dZ_i(t)}{dt} + (\gamma_i - \epsilon\omega^{-2})Z_i(t)$$
$$\le -\frac{2\epsilon}{Z_i(t)} \sum_{m=1}^{n_i} \int_\Omega (\nabla u_{im}(x,t))^2 dx + 2 \sum_{j=1, j\neq 1}^{N} a_{ij} Z_j(t)$$
$$+ 2\sum_{j=1}^{N} b_{ij} Z_j(t-\tau). \tag{21}$$

Multiply (21) by the integrating factor $\exp(\gamma_i - \epsilon\omega^{-2})$ and then integrate from $t = 0$ to t, we obtain

$$Z_i(t) \le M_i e^{-(\gamma_i - \epsilon\omega^{-2})t} + \sum_{j=1, j\neq i}^{N} \int_0^t e^{-(\gamma_i - \epsilon\omega^{-2})(t-s)} a_{ij} Z_j(s) ds$$
$$- 2\epsilon \int_0^t \int_\Omega Z_i^{-1}(s) \sum_{m=1}^{n_i} e^{-(\gamma_i - \epsilon\omega^{-2})(t-s)} (\nabla u_{im}(x,s))^2 dx ds \tag{22}$$
$$+ \sum_{j=1}^{N} \int_0^t e^{-2(\gamma_i - \epsilon\omega^{-2})(t-s)} b_{ij} Z_j(s-\tau) ds.$$

By means of arguments similar to those used to derive (12), we obtain

$$e^{\sigma t} Z_i(t) \le M_i - 2\epsilon \int_0^t \int_\Omega Z_i^{-1}(s) \sum_{m=1}^{n_i} e^{-\xi_i(t-s)} e^{\sigma s} (\nabla u_{im}(x,s))^2 dx ds$$
$$+ \xi_i^{-1} \sum_{j=1}^{N} (a_{ij}(1-\delta_{ij}) + b_{ij} e^{\sigma\tau}) \sup_{-\tau \le s \le t} (e^{\sigma s} Z_j(s)), \tag{23}$$

where δ_{ij} is the Dirac delta function. For convenience, define

$$f_i(t) = \sup_{-\tau \le s \le t} (e^{\sigma s} Z_i(s)).$$

Then similar to the derivation of (14), we may show that

$$f_i(t) \leq 2M_i + \xi_i^{-1} \sum_{j=1}^{N} (a_{ij}(1-\delta_{ij}) + b_{ij} e^{\sigma\tau}) f_j(t) = 2M_i + \sum_{j=1}^{N} s_{ij} f_j(t). \tag{24}$$

From (24), it is easy to see that

$$f_i(t) \leq K_1 \sum_{i=1}^{N} M_i$$

for some positive constant K_1. Indeed, $\rho(S) < 1$ implies $(I-S)^{-1}$ exists and is non-negative [7], thus from (24), we have

$$(f_1(t), \ldots, f_N(t))^T \leq 2(M_1, \ldots, M_N)^T + S(f_1(t), \ldots, f_N(t))^T,$$

so that

$$(f_1(t), \ldots, f_N(t))^T \leq 2(I-S)^{-1}(M_1, \ldots, M_N)^T,$$

or

$$f_i(t) \leq \sum_{j=1}^{N} s_{ij}^* M_j \leq K_1 \sum_{j=1}^{N} M_j,$$

where $(s_{ij}^*) = 2(I-S)^{-1}$ and $K_1 = \max_{i,j}\{s_{ij}^*\}$.

Note that from $f_i(t) \leq K_1(M_1 + \ldots + M_N)$, we have

$$Z_i(t) \leq e^{-\sigma t} f_i(t) \leq K_1 e^{-\sigma t} \sum_{j=1}^{N} M_j \equiv K_2 e^{-\sigma t},$$

where $K_2 = K_1(M_1 + \ldots + M_N)$, and

$$\begin{aligned} M_i + \sum_{j=1}^{N} s_{ij} f_j(t) &\leq M_i + K_3 \sum_{j=1}^{N} f_j(t) \\ &\leq M_i + NK_1K_3 \sum_{j=1}^{N} M_j \leq (1 + NK_1K_3) \sum_{j=1}^{N} M_j, \end{aligned} \tag{25}$$

where $K_3 = \max_{ij}\{s_{ij}\}$. Substituting these inequalities into (23), we obtain

$$\begin{aligned} e^{\sigma t} Z_i(t) + 2\epsilon \int_0^t \int_\Omega \frac{e^{\sigma t}}{K_2} \sum_{m=1}^{n_i} e^{-\xi_i(t-s)} e^{\sigma s} (\nabla u_{im}(x,s))^2 dx ds \\ \leq (1 + NK_1K_3) \sum_{j=1}^{N} M_j \end{aligned} \tag{26}$$

A lower bound similar to (16) can be derived for the integral term, so that for any $0 < a < t$, there is η in $[t-a, t]$ such that an inequality similar to (17) holds:

$$\sum_{i=1}^{N} e^{\sigma t} Z_i(t) + \frac{2\epsilon a}{K_2} \sum_{i=1}^{N} \sum_{m=1}^{n_i} e^{-\xi_i a} \int_\Omega e^{\sigma\eta} (\nabla u_{im}(x,\eta))^2 dx$$
$$\leq N(1 + NK_1K_3) \sum_{i=1}^{N} M_i.$$

It is clear now by arguments similar to those in the proof of Theorem 1, we may show that

$$\lim_{t\to\infty} p_i(t) \leq \lim_{t\to\infty} W_i^{1/2}(t) = \lim_{t\to\infty} Z_i(t) = 0$$

and

$$\lim_{t\to\infty} \|\nabla u_i(x,t,\phi)\|_{L^2} = 0.$$

The proof is complete.

