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MONOTONICITY AND SUMMABILITY OF SOLUTIONS OF A SECOND ORDER NONLINEAR DIFFERENCE EQUATION

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Abstract. The authors consider the nonlinear difference equation

$$(*) \qquad \Delta(p_{n-1}f(y_{n-1})\Delta y_{n-1}) = q_n g(y_n)$$

and obtain results on the asymptotic behavior of solutions of (*) including sufficient conditions for all solutions to bounded. Some results comparing the behavior of solutions of (*) to solutions of the linear equation

$$\Delta(p_{n-1}\Delta y_{n-1}) = q_n y_n$$

are also obtained. The nonlinear limit-point/limit-circle problem is introduced, and a sufficient condition for all solutions of (*) to be of the nonlinear limit-point type is proved.

1. Introduction. In this paper we study the asymptotic behavior of solutions of nonlinear difference equations of the type

$$(1) \qquad \Delta(p_{n-1}f(y_{n-1})\Delta y_{n-1}) = q_n g(y_n)$$

where Δ denotes the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{p_n\}$ and $\{q_n\}$ are real sequences, and the functions $f, g : \mathbb{R} \to \mathbb{R}$ are

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continuous. By a solution of (1) we mean a real sequence $\{y_n\}$ satisfying (1) for $n \geq n_0$ for some $n_0 \geq 0$. We consider only such solutions which are nontrivial for all large n.

The following conditions are assumed to hold throughout the remainder of this paper . There is an $N_0 \in \mathbb{N} = \{1, 2, \ldots\}$ such that:

(i) $p_n > 0$ and $q_n \geq 0$ for $n \geq N_0$ and $q_n \not\equiv 0$ for $n \geq N_1$ for any $N_1 \geq N_0$;

(ii) $f(r) > 0$ and $rg(r) > 0$ for all $r \neq 0$.

A solution $\{y_n\}$ of (1) is *bounded* if there exists $M > 0$ such that $|y_n| \leq M$ for all $n \geq N_0$, and $\{y_n\}$ is *nonoscillatory* if the terms y_n are eventually all of the same sign.

The qualitative behavior of solutions of special cases of equation (1), especially when $q_n \leq 0$, have been examined by a number of authors, for example, see [2, 3, 6-15] and the references contained therein. For additional references, the reader is referred to the recent monograph by Agarwal [1]. In this paper we study the asymptotic behavior of the solutions of equation (1), and in Section 2, we give some sufficient conditions for all solutions to be bounded. This extends a results for linear equations obtained in [3]. Section 3 contains some results on the asymptotic relationships between the solutions of (1) and the solutions of the second order linear difference equation

$$\Delta(p_{n-1}\Delta u_{n-1}) = q_n u_n. \tag{2}$$

In Section 4, we introduce the study of nonlinear limit-point/limit-circle problems for difference equations.

2. Monotonicity. We begin with two propositions giving some basic information about the behavior of solutions of (1).

Proposition 1. *Every solution of equation* (1) *is nonoscillatory.*

Proof. Let $\{y_n\}$ be an oscillatory solution of (1). Choose an integer $N \geq N_0$ such that $y_{N-1} \leq 0$ and $y_N > 0$. Now from (1)

$$\Delta y_{N-1}[\Delta(p_{N-1}f(y_{N-1})\Delta y_{N-1})] = (\Delta y_{N-1})q_N g(y_N) > 0,$$

so

$$\Delta y_{N-1}[p_N f(y_N)\Delta y_N - p_{N-1} f(y_{N-1})\Delta y_{N-1}] > 0.$$

Hence,

$$p_N f(y_N)\Delta y_N \Delta y_{N-1} > p_{N-1} f(y_{N-1})(\Delta y_{N-1})^2 > 0,$$

from which it follows that $\Delta y_N > 0$. Thus, $y_{N+1} > y_N > 0$. Repeating the above argument by taking equation (1) at $n = N+1$ and multiplying by Δy_N, we obtain $\Delta y_{N+1} > 0$ so $y_{N+2} > 0$. Continuing this process, we have $y_n > 0$ for $n \geq N$, which is a contradiction.

