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ON MULTIPLIERS FROM L_1 TO M_1

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Abstract. The multipliers of L_1 to M_1 are identified with those bounded measurable functions $f(x)$ such that $\sum_{n=-\infty}^{\infty} M_n < \infty$, where M_n is the essential supremum of $|f(x)|$ restricted to $[n, n+1]$.

N. Wiener [4] based his theory of the general Tauberian theorem on two Banach spaces, L_1 and M_1. The L_1-theory can be generalized to arbitrary locally compact groups, and consequently it generated great interest and caused the publication of a large number of research papers. The M_1-theory, however, is restricted to the real line, and people no longer pay much attention to it. In this paper we intend to investigate this space and to find all bounded linear operators $T : L_1 \to M_1$ such that

$$T(\varphi * \psi) = \varphi * (T\psi)$$

for all $\varphi, \psi \in L_1$. These operators T are called the *multipliers* of L_1 to M_1. For the basic properties of multipliers, see Larsen [1].

First let us review the definition of the Banach space M_1. Let a, b be two real numbers, where $b > 0$. For each integer n we denote the interval $[a+nb, a+(n+1)b]$ by I_n. Then the space M_1 is defined to be the set of all (real- or complex-valued) continuous functions on the real line $\mathbf{R}$ such that

$$(1) \qquad \|f\|_{a,b} = \sum_{n=-\infty}^{\infty} \sup\{|f(x)| : x \in I_n\} < \infty.$$

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Wiener showed that the space M_1 is independent of the choice of a and b, and all $\|f\|_{a,b}$ are equivalent norms to make M_1 into normed linear spaces. We shall prove its completeness in Theorem 1.

It is easily seen that $\|f\|_1 \leq \|f\|_{a,b}$, which means that $M_1 \subset L_1$. We shall insert a Banach space N_1 between L_1 and M_1 in the following way: N_1 consists of all functions $f \in L_\infty$ on $\mathbf{R}$ satisfying (1), where $\sup\{|f(x)| : x \in I_n\}$ is interpreted as the essential supremum of $|f(x)|$ when restricted to I_n. Two functions in M_1 which are equal almost everywhere are considered as the same element of N_1, as usual.

Theorem 1. (1) *N_1 is a Banach space with respect to the norm $\|f\|_{a,b}$.*

(2) *The space N_1 is independent of the choices of a and b, and all the norms $\|f\|_{a,b}$ are equivalent.*

(3) *M_1 is a closed subspace of N_1, and is therefore complete.*

Proof. (1) There is no difficulty in showing that N_1 is a normed linear space. We have only to show its completeness. This is equivalent to the condition that an ablsolutely convergent series in N_1 is always convergent in norm. Thus let $\sum_{k=1}^{\infty} f_k$ be an absolutely convergent series in N_1. This means that the series

$$\sum_{k=1}^{\infty} \|f_k\|_{a,b} = \sum_{k=1}^{\infty} \sum_{n=-\infty}^{\infty} \sup\{|f_k(x)| : x \in I_n\} < \infty.$$

As the order of summation may be interchanged for a series with non-negative terms, in each interval I_n the series $\sum_{k=1}^{\infty} f_k$ converges in $L_\infty(I_n)$. Define its sum to be the function f. Then f is defined almost everywhere on $\mathbf{R} = \cup I_n$ with

$$\sup\{|f(x)| : x \in I_n\} \leq \sum_{k=1}^{\infty} \sup\{|f_k(x)| : x \in I_n\}.$$

This inequality show that $f \in N_1$. As

$$\left\| f - \sum_{k=1}^{m} f_k \right\|_{a,b} \leq \sum_{k=m+1}^{\infty} \|f_k\|_{a,b},$$

$\sum f_k$ converges to f with respect to the norm in N_1.

