BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 23, Number 4, December 1995

A DISCRETE BOUNDARY VALUE PROBLEM WITH DELAY

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Abstract. By means of the Banach contraction principle, we show that a unique solution for a discrete boundary value problem with delay exists if the spectral radius of an associated matrix is less than one. Several other sufficient conditions for the existence of unique solutions are also given.

1. Introduction. Boundary value problems for difference equations arise in numerical computations of boundary value problems for differential equations. They are also studied in various branches of mathematics such as the theory of stochastic processes. One particular boundary value problem

$$\Delta^2 x_{k-1} + g(k, x_k) = 0, \ k = 1, 2, \ldots, n$$

$$x_0 = A, \ x_{n+1} = B$$

where g satisfies a Lipschitz condition has been studied by many authors (for background materials, see [4], see also [1,3]). Here we are concerned with a discrete boundary value problem of the form

(1) $$\Delta(r_{k-1}\Delta x_{k-1}) + f(k, x_k, x_{k-m}) = 0, \ k = 1, 2, \ldots, n$$

(2) $$x_i = a_i, \ -m+1 \le i \le 0$$

(3) $$\Delta x_n = b$$

Received by the editors March 26, 1994 and in revised form August 9, 1994.

Key words and phrases: Boundary value problems, Banach contraction principle, spectral radius, least positive eigenvalue.

* The first author is partially supported by NNSF of China. The second author is partially supported by the NSC of R.O.C.

where $a_{-m+1}, \ldots, a_0$ and b are given numbers; $r_0, \ldots, r_n$ are positive constants, and the delay m is a positive integer. To avoid trivial cases, we shall also assume that $n \geq m+1$.

Under the assumption that the function f satisfies a Lipschitz condition of the form

$$|f(k,x,y) - f(k,u,v)| \leq p_k|x-u| + q_k|y-v|, \; 1 \leq k \leq n,$$

we shall show that if the "least positive eigenvalue" of a related linear eigenvalue problem are greater than one, then the above stated nonlinear boundary value problem has a unique solution. This will be proved by means of the Banach contraction principle, so that numerical computation of the unique solution is possible.

General conditions for the least positive eigenvalue of the above mentioned eigenvalue problem to be greater than one are given. As we shall see, these conditions can be easily verified numerically. However, the eigenvalue problem approach is better posed and can be handled by many commercially available packages.

2. Auxiliary boundary problems. Let us first consider two auxiliary linear boundary value problems of the form

$$-\Delta(r_{i-1}\Delta x_{i-1}) = 0, \; i = 1, 2, \ldots, n \tag{4}$$

$$x_0 = a, \; \Delta x_n = b \tag{5}$$

and

$$-\Delta(r_{i-1}\Delta x_{i-1}) = g_i, \; i = 1, 2, \ldots, n \tag{6}$$

$$x_0 = 0, \; \Delta x_n = 0 \tag{7}$$

where $r_0, \ldots, r_n$ are positive numbers, while $a, b, g_1, \ldots, g_n$ are arbitrary. The linear problem (4)-(5) has the unique solution $\{x_i\}$ defined by

$$x_k = a + br_n\Gamma_k, \; 0 \leq k \leq n \tag{8}$$

where

$$\Gamma_k = \sum_{i=0}^{k-1} \frac{1}{r_i}, \quad 0 \le k \le n+1 \tag{9}$$

(here and in the sequel, we assume that an empty sum equals zero).

The linear problem (6)-(7) can be viewed as a matrix problem of the form

$$Hx = g \tag{10}$$

where $H = (h_{ij})$ is defined by

$$\begin{aligned} h_{nn} &= r_{n-1}, \\ h_{n,n-1} &= -r_{n-1}, \\ h_{i,j} &= r_{i-1} + r_i \quad \text{if} \quad 1 \le i = j \le n-1, \\ h_{ij} &= -r_{i-1} \quad \text{if} \quad i - j = 1, \\ h_{ij} &= -r_i \quad \text{if} \quad i - j = -1, \\ h_{ij} &= 0 \text{ otherwise.} \end{aligned} \tag{11}$$

The vector x is the column vector $\mathrm{col}(x_1, \ldots, x_n)$ and g is the column vector $\mathrm{col}(g_1, \ldots, g_n)$. The matrix H is invertible and we can verify easily that its inverse is given by the matrix $G = (g_{ij})$ where

$$g_{ij} = \begin{cases} \Gamma_i & 1 \le i \le j \le n \\ \Gamma_j & 1 \le j \le i \le n \end{cases} \tag{12}$$