Proposition 2. *Let $\{y_n\}$ be any solution of (1). Then $\{y_n\}$ is either eventually increasing or eventually decreasing.*

Proof. Since equation (1) is nonoscillatory, we may assume y_n is of one sign, say $y_n > 0$ for all $n \geq N$ for some $N \geq N_0$. Consider the sequence $\{F_n\}$ defined by

$$F_n = y_n p_n f(y_n)\Delta y_n, \quad n \geq N. \tag{3}$$

Then

$$\begin{aligned} \Delta F_n &= y_{n+1}\Delta(p_n f(y_n)\Delta y_n) + p_n f(y_n)(\Delta y_n)^2 \\ &= y_{n+1} q_{n+1} g(y_{n+1}) + p_n f(y_n)(\Delta y_n)^2 \\ &\geq 0. \end{aligned} \tag{4}$$

Suppose there exists an integer $N_1 \geq N$ such that $\Delta y_{N_1} > 0$. Since y_n, p_n, and $f(y_n)$ are positive, (3) and (4) imply that $\Delta y_n > 0$ for $n \geq N_1$, which in turn implies $\{y_n\}$ is increasing for $n > N_1$.

If there does not exist an integer $N_1 \geq N$ such that $\Delta y_{N_1} > 0$, then it must be the case that $\Delta y_n \leq 0$ for $n \geq N_1$. Now for any $n_1 \geq N_1$ there exists $n_2 > n_1$ such that $q_{n_2} > 0$. Hence

$$\begin{aligned}
0 &\geq p_{n_2} f(y_{n_2})\Delta y_{n_2} \\
&= p_{n_2-1} f(y_{n_2-1})\Delta y_{n_2-1} + q_{n_2} g(y_{n_2}) \\
&> p_{n_2-1} f(y_{n_2-1})\Delta y_{n_2-1} \\
&= p_{n_2-2} f(y_{n_2-2})\Delta y_{n_2-2} + q_{n_2-1} g(y_{n_2-1}) \\
&\geq p_{n_2-2} f(y_{n_2-2})\Delta y_{n_2-2} \\
&= p_{n_2-3} f(y_{n_2-3})\Delta y_{n_2-3} + q_{n_2-2} g(y_{n_2-2}) \\
&\geq \cdots \\
&\geq p_N f(y_N)\Delta y_N.
\end{aligned}$$

Therefore, $\Delta y_n < 0$ for every n with $N \leq n \leq n_2 - 1$. Since such arbitrarily large n_2 always exist, we have $\Delta y_n < 0$ for all $n \geq N$, i.e., $\{y_n\}$ is decreasing.

Remark 1. Proposition 2 generalizes to nonlinear equations some results in [3] that are for linear equations (see [3; Lemmas 1 and 2]).

We now divide the set of all solutions of equation (1) (or (2)) into two classes. A solution is said to belong to Class $\mathcal{A}$ if it is eventually either positive and increasing or negative and decreasing; a solution belongs to Class $\mathcal{B}$ if it is eventually either positive and decreasing or negative and increasing.

Clearly, all solutions of Class $\mathcal{B}$ are bounded; for solutions of (2) of Class $\mathcal{A}$ we recall the following known results.

Proposition 3. ([3, Theorem 4]) *Every (Class $\mathcal{A}$) solution of* (2) *is bounded if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \sum_{k=1}^{n} q_k < \infty. \tag{5}$$

Proposition 4. ([3, Lemma 6]) *For every (Class $\mathcal{A}$) solution $\{u_n\}$ of* (2), *the sequence $\{p_n \Delta u_n\}$ is bounded if and only if*

$$\sum_{n=1}^{\infty} q_{n+1} \sum_{k=1}^{n} \frac{1}{p_k} < \infty. \tag{6}$$

Proposition 5. ([3, Theorem 6]) *Assume that equation* (2) *has unbounded solutions. Then every solution of Class $\mathcal{B}$ tends to zero as $n \to \infty$ if and only if*

$$\sum_{n=1}^{\infty} q_{n+1} \sum_{k=1}^{n} \frac{1}{p_k} = \infty. \tag{7}$$

Proposition 6. *If condition* (6) *holds, then for every unbounded solution $\{v_n\}$ of* (2) *we have*

