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REAL ANALYTIC REGULARITY OF THE BERGMAN KERNEL FUNCTION

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Abstract. In this note we prove the real analytic regularity of the Bergman kernel function in joint variables on pseudoconvex circular domains.

1. Introduction. In several complex variables the Bergman kernel function $K_\Omega(z, w)$ associated with a smooth bounded domain Ω has been shown to be closely related to the extension problem of a biholomorphic mapping between two such domains which in turn has an important consequence concerning the classification of domains in higher dimensional space. Thus, understanding the boundary behavior of the Bergman kernel function has become an extremely important issue.

Now let Ω be a smooth bounded pseudoconvex domain in C^n with $n \geq 2$. It was first proved by Kerzman [10] that the Bergman kernel function $K_\Omega(z, w)$ associated with a smooth bounded strictly pseudoconvex domain Ω can be extended smoothly to $\overline{\Omega} \times \overline{\Omega} - \Delta(b\Omega)$, where $\Delta(b\Omega) = \{(z, z) | z \in b\Omega\}$. Later it was generalized independently by Bell [2] and Boas [4] to the follows.

Theorem A. *Let Ω be a smooth bounded pseudoconvex domain in C^n. Let Γ_1 and Γ_2 be two disjoint open subsets of $b\Omega$ consisting of points of finite type in the sense of D'Angelo [7]. Then the Bergman kernel function associated with Ω extends smoothly to $\Gamma_1 \times \Gamma_2$.*

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Theorem B. *Let Ω be a smooth bounded pseudoconvex domain in C^n. Suppose that condition R holds on Ω. Let Γ_1 and Γ_2 be two disjoint open subsets of $b\Omega$ such that Γ_1 consists of points of finite type. Then the Bergman kernel function associated with Ω extends smoothly to $\Gamma_1 \times \Gamma_2$.*

Condition R here means that the orthogonal projection P from $L^2(\Omega)$ onto the subspace of sqaure-integrable holomorphic functions maps $C^\infty(\overline{\Omega})$ continuously into itself.

But, in general, the smooth extension phenomenum of the Bergman kernel function in joint variables is false. A counterexample has been recently discovered by the author [6].

In fact, we showed in Chen [6] that if there is a complex variety V sitting in the boundary $b\Omega$ of Ω, then

$$K_\Omega(z,w) \notin C^\infty(\overline{\Omega} \times \overline{\Omega} - \Delta(b\Omega)).$$

The purpose of this article is to investigate the real analytic extension problem of the Bergman kernel function in joint variables. So far very little was known in this direction. The following result was obtained by Bell [3].

Theorem C. *Let Ω be defined as above, and let a and b be two distinct boundary points. Suppose that b is a point of finite type in the sense of D'Angelo and that the boundary of Ω is real analytic near a and that the $\overline{\partial}$-Neumann problem on Ω satisfies local condition Q at a, then there exists disjoint balls B_a and B_b centered at a and b respectively such that $K(z,w)$ extends to be in $C^\infty(B_a \times (\overline{\Omega} \cap B_b))$ as a function which is holomorphic in z and antiholomorphic in w on $B_a \times (\Omega \cap B_b)$.*

Here we follow the definition given in Bell [3]. By saying that the $\overline{\partial}$-Neumann problem on Ω satisfies local condition Q at p we mean that the boundary $b\Omega$ is real analytic and of finite type near p and that N_α extends to be real analytic at p whenever α is a form in $L^2_{0,1}(\Omega)$ which is supported away from p, where N is the Neumann operator. Based on the works by Tartakoff [12] and Treves [13] we know that the domain Ω satisfies local condition Q at every real analytic strictly pseudoconvex point, and at some weakly pseudoconvex points on certain model hypersurface. For instance, see Derridj and Tartakoff [8] [9].

In this note we shall first prove an analytic version of Theorem B.

Theorem 1. *Let Ω be a smooth bounded pseudoconvex domain in C^n with $n \geq 2$. Let p and q be two distinct boundary points. Suppose that condition R holds on Ω, and that Ω satisfies local condition Q at p, then the same conclusion for the kernel function $K(z, w)$ as stated in Theorem C holds near (p, q).*

Note that if q is of finite type, then Theorem 1 is reduced to Theorem C. Here we allow q to be of infinite type.