(2) The same proof which Wiener [4, p. 73] gave for M_1 works for N_1 too. Thus, let a, b, a' and b' be real numbers where b and b' are positive. Find an integer ν such that $\nu b > b'$. Then each interval of length b' is always covered by $\nu + 1$ consecutive intervals of length b. Then

$$\|f\|_{a,b} \leq (\nu+1)\|f\|_{a',b'}.$$

(3) Let $\{f_k\}$ be a sequence in M_1 converging in N_1 to a function $f \in N_1$. Since the convergence is uniform in every interval I_n, we see that f is continuous and is therefore in M_1.

The norms $\|f\|_{a,b}$ are unfortunately not translation invariant. By this we mean that if for $r \in \mathbf{R}$ we denote by λ_r the translation operator

$$(\lambda_r f)(x) = f(x-r), \qquad f \in N_1$$

then $\|f\|_{a,b} = \|\lambda_r f\|_{a,b}$ is not always true. We can, however, replace the norm with

$$\|f\|_{,b} = \sup_{a \in \mathbf{R}} \|f\|_{a,b}.$$

It is easily verified that $\|f\|_{,b}$ is again an equivalent norm, and $\|\lambda_r f\|_{,b} = \|f\|_{,b}$. In the sequel we shall write

$$\|f\| = \|f\|_{,1}, \qquad f \in N_1$$

and use it as our canonical norm for both N_1 and M_1.

In addition to the Banach space structure, L_1 is also a Banach algebra with the convolution

$$\varphi * \psi(x) = \int_{-\infty}^{\infty} \varphi(y)\psi(x-y)dy \qquad \varphi, \psi \in L_1$$

as multiplication.

Theorem 2. *M_1 and N_1 are dense ideals in L_1. Further, if $\varphi \in L_1$ and $f \in N_1$, then $\varphi * f \in M_1$, and*

$$\|\varphi * f\| \leq \|\varphi\|_1 \|f\|.$$

Proof. We have $M_1 \subset N_1$. If $f \in N_1$, then

$$\|f\|_1 = \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f(x)|dx \leq \|f\|_{0,1} \leq \|f\|.$$

Hence $f \in L_1$. But as M_1 contains all continuous functions with compact supports, we see that M_1 is dense in L_1.

Now let $\varphi \in L_1$ and $f \in N_1$. Since $f \in L_\infty$, it is a well-known result (see e. g. Rudin [2, p. 4]) that $\varphi * f$ is continuous. For any $a \in \mathbf{R}$,

$$\begin{aligned}\|\varphi * f\|_{a,1} &= \sum_{n=-\infty}^{\infty} \sup\left\{\left|\int_{-\infty}^{\infty} \varphi(y) f(x-y)dy\right| : x \in [a+n, a+n+1]\right\} \\ &\leq \int_{\infty}^{\infty} |\varphi(y)| \sum_{n=-\infty}^{\infty} \sup\{|f(x-y)| : x \in [a+n, a+n+1]\}dy \\ &\leq \int_{-\infty}^{\infty} |\varphi(y)| \cdot \|f\|_{a+y,1}dy \leq \|\varphi\ |_1 \|f\|.\end{aligned}$$

Hence $\varphi * f \in M_1$.

Fix an $f \in N_1$. Define the operator $T_f : L_1 \to M_1$ as follows:

$$T_f \varphi = \varphi * f. \qquad \varphi \in L_1$$

It is easily seen that T_f is a multiplier from L_1 to M_1. Now we want to decide its norm.

Theorem 3. *The norm of T_f is given by*

$$\|T_f\| = \|f\|.$$

To prove this theorem, we need the following result:

Lemma 4. *Let f be a locally integrable function on $\mathbf{R}$. For each number $l > 0$ let*

$$\phi_l(x) = \frac{1}{2l}\chi_{[-l,l]}(x),$$

where χ denotes the characteristic function. Then

$$\lim_{l\to 0} \phi * f(x) = f(x) \tag{2}$$

for almost all $x \in \mathbf{R}$.

This Lemma was given in Stein and Weiss [3, p. 12] as Formula (1.23). Actually (2) is valid at every Lebesgue point x of f.