$$v_n \sim K \sum_{s=1}^{n-1} \frac{1}{p_s} \quad (K \neq 0). \tag{8}$$

Proof. It follows from (2) and Proposition 4 that every positive unbounded solution $\{v_n\}$ of (2) is of Class $\mathcal{A}$, $\{p_n \Delta v_n\}$ is monotone increasing, and

$$\lim_{n\to\infty} p_n \Delta v_n = K < +\infty.$$

If

$$\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty,$$

then $\{v_n\}$ is bounded, and this is a contradiction. Thus, we can apply L'Hôpital's rule to obtain

$$\lim_{n\to\infty} \frac{v_n}{\sum_{s=1}^{n-1} \frac{1}{p_s}} = \lim_{n\to\infty} p_n \Delta v_n = K,$$

so (8) holds.

Next, we extend some of the above results to the nonlinear equation (1).

Theorem 1. *Suppose condition* (5) *holds,*

$$f/g \quad \textit{is nondecreasing,} \tag{9}$$

and there exists $M > 0$ such that

$$g(r) \quad \text{is nondecreasing for } |r| \geq M. \tag{10}$$

Then every solution of equation (1) *is bounded.*

Proof. We can consider without loss of generality only Class $\mathcal{A}$ solutions since every solution in Class $\mathcal{B}$ is bounded. Let $\{y_n\}$ be a Class $\mathcal{A}$ solution of (1) and suppose $y_n > 0$ and $\Delta y_n > 0$ for all $n \geq N_1$ for some $N_1 \geq N_0$. (The case $y_n < 0$, $\Delta y_n < 0$ for all large n is similar, and the proof will be omitted.) If $\{y_n\}$ is unbounded, then there exists an integer $N \geq N_1$ such that $y_n \geq M$ for all $n \geq N$. Let $\{u_n\}$ be a solution of (2) that belongs to Class $\mathcal{A}$ for all $n \geq N-1$ and satisfies

$$u_N f(y_{N-1})\Delta y_{N-1} - g(y_N)\Delta u_{N-1} < 0. \tag{11}$$

From equations (1) and (2), we have

$$\frac{\Delta(p_{n-1}f(y_{n-1})\Delta y_{n-1})}{g(y_n)} = q_n = \frac{\Delta(p_{n-1}\Delta u_{n-1})}{u_n}$$

for $n \geq N$, so

$$u_n\Delta(p_{n-1}f(y_{n-1})\Delta y_{n-1}) = g(y_n)\Delta(p_{n-1}\Delta u_{n-1}).$$

Since

$$u_N\Delta(p_{n-1}f(y_{n-1})\Delta y_{n-1}) \leq u_n\Delta(p_{n-1}f(y_{n-1})\Delta y_{n-1}),$$

a summation from N to $n > N$ yields

$$u_N p_n f(y_n)\Delta y_n \leq u_N p_{N-1} f(y_{N-1})\Delta y_{N-1} + \sum_{s=N}^{n} g(y_s)\Delta(p_{s-1}\Delta u_{s-1}).$$

Using condition (10), we obtain

$$\begin{aligned}
&u_N p_n f(y_n)\Delta y_n \\
\leq &u_N p_{N-1} f(y_{N-1})\Delta y_{N-1} + g(y_n)\sum_{s=N}^{n}\Delta(p_{s-1}\Delta u_{s-1}) \\
\leq &u_N p_{N-1} f(y_{N-1})\Delta y_{N-1} + g(y_n)\Big[p_n\Delta u_n - p_{N-1}\Delta u_{N-1}\Big] \\
\leq &p_{N-1}\Big[u_N f(y_{N-1})\Delta y_{N-1} - g(y_N)\Delta u_{N-1}\Big] + g(y_n)p_n\Delta u_n,
\end{aligned}$$

and then from (9) and (11), we have

$$\frac{f(y_N)}{g(y_N)}\Delta y_n \leq \frac{f(y_n)}{g(y_n)}\Delta y_n \leq \frac{\Delta u_n}{u_N}.$$

Summing the above inequality yields a contradiction to Proposition 3.