Theorem 1 will be applied to prove the real analytic extension of the Bergman kernel function in joint variables. Let Ω be a smoothly bounded domain in C^n. Ω is said to be invariant under S^1-action if $z = (z_1, \ldots, z_n) \in \Omega$ implies that $e^{i\theta} \cdot z \in \Omega$ for $\theta \in R$. Here $e^{i\theta} \cdot z$ means that we multiply some or all of the components $z_i's$ by $e^{i\theta}$. Hence for each z near the boundary $b\Omega$, if the mapping

$$\Lambda : S^1 \to \overline{\Omega}$$

$$e^{i\theta} \mapsto e^{i\theta} \cdot z,$$

generates a curve through z in $\overline{\Omega}$, then the differential mapping Λ_* of Λ will induce a tangential vector T_z at z by

$$T_z = \Lambda_* \left(\frac{\partial}{\partial \theta} |_{\theta=0} \right).$$

In particular, if $z \in b\Omega$, then T_z will be tangent to the boundary $b\Omega$. So now if we let z vary in the boundary $b\Omega$, we obtain a globally defined tangential vector field T. Here we recall briefly the definition of symmetry. We shall say that the domain Ω has rotational symmetry near a boundary point z if the domain Ω is invariant under the S^1-action, and if the tangential vector field T generated by the S^1-action points in the missing direction, namely, $T_z \notin T_z^{1,0}(b\Omega) \oplus T_z^{0,1}(b\Omega)$. Then we prove

Theorem 2. *Let Ω be a smooth bounded pseudoconvex domain in C^n with $n \geq 2$. Let the boundary $b\Omega$ be real analytic near two distinct boundary points p and q. Suppose that condition R holds on Ω, and that Ω satisfies local condition Q at p, and that Ω has rotational symmetry at q, then there exists two disjoint balls B_p and B_q centered at p and q respectively such that the Bergman kernel function $K(z,w)$ extends real analytically to be in $C^\omega(B_p \times B_q)$ which is holomorphic in z and antiholomorphic in w.*

In Theorem 2, instead of assuming local condition Q at q, we impose an easier and more geometric condition near q. One should compare the above theorem with the Theorem 2 in Bell [3].

Then an immediate consequence of Theorem 2 is the following

Theorem 3. *Let Ω be a smooth bounded pseudoconvex domain with real analytic boundary in C^n with $n \geq 2$. Suppose that Ω is either Reinhardt or circular with $\sum_{j=1}^n z_j \frac{\partial r}{\partial z_j}(z) \neq 0$ near the boundary, where $r(z)$ is a real analytic defining function for Ω. Then the Bergman kernel function $K(z,w)$ associated with Ω extends real analytically across the boundary in joint variables near any pair of boundary points (p,q) with $p \neq q$, provided that Ω is strictly pseudoconvex at one of these two points.*

To conclude the introduction we give an example below.

Example 4. One can easily construct a smooth bounded convex complete Reinhardt domain Ω in C^2 such that Ω has real analytic boundary and is Levi flat on $q = (z,w) \in b\Omega$ with $|w| < \delta$ for some $\delta > 0$, and that the boundary $b\Omega$ is real analytic and of strict pseudoconvexity near $p = (0,w_0) \in b\Omega$ for some w_0. Then Theorem 2 implies that the kernel function $K(z,w)$ extends real analytically across the boundary in joint variables near (p,q). Note that in this example the point q is of infinite type.

2. Proofs of Theorems 1 and 2. Let Ω be a smooth bounded pseudoconvex domain in C^n with $n \geq 2$. Suppose that condition R holds on Ω. Let p and q be two distinct boundary points such that the boundary $b\Omega$ is real analytic and of finite type near p. Let us assume that Ω satisfies local condition Q at p.

We shall now prove Theorem 1 first by following the line developed in Bell [2]. Let $\epsilon > 0$ be a small positive number such that $B(p;2\epsilon)$ and $B(q;2\epsilon)$ are disjoint. Construct s smooth bounded finite type pseudoconvex subdomain $D \subseteq \Omega \cap B(p;\epsilon)$ whose boundary coincides with $b\Omega$ near p, namely, for some small η with $\epsilon > \eta > 0$, we have $D \cap B(p;\eta) = \Omega \cap B(p;\eta)$. We may also assume that η is so small that the boundary bD is real analytic near $bD \cap \overline{B(p;\eta)}$. Thus, the Bergman kernel function $K_D(z,w)$ associated with D extends smoothly to $\overline{D} \times \overline{D} - \{(z,z)|z \in bD\}$.

Let χ be the cut-off function which is equal to zero on $B(p;\frac{\eta}{2})$ and equal to one on $C^n - B(p;\eta)$. Let s be a positive integer, and let Φ_D^s be the operator constructed in Bell [1] that satisfies the following properties:

(i) Φ_D^s is a bounded operator in $W^s(D)$ norm from $H^s(D)$ into $W_0^s(D)$. Here $W_0^s(D)$ denotes the closure of $C_0^\infty(D)$ in $W^s(D)$,

(ii) Φ_D^s is a linear differential operator with coefficients in $C^\infty(\overline{D})$,

(iii) $P_D\Phi_D^s = P_D$ on $L^2(D)$.