Next, consider the linear eigenvalue problem

$$-\Delta(r_{i-1}\Delta x_{i-1}) = \lambda(p_i x_i + q_i x_{i-m}), \quad 1 \le i \le n \tag{13}$$

$$x_{-m+1} = x_{-m+2} = \ldots = x_0 = 0 \tag{14}$$

$$\Delta x_n = 0, \tag{15}$$

where $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ are nonnegative numbers which are not all zero. Again, this problem can be viewed as a matrix eigenvalue problem of the form

$$Hx = \lambda(\text{diag}(p) + \text{diag}(q)D)x \tag{16}$$

where diag(p) is the diagonal matrix with diagonal entries $p_1, \ldots, p_n$, diag(q) is the diagonal matrix with diagonal entries $q_1, \ldots, q_n$, $x = \text{col}(x_1, \ldots, x_n)$, H is defined by (11), and $D = (d_{ij})$ is the n by n block matrix

$$\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \tag{17}$$

where I is the $(n-m)$ by $(n-m)$ identity matrix.

By means of the inverse matrix G of H, we can write (16) in the equivalent form

$$x = \lambda G(\text{diag}(p) + \text{diag}(q)D)x, \tag{18}$$

or,

$$x_i = \lambda \sum_{j=1}^{n} g_{ij}p_j x_j + \lambda \sum_{j=m}^{n} g_{ij}q_j x_{j-m}, \quad 1 \le i \le n+1.$$

Clearly, $\lambda = 0$ cannot be an eigenvalue of (18). Furthermore, since the components of $G(\text{diag}(p) + \text{diag}(q)D)$ are nonnegative, according to the Perron-Frobenius theory of nonnegative matrices, we know that the matrix

$$G(\text{diag}(p) + \text{diag}(q)D) \tag{19}$$

has a simple nonnegative (and hence positive) eigenvalue which exceeds the moduli of all other eigenvalues and that there corresponds a nonnegative eigenvector u with nonnegative components $u_1, \ldots, u_n$.

Lemma 1. *The eigenvalue problem* (13) – (15) *has a simple positive eigenvalue* $\lambda(p,q)$ *which is smaller in magnitude than all the other eigenvalues, and there corresponds an eigensolution* $\{u_k\}_{-m+1}^{n+1}$ *which satisfies* $u_k > 0$ *for* $1 \le k \le n$, *and* $\Delta u_k > 0$ *for* $0 \le k \le n-1$.

We have already explained the existence of $\lambda(p,q)$ and a corresponding nonnegative eigensolution $\{u_k\}_{1-m}^{n+1}$. We need to show that the nonnegative eigenvector corresponding to $\lambda(p,q)$ has positive components. This can be seen from maximum principle for difference equations (see for example

Cheng [2]), but an elementary argument is just as easy. First note from (13) – (15) that

$$\Delta(r_{i-1}\Delta u_{i-1}) \leq 0, \ 1 \leq i \leq n$$

$$u_0 = 0 = \Delta u_n.$$

If $u_1 = 0$, then

$$r_1\Delta u_1 - r_0\Delta u_0 = r_1\Delta u_0 \leq 0,$$

so that $u_2 = 0$. By induction, we see that $u_3 = u_4 = \ldots = u_{n+1} = 0$, which is impossible. if $u_n = 0$, then

$$r_n\Delta u_n - r_{n-1}\Delta u_{n-1} = r_{n-1}u_{n-1} \leq 0,$$

so that $u_{n-1} = 0$. By induction, we see that $u_{n-2} = u_{n-3} = \ldots = u_1 = 0$, which is impossible. Finally, if $u_i = 0$ where $2 \leq i \leq n-1$, then

$$r_i\Delta u_i - r_{i-1}\Delta u_{i-1} = r_i u_{i+1} + r_{i-1}u_{i-1} \leq 0$$

so that $u_{i-1} = u_{i+1} = 0$. By induction, we see that $u_1 = u_2 = \ldots = u_n = 0$, which is impossible.

To complete the proof, we need to show that $\Delta u_k > 0$ for $0 \leq k \leq n-1$. It is clear that $\Delta u_0 > 0$. If $\Delta u_1 \leq 0$, then from (13) and the fact that $u_k > 0$ for $1 \leq k \leq n+1$, $r_2\Delta u_2 - r_1\Delta u_1 < 0$, so that $\Delta u_2 < 0$ and inductively, $\Delta u_3 < 0, \ldots, \Delta u_n < 0$, contrary to our assumption that $\Delta u_n = 0$. Similarly, $\Delta u_k > 0$ for $2 \leq k \leq n-1$.

3. Comparison theorems. We shall need several comparison theorems for difference equations with delay. As before, m shall denote an integer greater than or equal to 1.