Remark 2. Theorem 1 remains valid if we replace the conditions (9) and (10) with

$$f \text{ is nondecreasing,}$$

and there exists $C > 0$ such that

$$0 < |g(r)| < C \text{ for all } r \neq 0.$$

In the proof, $g(y_N)$ in (11) is replaced by C.

Remark 3. Theorem 1 generalizes the sufficiency part of [3; Theorem 4] (see Proposition 3 above) to nonlinear equations.

The following example shows that condition (9) cannot be eliminated from the hypotheses of Theorem 1.

Example. Consider the equation

$$\Delta(n(n-1)y_{n-1}^2)\Delta y_{n-1}) = \frac{n(4n^2+3n+1)}{(n+1)^5}y_n^5.$$

This equation has the unbounded solution $\{y_n\} = \{n+1\}$, while the corresponding linear equation has bounded solutions (see Proposition 3). Here, $f(r)/g(r) = 1/r^3$ and so condition (9) does not hold.

3. Asymptotic Comparison. In this section we prove some asymptotic relationships between the solutions of the nonlinear equation (1) and the linear equation (2).

Theorem 2. *Suppose that*

$$rf'(r) \geq 0 \textit{ for all } r \in \mathbb{R} \tag{12}$$

and there exists $M_1 > 0$ such that

$$|g(r)| - |r|f(r) \leq 0 \textit{ for } |r| \geq M_1. \tag{13}$$

Then for every Class $\mathcal{A}$ solution $\{y_n\}$ of (1), *there exists a Class $\mathcal{A}$ solution $\{u_n\}$ of* (2) *such that $L \equiv \lim_{n\to\infty}(y_n/u_n)$ exists and is finite.*

Proof. Let $\{y_n\}$ be a Class $\mathcal{A}$ solution of (1) with $y_n > 0$ and $\Delta y_n > 0$ for all $n \geq N_1 \geq N_0$. If $\{y_n\}$ is bounded, then the proof is obvious. Hence, assume $\{y_n\}$ is unbounded, i.e., $\lim_{n\to\infty} y_n = +\infty$. Then there exists an integer N such that $y_n > M_1$ for $n > N \geq N_1$. Let $\{u_n\}$ be a solution of (2) belonging to Class $\mathcal{A}$ for all $n \geq N-1$ and satisfying

$$u_{N-1}\Delta y_{N-1} - y_{N-1}\Delta u_{N-1} \leq 0. \tag{14}$$

For $n \geq N$, define

$$F_n = p_{n-1}f(y_{n-1})[u_{n-1}\Delta y_{n-1} - y_{n-1}\Delta u_{n-1}].$$

Then

$$\Delta F_n = q_n u_n[g(y_n) - y_n f(y_n)] + p_{n-1}\Delta u_{n-1}[f(y_{n-1}) - f(y_n)]y_n.$$

Since $\{y_n\}$ and $\{u_n\}$ belong to Class $\mathcal{A}$, conditions (12) and (13) imply that $\Delta F_n \leq 0$ for $n \geq N$. By (14), $F_n \leq F_N \leq 0$ for $n > N$, so $u_{n-1}\Delta y_{n-1} - y_{n-1}\Delta u_{n-1} \leq 0$ for $n > N$. Hence, $\{\frac{y_n}{u_n}\}$ is a positive nonincreasing sequence for $n > N$ and thus converges to a finite limit. The proof for an eventually negative Class $\mathcal{A}$ solution is similar.

In the next theorem we add an additional condition to the hypotheses of Theorem 2 and obtain that the limit L in the conclusion of Theorem 2 is different from zero.