Let ζ be a point in $Y = B(p;\frac{\eta}{3}) \cap \Omega$, and let $h_\zeta^\alpha = \frac{\partial^\alpha}{\partial \bar{\zeta}^\alpha} K_D(z,\zeta)$. Notice that both $\|\chi h_\zeta^\alpha\|_{W^s(D)}$ and $\|\Phi_D^s(\chi h_\zeta^\alpha)\|_{W^s(\Omega)}$ are uniformly bounded as ζ varies in Y. The latter norm is taken by extending the function $\Phi_D^s(\chi h_\zeta^\alpha)$ by zero on $\Omega - D$.

The function $(1-\chi)h_\zeta^\alpha$ is in $C^\infty(\overline{\Omega})$ when it is extended to Ω via extension by zero, and $\overline{\partial}[(1-\chi)h_\zeta^\alpha]$ is a (1,0)-form on Ω whose coefficients are uniformly bounded in $W^s(\Omega)$ norm as ζ varies in Y. Thus, by Kohn's theorem [11], there is a function ν_ζ^α in $W^s(\Omega)$ such that

$$\overline{\partial} v_\zeta^\alpha = \overline{\partial}[(1-\chi)h_\zeta^\alpha]$$

and that

$$\|v_\zeta^\alpha\|_{W^s(\Omega)} \leq C\|\overline{\partial}[(1-\chi)h_\zeta^\alpha]\|_{W^s(\Omega)},$$

where the constant C is independent of α and ζ. Then it was shown in Bell [2] that the Bergman kernel function $K_\Omega(z,w)$ for Ω satisfies

$$\frac{\partial^\alpha}{\partial\overline{\zeta}^\alpha}K_\Omega(z,\zeta)=P_\Omega(\Phi_D^s(\chi h_\zeta^\alpha))+P_\Omega v_\zeta^\alpha+(1-\chi)h_\zeta^\alpha-v_\zeta^\alpha. \tag{2.1}$$

Next, if $\eta>0$ is sufficiently small, notice that for each multiindex β, by Theorem C there exists a constant $C_\beta>0$ such that

$$\left|\frac{\partial^{s+t}}{\partial z^s\partial\overline{z}^t}(\chi h_\zeta^\alpha)\right|\le C_\beta C_\beta^{|\alpha|}|\alpha|!, \tag{2.2}$$

for any $(z,\zeta)\in D\times Y$ and that

$$\left|\frac{\partial^{s+t}}{\partial z^s\partial\overline{z}^t}\overline{\partial}[(1-\chi)h_\zeta^\alpha]\right|\le C_\beta C_\beta^{|\alpha|}|\alpha|!, \tag{2.3}$$

for any $(z,\zeta)\in\Omega\times Y$, where s and t are two multiindices with $s+t=\beta$.

Now, given any $k\in N$, let the integer $s\ge k$ be so chosen that P_Ω maps $W^s(\Omega)$ continuously into $W^k(\Omega)\cap H(\Omega)$. We shall estimate the L^2-norm of $\frac{\partial^{\beta+\alpha}}{\partial z^\beta\partial\overline{\zeta}^\alpha}K_\Omega(z,\zeta)$ on $U\times Y$ for $|\beta|\le k$, where $U=B(q;\epsilon)\cap\Omega$.

Notice that $(1-\chi)h_\zeta^\alpha\equiv 0$ for any $(z,\zeta)\in U\times Y$, and that the L^2-norm on $U\times Y$ is clearly dominated by the L^2-norm on $\Omega\times Y$. Thus the estimates for the remaining three terms in (2.1) can be obtained as follows. By (2.2) we get

$$\begin{aligned}\left\|\frac{\partial^\beta}{\partial z^\beta}P_\Omega(\Phi_D^s(\chi h_\zeta^\alpha))\right\|_{L^2(\Omega\times Y)}^2&=\int_Y\int_\Omega\left|\frac{\partial^\beta}{\partial z^\beta}P_\Omega(\Phi_D^s(\chi h_\zeta^\alpha))\right|^2d\nu_z d\nu_\zeta\\&\lesssim\int_Y\|\Phi_D^s(\chi h_\zeta^\alpha)\|_{W^s(\Omega)}^2d\nu_\zeta\\&=\int_Y\|\Phi_D^s(\chi h_\zeta^\alpha)\|_{W^s(D)}^2d\nu_\zeta\\&\le C_k C_k^{|\alpha|}|\alpha|!\end{aligned}$$

for some $C_k>0$. C_k depends on k. The notation $A\lesssim B$ means that $A\le CB$ for some constant C independent of A and B. The second and fourth terms in (2.1) can be estimated similarly, and by (2.3) we have