Proof of Theorem 3. It follows from Theorem 2 that $\|T_f\| \le \|f\|$. Now we are going to prove the reverse inequality.

Let $\epsilon > 0$ be given. First select a real number a such that $\|f\|_{a,1} > \|f\| - \epsilon$. For each integer n write $I_n = [a+n, a+n+1]$ and $M_n = \sup\{|f(x)| : x \in I_n\}$. Select a positive integer m such that

$$\sum_{n=-m}^{m-1} M_n > \|f\|_{a,1} - \epsilon.$$

For each integer n between $-m$ and $m-1$, we can find a point $x_n \in I_n$ where (2) is valid, and where $|f(x_n)| > M_n - \epsilon/(4m)$. Then we can find a positive number l which is small enough such that

$$|\phi_l * f(x_n)| > M_n - \frac{\epsilon}{2m}$$

for every n. But then

$$\|T_f\| \ge \|\phi_l * f\| \ge \sum_{n=-m}^{m-1} \sup\{|\phi_l * f(x)| : x \in I_n\} \ge \|f\|_{a,1} - 2\epsilon \ge \|f\| - 3\epsilon.$$

Since ϵ is arbitrary, this proves that $\|T_f\| \ge \|f\|$.

It is curious to say that the part (2) $\Rightarrow$ (3) of the proof of the following theorem follows the same line of reasoning as the proof of the previous one.

Theorem 5. *Let $T : L_1 \to M_1$ be a bounded linear operator. Then the following are equivalent:*

(1) *T is a multiplier;*

(2) *T is translation-invariant, i.e., $\lambda_r T = T\lambda_r$ for every $r \in \mathbf{R}$.*

(3) *There is a function $f \in N_1$ such that $T = T_f$.*

Proof. (1) $\Rightarrow$ (2). Since $M_1 \subset L_1$ with a larger norm, this (2) is a consequence of a similar result for multipliers on L_1. (See Larsen [1], p.2).

(2) $\Rightarrow$ (3). Let T be a translation-invariant operator, and let $x \in \mathbf{R}$. Then the mapping $\varphi \mapsto T_\varphi(x), \varphi \in L_1$, is a bounded linear functional on L_1. Hence there is a function $f_x \in L_\infty$ such that

$$T\varphi(x) = \int_{-\infty}^{\infty} \varphi(y) f_x(y) dy.$$

Now $T\varphi(x) = T\lambda_{-x}\varphi(0) = \lambda_{-x}T\varphi(0)$. This may be rewritten as

$$\int_{-\infty}^{\infty} \varphi(y) f_x(y) dy = \int_{-\infty}^{\infty} \varphi(y) f(y-x) dy,$$

where $f = f_0$. Since this is valid for all $\varphi \in L_1$, we have $f_x = \lambda_x f$, and $T\varphi = \varphi * f$. All we need to show now is that $f \in N_1$.

We know that $f \in L_\infty$. Assume that $f \notin N_1$. For any $G > 0$ there would be an integer $m > 0$ such that

$$\sum_{n=-m}^{m-1} M_n > G,$$

where $M_n = \sup |f(x)| : x \in [n, n+1]$. For each $\epsilon > 0$ and each integer n between $-m$ and $-m-1$, there is a point $x_n \in [n, n+1]$ where (2) is valid, and where $|f(x_n)| > M_n - \epsilon/(4m)$. Then we can find a positive number l which is small enough such that

$$|\phi_l * f(x_n)| > M_n - \frac{\epsilon}{2m}$$

for every n. But then

$$\|T_f\| \geq \|\phi_l * f\| \geq \|\phi_l * f\|_{0,1} > G - \epsilon.$$

Since ϵ is arbitrary, this proves that $\|T\| \geq G$, contradicting the boundedness of T.

(3) $\Rightarrow$ (1). This is a corollary to Theorem 2.

Finally we like to remark that all the results may be extended to $\mathbf{R}^n$ for any natural number n without any further difficulty.

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