Theorem 3. *Let conditions* (6), (12) *and* (13) *hold. If every Class $\mathcal{A}$ solution of* (1) *and* (3) *is unbounded, then the limit L in Theorem* 2 *is zero if and only if*

$$\lim_{r\to\pm\infty} f(r) = \infty.$$

Furthermore, in this case we have

$$L' = \lim_{n\to\infty} \frac{\Delta y_n}{\Delta u_n} = 0.$$

Proof. Let $\{y_n\}$ be an unbounded Class $\mathcal{A}$ solution of equation (1), and with no loss in generality, assume $y_n > M_1 > 0$ for all $n > N \geq N_0$. Let $\{u_n\}$ be an unbounded Class $\mathcal{A}$ solution of equation (3) satisfying (14). We will first show that $L = L' = 0$ if $\lim_{r\to\infty} f(r) = +\infty$. From equation (1) and condition (13), we have

$$\Delta(p_{n-1}f(y_{n-1})\Delta y_{n-1}) = q_n g(y_n) \leq \frac{q_n y_n}{u_n} f(y_n) u_n.$$

From the proof of Theorem 2, the sequence $\{\frac{y_n}{u_n}\}$ is nonincreasing for $n > N$, so from (2) we have

$$\Delta(p_{n-1}f(y_{n-1})\Delta y_{n-1}) \leq Aq_n f(y_n) u_n = Af(y_n)\Delta(p_{n-1}\Delta u_{n-1})$$

where $A = \frac{y_N}{u_N}$. Summing from N to n and applying condition (12), we obtain

$$p_n f(y_n)\Delta y_n \leq p_{N-1}f(y_{N-1})\Delta y_{N-1} + Af(y_n)\sum_{s=N}^{n} \Delta(p_{s-1}\Delta u_{s-1}),$$

or

$$p_n\Delta y_n \leq \frac{p_{N-1}f(y_{N-1})\Delta y_{N-1}}{f(y_n)} + A\Big[p_n\Delta u_n - p_{N-1}\Delta u_{N-1}\Big].$$

By (2) and Proposition 4, $\lim_{n\to\infty} p_n\Delta u_n$ exists and is finite. Since N can be chosen arbitrarily large, $\{p_n\Delta u_n\}$ is a Cauchy sequence, and $f(r) \to \infty$ as $r \to \infty$, a simple argument shows that

$$\lim_{n\to\infty} p_n\Delta y_n = 0.$$

Hence,

$$L' = \lim_{n\to\infty} \frac{\Delta y_n}{\Delta u_n} = \lim_{n\to\infty} \frac{p_n\Delta y_n}{p_n\Delta u_n} = 0.$$

Conversely, if $L = 0$, then we will show that $\lim_{r\to\infty} f(r) = \infty$. Suppose $\lim_{r\to\infty} f(r) = F < \infty$. The function $p_{n-1}f(y_{n-1})\Delta y_{n-1}$ is positive and nondecreasing; hence, for large n, we have

$$\frac{\Delta y_n}{\Delta u_n} = \frac{p_n f(y_n)\Delta y_n}{p_n \Delta u_n} \cdot \frac{1}{f(y_n)} \geq \frac{p_n f(y_n)\Delta y_n}{p_n \Delta u_n} \cdot \frac{1}{F} \geq h > 0. \tag{15}$$

Since $\{\frac{y_n}{u_n}\}$ is nonincreasing, we have

$$\frac{y_n}{u_n} - \frac{\Delta y_n}{\Delta u_n} = -\frac{u_{n+1}}{\Delta u_n}\Delta\Big(\frac{y_n}{u_n}\Big) \geq 0.$$

From (15), it follows that

$$\frac{y_n}{u_n} \geq \frac{\Delta y_n}{\Delta u_n} \geq h > 0,$$

and the desired contradiction $L \geq h > 0$ is reached. A similar argument holds when $y_n < -M_1 < 0$ for all $n > N$.

From Proposition 6 and Theorem 3 we have the following result.

Theorem 4. *Let conditions* (6), (12) *and* (13) *hold. For every unbounded solution* $\{y_n\}$ *of equation* (1) *the limit*

$$\lim_{n\to\infty} \frac{y_n}{\sum_{s=1}^{n-1} \frac{1}{p_s}}$$

exists and is finite. Moreover, it is different from zero if and only if the function f *is bounded.*

4. The nonlinear Limit-Point/Limit-Circle Problem. In this section we introduce the study of limit-point/limit-circle properties for nonlinear difference equations similar to what the second and third authors did for differential equations (see, for example, [4,5] and the references contained therein). We consider equation (1) with $f(r) \equiv 1$, namely,

$$\Delta(p_{n-1}\Delta y_{n-1}) - q_n g(y_n) = 0. \tag{16}$$

A solution $\{y_n\}$ of (16) will be said to be of nonlinear limit-circle type if

$$\sum_{n=1}^{\infty} y_n g(y_n) < \infty, \tag{17}$$

and will be said to be of nonlinear limit-point type otherwise, i.e., if

$$\sum_{n=1}^{\infty} y_n g(y_n) = \infty. \tag{18}$$

We begin with the following result.