$$\begin{aligned}\left\| \frac{\partial^\beta}{\partial z^\beta} P_\Omega v_\zeta^\alpha \right\|^2_{L^2(\Omega\times Y)} &\lesssim \int_Y \|v_\zeta^\alpha\|^2_{W^s(\Omega)} d\nu_\zeta \\ &\lesssim \int_Y \|\bar\partial[(1-\chi)h_\zeta^\alpha]\|^2_{W^s(\Omega)} d\nu_\zeta \\ &\leq C_k C_k^{|\alpha|}|\alpha|!.\end{aligned}$$

$$\begin{aligned}\left\| \frac{\partial^\beta}{\partial z^\beta} v_\zeta^\alpha \right\|^2_{L^2(\Omega\times Y)} &\lesssim \int_Y \|v_\zeta^\alpha\|^2_{W^s(\Omega)} d\nu_\zeta \\ &\leq C_k C_k^{|\alpha|}|\alpha|!.\end{aligned}$$

The above estimates show that the kernel function $K_\Omega(z,\zeta)$ extends to be in $C^\infty(B(p;\delta)\times(B(q;\epsilon)\cap\overline{\Omega}))$ as a function which is holomorphic in z and antiholomorphic in ζ for some small $\delta > 0$ with $\eta > \delta > 0$. This completes the proof of Theorem 1.

Next we prove Theorem 2. By assumption the domain Ω is invariant under S^1-action. Therefore, for each $z \in \overline{\Omega}$, there is a mapping Λ from S^1 to $\overline{\Omega}$ defined by

$$\Lambda : S^1 \to \overline{\Omega}$$

$$e^{i\theta} \mapsto e^{i\theta}\cdot z.$$

Here $e^{i\theta}\cdot z$ means that we multiply some of the coordinates of z by $e^{i\theta}$. Then the differential mapping Λ_* of Λ induces a tangential vector field T at z as follows,

$$T_z = \Lambda_*\left(\frac{\partial}{\partial\theta}|_{\theta=0}\right).$$

By our hypothesis we see that the vector field T is real, real analytic and

$$T_q \notin T_q^{1,0}(b\Omega)\oplus T_q^{0,1}(b\Omega).$$

It is also clear that for any smooth function f on $\overline{\Omega}$ we have

$$Tf(z) = \frac{\partial}{\partial\theta} f(e^{i\theta}\cdot z)|_{\theta=0}.$$

Now, since Ω satisfies local condition Q at p and Condition R holds on Ω, hence by Theorem 1 there exists disjoint balls B_p and B_q centered at p and q respectively such that for any multiindex α the following estimate holds

$$\left|\frac{\partial^\alpha}{\partial z^\alpha} K_\Omega(z,w)\right| \le C C^{|\alpha|} |\alpha|!, \tag{2.4}$$

for any $(z,w) \in B_p \times (\overline{\Omega} \cap B_q)$, where $C > 0$ is an universal constant.

We may assume that the radius of B_q is so small that the vector field T is transversal to $T^{1,0}(b\Omega) \oplus T^{0,1}(b\Omega)$ on $b\Omega \cap \overline{B}_q$, and that the boundary $b\Omega$ is real analytic near $b\Omega \cap \overline{B}_q$. Since Ω is invariant under S^1-action, it is not hard to see that we have

$$\begin{aligned} T_w K_\Omega(z,w) &= \frac{d}{d\theta} K_\Omega(z, e^{i\theta} \cdot w)|_{\theta=0} \\ &= \frac{d}{d\theta} K_\Omega(e^{i\theta} \cdot e^{-i\theta} \cdot z, e^{i\theta} \cdot w)|_{\theta=0} \\ &= \frac{d}{d\theta} K_\Omega(e^{-i\theta} \cdot z, w)|_{\theta=0} \\ &= -T_z K_\Omega(z,w). \end{aligned}$$

Hence it follows from the above observation that for any multiindex α and any $k \in N \cup \{0\}$ we obtain

$$\frac{\partial^\alpha}{\partial z^\alpha}(T_w^k K_\Omega(z,w)) = (-1)^k \frac{\partial^\alpha}{\partial z^\alpha}(T_z^k K_\Omega(z,w)).$$

Then by estimate (2.4) we get

$$\left|\frac{\partial^\alpha}{\partial z^\alpha}(T_w^k K_\Omega(z,w))\right| \le C C^{|\alpha|+k} (|\alpha| + k)!, \tag{2.5}$$

for all $(z,w) \in B_p \times (\overline{\Omega} \cap B_q)$.

From estimate (2.5) it is then standard to see that the Bergman kernel function extends real analytically across the boundary in joint variables near (p,q). For instance, see Chen [5]. The proof of Theorem 2 is now complete.

To prove Theorem 3 we simply observe that under the hypotheses of the theorem the domain Ω enjoys rotational symmetry at every boundary point, and that local condition Q holds at every strictly pseudoconvex point. Hence Theorem 2 applies.

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