Lemma 2. *Suppose $r_i > 0$ for $a-1 < i \leq b$ and suppose*

$$\Delta(r_{k-1}\Delta x_{k-1}) + S_k x_k = 0, \ a \leq k \leq b$$

has a solution $\{x_k\}_{a-1}^{b+1}$ which satisfies $x_{a-1} = 0$, $x_a > 0$ and $\Delta x_k \geq 0$ for $a \leq k \leq b$. Suppose further that $s_k \leq S_k$ for $a \leq k \leq b$ and that $\{y_k\}_{a-1}^{b+1}$ is *a solution of*

$$\Delta(r_{k-1}\Delta y_{k-1}) + s_k y_k = 0,\ a \le k \le b \tag{20}$$

determined by the conditions $y_{a-1} = 0$ *and* $y_a = x_a$. *Then* $y_k \ge x_k$ *for* $a+1 \le k \le b+1$ *and* $\Delta y_k/y_k \ge \Delta x_k/x_k$ *for* $a \le k \le b$.

Proof. Let $w_k = x_k r_k \Delta y_k - y_k r_k \Delta x_k$ for $a-1 \le k \le b$. Then $w_{a-1} = 0$ and

$$\begin{aligned}\Delta w_k &= r_k \Delta y_k \Delta x_k + x_{k+1}\Delta(r_k\Delta y_k) - r_k \Delta x_k \Delta y_k - y_{k+1}\Delta(r_k \Delta x_k)\\ &= x_{k+1}\Delta(r_k\Delta y_k) - y_{k+1}\Delta(r_k\Delta x_k)\\ &= (S_k - s_k)x_{k+1}y_{k+1},\ a \le k \le b-1.\end{aligned}$$

It is clear that $\Delta w_{a-1} = (S_a - s_a)x_a y_a \ge 0$. Thus $w_a = x_a r_a \Delta y_a - y_a r_a \Delta x_a \ge 0$ so that $\Delta y_a/y_a \ge \Delta x_a/x_a$ and $y_{a+1}/x_{a+1} \ge y_a/x_a = 1$. Assume by induction that $\Delta w_i \ge 0$ for $a-1 \le i \le n-1$, and $y_{i+1} \ge x_{i+1}$ for $a-1 \le i \le n$. Then as before, we see that $\Delta w_n = (S_{n+1} - s_{n+1})x_{n+1}y_{n+1} \ge 0$, $\Delta y_{n+1}/y_{n+1} \ge \Delta x_{n+1}/x_{n+1}$ and $y_{n+2}/x_{n+2} \ge y_{n+1}/x_{n+1} \ge 1$ as required. The proof is complete.

The following comparison theorem can now be proved.

Lemma 3. *Suppose* $r_i > 0$ *for* $1 \le i \le n$ *where* $n > m \ge 1$, *and suppose*

$$\Delta(r_{k-1}\Delta x_{k-1}) + S_k x_k + T_k x_{k-m} = 0,\ 1 \le k \le n \tag{21}$$

has a solution $\{x_k\}_{1-m}^{n+1}$ *which satisfies* $x_i = 0$ *for* $1-m \le i \le 0$, $x_0 = 0$, $x_1 > 0$ *and* $\Delta x_k \ge 0$ *for* $1 \le k \le n$. *Suppose further that* $s_k \le S_k$ *and* $t_k \le T_k$ *for* $1 \le k \le n$ *and that* $\{y_k\}_{1-m}^{n+1}$ *is a solution of*

$$\Delta(r_{k-1}\Delta y_{k-1}) + s_k y_k + t_k y_{k-m} = 0,\ 1 \le k \le n \tag{22}$$

determined by the conditions $y_{1-m} = 0, \ldots, y_0 = 0$ *and* $y_1 = x_1$. *Then* $y_k \ge x_k$ *for* $1 \le k \le n+1$ *and* $\Delta y_k/y_k \ge \Delta x_k/x_k$ *for* $1 \le k \le n$.

Proof. For $1 \le k \le m$, since $x_{k-m} = 0 = y_{k-m}$, we see from Lemma 2 that $y_k \ge x_k$ for $1 \le k \le m+1$ and

(23) $$\Delta y_k/y_k \geq \Delta x_k/x_k,\ 1 \leq k \leq m.$$

Let $w_k = x_k r_k \Delta y_k - y_k r_k \Delta x_k$ for $m \leq k \leq n$. Then from (23), we have $w_m \geq 0$. Also,

$$\begin{aligned}
&\Delta w_k\\
=&T_{k+1}x_{k+1-m}y_{k+1} - t_{k+1}x_{k+1}y_{k+1-m} + S_{k+1}x_{k+1}y_{k+1} - s_{k+1}x_{k+1}y_{k+1}\\
=&x_{k+1}y_{k+1}\left\{T_{k+1}\frac{x_{k+1-m}}{x_{k+2-m}}\cdots\frac{x_k}{x_{k+1}} - t_k\frac{y_{k+1-m}}{y_{k+2-m}}\cdots\frac{y_k}{y_{k+1}} + S_{k+1} - s_{k+1}\right\}
\end{aligned}$$

for $m \leq k \leq n-1$.