Theorem 5. *Suppose that*

(19) $g(x)$ *is bounded away from zero if* x *is bounded away from zero.*

If $\{y_n\}$ *is a nonlinear limit-circle solution of* (16), *then* $y_n \to 0$ *as* $n \to \infty$ *and* $y_n \Delta y_n < 0$ *for all large* n.

Proof. Let $\{y_n\}$ be a nonlinear limit-circle solution of equation (16). By Propositions 1 and 2, $\{y_n\}$ is nonoscillatory and eventually monotonic. Now (17) implies $y_n g(y_n) \to 0$ as $n \to \infty$, and it follows from (19) that $y_n \to 0$ as $n \to \infty$. Clearly, $y_n \Delta y_n < 0$ for all large n. A similar proof holds if $\{y_n\}$ is eventually negative.

The next result is a necessary condition for a solution of equation (16) to be of the nonlinear limit-circle type.

Theorem 6. *Suppose there exist constants* $A > 0, B \geq 0$, *and* $K > 0$ *such that*

$$r^2 \leq Arg(r) + B, \tag{20}$$

$$\left| (p_n)^{\frac{1}{2}} \frac{(\Delta q_n)}{(q_n)^{\frac{1}{2}} q_{n+1}} \right| \leq K, \tag{21}$$

and

$$\sum_{n=1}^{\infty} p_n \frac{(\Delta q_n)^2}{q_n q_{n+1}^2} < \infty. \tag{22}$$

If $\{y_n\}$ *is a limit-circle solution of equation* (16), *then*

$$\sum_{n=1}^{\infty} p_n \frac{(\Delta y_n)^2}{q_n} < \infty. \tag{23}$$

Proof. Let $\{y_n\}$ be a nonlinear limit-circle solution of equation (16). Then $\{y_n\}$ is nonoscilatory and eventually monotonic, say $y_n > 0$ and $\Delta y_n <$

0 for $n \geq N \geq N_0$. Multiplying equation (16) by $\frac{y_n}{q_n}$ and summing from $N+1$ to n by parts yields

$$\begin{aligned}(24)\qquad &\frac{y_n p_n \Delta y_n}{q_n} - \frac{p_N y_N \Delta y_N}{q_N} - \sum_{s=N+1}^{n} \frac{p_{s-1}(\Delta y_{s-1})^2}{q_{s-1}} \\ &+ \sum_{s=N+1}^{n} \frac{p_{s-1} y_s \Delta y_{s-1} \Delta q_{s-1}}{q_s q_{s-1}} - \sum_{s=N+1}^{n} y_s g(y_s) = 0.\end{aligned}$$

By the Schwarz inequality, we have

$$\begin{aligned}&\left| \sum_{s=N+1}^{n} \frac{p_{s-1} y_s \Delta y_{s-1} \Delta q_{s-1}}{q_s q_{s-1}} \right| \\ &\leq \left[\sum_{s=N+1}^{n} \frac{p_{s-1}(\Delta y_{s-1})^2}{q_{s-1}} \right]^{\frac{1}{2}} \left[\sum_{s=N+1}^{n} \frac{p_{s-1} y_s^2 (\Delta q_{s-1})^2}{q_s^2 q_{s-1}} \right]^{\frac{1}{2}}\end{aligned}$$

Now (20)-(22) and the fact that $\{y_n\}$ is a nonlinear limit-circle solution of (16) imply that the second sum on the right hand side of the last inequality is bounded. Also, the first term on the left hand side of (24) is nonpositive. Hence, we have

$$H_n \leq K_1 H_n^{\frac{1}{2}} + K_2$$

for some constants $K_1 > 0, K_2 > 0$ where

$$H_n = \sum_{s=N+1}^{n} p_{s-1} \frac{(\Delta y_{s-1})^2}{q_{s-1}}.$$

Clearly, this implies (23).