Theorem 3. *Let $V = (v_{ij})$ be the matrix defined by*

$$v_{ii} = \frac{b_{ii} + \sum_{i=1}^{N} (a_{ij}(1-\delta_{ij}) + b_{ij})}{2\gamma_i},$$

and

$$v_{ij} = \frac{a_{ij} + b_{ij}}{2\gamma_i}, \quad i \neq j.$$

Suppose $\gamma_i > 0$ for $1 \leq i \leq N$, and suppose the spectral radius $\rho(V)$ is less than 1*, then the trivial solution of* (1-3) *is asymptotically $\aleph^{1,2}$-stable.*

Proof. Since $\rho(V) < 1$, by continuity, there are positive numbers ϵ and σ such that $\xi_i \equiv (\gamma_i - \epsilon\omega^{-2} - \sigma) > 0$ and the spectral radius $\rho(Y)$ of the matrix Y is less than 1, where $Y = (y_{ij})$ is defined by

$$y_{ii} = \frac{b_{ii}e^{2\sigma\tau} + \sum_{i=1}^{N} (a_{ij}(1-\delta_{ij}) + b_{ij})}{2\gamma_i}$$

and

$$y_{ij} = v_{ij} = \frac{a_{ij} + b_{ij}e^{2\sigma\tau}}{2\gamma_i}, \ i \neq j.$$

In view of (9), we have

$$\begin{aligned} p_i^2(t) \leq M_i^2 e^{-2(\gamma_i - \epsilon\omega^{-2})t} &- 2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2(\gamma_i - \epsilon\omega^{-2})(t-s)} (\nabla u_{im}(x,s))^2 dxds \\ &+ 2 \sum_{j=1, j\neq i}^{N} \int_0^t e^{-2(\gamma_i - \epsilon\omega^{-2})(t-s)} a_{ij}\big(p_i^2(s) + p_j^2(s)\big)ds \\ &+ 2 \sum_{j=1}^{N} \int_0^t e^{-2(\gamma_i - \epsilon\omega^{-2})(t-s)} b_{ij}\big(p_i^2(s) + p_j^2(s-\tau)\big)ds, \end{aligned}$$

where $p_i(t)$ is defined by (4) and $M_i = \sup_{-\tau \leq t \leq 0} p_i(t)$. Thus it follows that

$$\begin{aligned} &e^{2\sigma t} p_i^2(t) \\ \leq &M_i^2 e^{-2\xi_i t} - 2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2\xi_i(t-s)} e^{2\sigma s} (\nabla u_{im}(x,s))^2 dxds \\ &+ \sum_{j=1}^{N} \int_0^t e^{-2\xi_i(t-s)} (a_{ij}(1-\delta_{ij}) + b_{ij}) e^{2\sigma s} p_i(s) ds \\ &+ \sum_{j=1, j\neq i}^{N} \int_0^t e^{-2\xi_i(t-s)} a_{ij} e^{2\sigma s} p_i(s) ds \\ &+ \sum_{j=1}^{N} \int_0^t e^{-2\xi_i(t-s)} b_{ij} e^{2\sigma\tau} e^{2\sigma(s-\tau)} p_j(s-\tau) ds \\ \leq &M_i^2 - 2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2\xi_i(t-s)} e^{2\sigma s} (\nabla u_{im}(x,s))^2 dxds \\ &+ \sum_{j=1}^{N} \frac{a_{ij}(1-\delta_{ij}) + b_{ij}}{2\xi_i} Q_i(t) + \sum_{j=1}^{N} \frac{a_{ij}(1-\delta_{ij}) + b_{ij}e^{2\sigma\tau}}{2\gamma_i} Q_j(t), \end{aligned}$$

where

$$Q_i(t) \equiv \sup_{-\tau \leq s \leq t} \big(e^{2\sigma s} p_i(s)\big).$$

By means of arguments similar to those used to derive (14) (cf. (24)), we obtain

$$Q_i(t) \le 2M_i^2 + \sum_{j=1}^{N} \frac{a_{ij}(1-\delta_{ij}) + b_{ij}}{2\xi_i} Q_i(t) + \sum_{j=1}^{N} \frac{a_{ij}(1-\delta_{ij}) + b_{ij}e^{2\sigma\tau}}{2\gamma_i} Q_j(t)$$

$$= 2M_i^2 + \sum_{j=1}^{N} y_{ij} Q_j(t).$$

As in the proof of theorem 2 (cf. (25)), we may show that

$$2M_i^2 + \sum_{j=1}^{N} y_{ij} Q_j(t) \le K^* \sum_{i=1}^{N} M_i^2,$$

for some positive constant K^*. Thus we obtain

$$e^{\sigma t} p_i^2(t) + 2\epsilon \int_0^t \int_\Omega \sum_{m=1}^{n_i} e^{-2\xi_i(t-s)} e^{2\sigma s} (\nabla u_{im}(x,s))^2 dx ds \le K^{**} \sum_{i=1}^{N} M_i^2,$$

for some positive constant K^{**}. The rest of the proof is similar to that of Theorem 1 (see explanations after (26)). This concludes our proof.

We close this paper by remarking that, in view of the proof of Corollary 1, under the assumptions of Theorem 2 or Theorem 3, the conclusion of Corollary 1 still holds.

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