Since (23) implies $y_k/y_{k+1} \leq x_k/x_{k+1}$ for $1 \leq k \leq m$, and since $s_k \leq S_k$, $t_k \leq T_k$ for $1 \leq k \leq n$, we have $w_{m+1} \geq 0$. As in the proof of Lemma 2, we see that $y_{m+2} \geq x_{m+2}$ and $\Delta y_{m+1}/y_{m+1} \geq \Delta x_{m+1}/x_{m+1}$. An induction argument similar to that in the proof of Lemma 2 can now be used to complete the proof.

In view of Lemma 3, the following is clear.

Lemma 4. *Let $\lambda(p,q)$ be the least positive eigenvalue of the eigenvalue problem (13) – (15) as defined in Lemma 1, then for any $0 < \mu < \lambda(p,q)$, the difference equation*

$$-\Delta(r_{i-1}\Delta x_{i-1}) = \mu(p_i x_i + q_i x_{i-m}),\ 1 \leq i \leq m$$

has a solution $\{u_i^*\}_{-m+1}^{n+1}$ such that $u_i^* = 0$ for $1-m \leq i \leq 0$, $u_1^* > 0$ and $\Delta u_k^* \geq 0$ for $1 \leq k \leq n$.

Note that in view of (18), it is easily verified that the solution $\{u_i^*\}_{1-m}^{n+1}$ satisfies

$$\frac{u_i^*}{\mu} = \sum_{j=1}^{n} g_{ij}p_j u_j^* + \sum_{j=m}^{n} g_{ij}q_j u_{j-m}^* + r_n\Gamma_i\Delta u_n^*,\ 1 \leq i \leq n+1.$$

Thus

(24) $$\frac{1}{\mu} \geq \frac{1}{\mu_i^*}\sum_{j=1}^{n} g_{ij}p_j u_j^* + \frac{1}{u_i^*}\sum_{j=m}^{n} g_{ij}q_j u_{j-m}^*,\ 1 \leq i \leq n+1.$$

Furthermore, by means of the solution $\{u_i^*\}_{-m+1}^{n+1}$, we may define a complete metric space

$$\Omega = \{x = col(x_{-m+1}, \dots, x_{n+1}) | x_i = a_i, -m+1 \le i \le 0\}$$

with distance

$$\|x - y\| = \max_{1 \le i \le n+1} \frac{|x_i - y_i|}{u_i^*}.$$

4. Existence theorems. The boundary value problem (1)-(3) can be viewed as a fixed point problem in the metric space Ω. To this end, let $x \in \Omega$, let T be a mapping from Ω into Ω defined by

$$(Tx)_i = \begin{cases} \sum_{j=1}^n g_{ij}(p_j u_j + q_j u_{j-m}) + a_0 + r_n \Gamma_i b, & 1 \le i \le n+1 \\ a_i, & -m+1 \le i \le 0 \end{cases}$$

Then it is easy to verify that a fixed point of T in Ω is a solution of (1)-(3). We assert further that T is a contraction mapping on Ω if $\lambda(p,q) > 1$. To see this, let $x, y \in \Omega$, then $|(Tx)_i - (Ty)_i| = 0$ for $-m+1 \le i \le 0$ and

$$|(Tx)_i - (Ty)_i| \le \sum_{j=1}^n g_{ij} |f(j, x_j, x_{j-m}) - f(j, y_j, y_{j-m})|, \ 1 \le i \le n+1.$$

Thus for $1 \le i \le n+1$, we have

$$\begin{aligned} &\frac{|(Tx)_i - (Ty)_i|}{u_i^*} \\ \le &\frac{1}{u_i^*} \sum_{j=1}^n g_{ij} p_j \frac{|x_j - y_j|}{u_j^*} u_j^* + \frac{1}{u_i^*} \sum_{j=m}^n g_{ij} q_j \frac{|x_{j-m} - y_{j-m}|}{u_{j-m}^*} u_{j-m}^* \\ \le &\frac{\|x-y\|}{u_i^*} \sum_{j=1}^n g_{ij} p_j u_j^* + \frac{\|x-y\|}{u_i^*} \sum_{j=m}^n g_{ij} q_j u_{j-m}^* \le \frac{1}{\mu} \|x-y\|, \end{aligned}$$

where the last inequality holds in view of (24). If we choose μ so close to $\lambda(p,q)$ that $\mu > 1$, then $\|Tx - Ty\| \le \|x - y\|/\mu$ implies T is a contraction mapping. The proof is complete. We summarize these as follows.