Our final result given sufficient conditions for equation (16) to be of the nonlinear limit-point type.

Theorem 7. *Suppose that* (20)-(22) *hold and*

$$q_n \Delta p_n + p_n \Delta q_n \leq 0$$

for all $n \geq N \geq N_0$. *Then equation* (16) *is of nonlinear limit-point type.*

Proof. Define

$$V_n = F(y_n) - \frac{p_{n-1}(\Delta y_{n-1})^2}{2q_{n-1}}$$

where $\Delta F(y_n) = g(y_n)\Delta y_{n-1}$. Then

$$\begin{aligned}
\Delta V_n &= g(y_n)\Delta y_{n-1} - \frac{p_n}{2q_n}\Delta((\Delta y_{n-1})^2) - \frac{(\Delta y_{n-1})^2}{2}\Delta\Big(\frac{p_{n-1}}{q_{n-1}}\Big) \\
&= g(y_n)\Delta y_{n-1} - \frac{p_n}{2q_n}\Delta^2 y_{n-1}\Big[\Delta y_n + \Delta y_{n-1}\Big] - \frac{(\Delta y_{n-1})^2}{2}\Delta\Big(\frac{p_{n-1}}{q_{n-1}}\Big) \\
&= g(y_n)\Delta y_{n-1} - \frac{p_n}{2q_n}\Big[\Delta y_n + \Delta y_{n-1}\Big]\Big[\frac{q_n g(y_n)}{p_n} - \frac{\Delta y_{n-1}\Delta p_{n-1}}{p_n}\Big] \\
&\quad - \frac{(\Delta y_{n-1})^2}{2}\Delta\left(\frac{p_{n-1}}{q_{n-1}}\right) \\
&= -\frac{g(y_n)}{2}\Delta^2 y_{n-1} + \frac{\Delta y_{n-1}\Delta p_{n-1}}{2q_n}\Big[\Delta y_n + \Delta y_{n-1}\Big] \qquad (25) \\
&\quad - \frac{(\Delta y_{n-1})^2}{2}\Delta\left(\frac{p_{n-1}}{q_{n-1}}\right) \\
&= -\frac{\Delta^2 y_{n-1}}{2q_n}\Big[p_n\Delta^2 y_{n-1} + \Delta y_{n-1}\Delta p_{n-1}\Big] \\
&\quad + \frac{\Delta y_{n-1}\Delta p_{n-1}}{2q_n}\Big[\Delta^2 y_{n-1} + 2\Delta y_{n-1}\Big] - \frac{(\Delta y_{n-1})^2}{2}\Delta\Big(\frac{p_{n-1}}{q_{n-1}}\Big) \\
&= -\frac{p_n}{2q_n}\Big(\Delta^2 y_{n-1}\Big)^2 + \frac{(\Delta y_{n-1})^2}{2q_n q_{n-1}}\Big[q_{n-1}\Delta p_{n-1} + p_{n-1}\Delta q_{n-1}\Big]
\end{aligned}$$

for $n \geq N$. Now choose a solution $\{y_n\}$ of (16) such that

$$V_N = F(y_N) - \frac{p_{N-1}(\Delta y_{N-1})^2}{2q_{N-1}} \leq -L < 0$$

and suppose that $\{y_n\}$ is a nonlinear limit-circle type solution. Summing (25) for $n \geq N$, we obtain $V_n \leq V_N \leq -L$ and summing again, we have

$$-\sum_{s=N}^{n-1} \frac{p_{s-1}(\Delta y_{s-1})^2}{2q_{s-1}} \leq \sum_{s=N}^{n-1} V_s \leq -L(n-N) \to -\infty$$

as $n \to \infty$ which contradicts Theorem 6.

Remark 4. From Theorem 7, we see that a necessary condition for the solution $\{y_n\}$ to be a nonlinear limit-circle type solution of (16) is that it satisfies

$$F(y_n) \geq \frac{p_{n-1}(\Delta y_{n-1})^2}{2q_{n-1}}$$

for all large n.

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