Theorem 1. *If the least positive eigenvalue $\lambda(p,q)$ of the boundary value problem (13) – (15) satisfies $\lambda(p,q) > 1$, then the boundary value problem (1) – (3) has a unique solution.*

Since $\lambda^{-1}(p,q)$ is the spectral radius of the matrix (19), in practical situations, we need only to calculate the largest positive eigenvalue of the matrix (19) and check if it is less than one. If this is the case, a standard iteration scheme will then yield a solution of the boundary value problem (1) – (3).

Theorem 1 is of theoretical interest also. As an example, consider the boundary value problem

$$\Delta^2 x_{k-1} + f(x_{k-m}) = 0,\ 1 \le k \le n,$$

$$x_{1-m} = x_{2-m} = \ldots = x_0 = 0,\ \Delta x_n = 0,$$

where f satisfies a Lipschitz condition of the form

$$|f(u) - f(v)| \le \beta |u - v|.$$

The corresponding matrix equation (18) becomes

$$x = \lambda \beta G D x.$$

As a consequence of Theorem 1, if τ is the least positive eigenvalue of the matrix GD (which can easily be calculated using standard numerical eigenvalue solvers), then the condition $\beta < 1/\tau$ is sufficient for the above boundary value problem to have a unique solution.

Theorem 2. *Suppose* $p_1, \ldots, p_n$, *and* $q_1, \ldots, q_n$ *are nonnegative numbers which are not all zero. Suppose further that difference equation*

$$\Delta(r_{k-1}\Delta x_{k-1}) + p_k x_k + q_k x_{k-m} = 0,\ 1 \le k \le n \tag{25}$$

has a solution $\{x_k\}_{-m+1}^{n+1}$ *which satisfies* $x_i = 0$ *for* $-m+1 \le i \le 0$ *and* $\Delta x_i > 0$ *for* $0 \le i \le n$. *Then the least positive eigenvalue* $\lambda(p,q)$ *of the boundary value problem* (13) – (15) *is greater than one.*

Proof. Assume to the contrary that $\lambda(p,q) \le 1$. Then by Lemma 3, the solution of the equation

$$\Delta(r_{k-1}\Delta y_{k-1}) + \lambda(p,q)(p_k x_k + q_k x_{k-m}) = 0,\ 1 \le k \le n$$

determined by the conditions $y_i = 0$ for $1 - m \le i \le 0$ and $y_1 > 0$ will satisfy $\Delta y_n \ge \Delta x_n > 0$. Since the eigenvalue $\lambda(p,q)$ is simple, $\{y_k\}$ is also an eigensolution (with $\Delta y_n = 0$) corresponding to $\lambda(p,q)$, which is a contradiction. The proof is complete.

The above Theorem is of theoretical interest. It may also be used as a straight-forward test for the existence of solution to the boundary value problem (1)-(3), since we can calculate inductively the solution of (25) and its differences. This procedure, however, is well known to be unstable in various circumstances and is not recommended when n is large.

As a final example, suppose $p_k = q_k = \alpha$ for $1 \le k \le n$ and $2\alpha n\Gamma_n < 1$, we assert that the least positive eigenvalue $\lambda(p,q) > 1$. Indeed, note that the eigenvalue problem

$$\Delta(r_{k-1}\Delta x_{k-1}) + \lambda(\alpha x_k + \alpha x_{k-m}) = 0,\ 1 \le k \le n$$

$$x_{1-m} = x_{2-m} = \ldots = x_0 = 0,\ \Delta x_n = 0$$

is equivalent to

$$x = \lambda G(\alpha I + \alpha D)x. \tag{26}$$

Let $x = \text{col}(x_1, \ldots, x_n)$ be the positive eigenvector of (26) corresponding to $\lambda(p,q)$. Let $x_i = \max_{1\le j\le n}\{x_j\}$. Then

$$x_i \le x_i\lambda(p,q)n\alpha \max\ G(I+D)$$

so that

$$1 \le 2\lambda(p,q)n\alpha \max\ G.$$

Since $\max G = \Gamma_n$, if $2n\alpha\Gamma_n < 1$, then $\lambda(p,q) > 1$.

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