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# DERIVATIONS COCENTRALIZING LIE IDEALS

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**Abstract.** Let $R$ be a prime ring and $U$ be a noncentral Lie ideal of $R$. Suppose that $d$ and $\delta$ are derivations on $R$ such that $ud(u)-\delta(u)u$ is central for all $u \in U$. Then either $d=\delta=0$ or $R$ is an order of a 4-dimensional central simple algebra.

Let $R$ be a prime ring with center $\mathcal{Z}$. A well-known theorem of Posner [13] states that $R$ must be commutative if it admits a nonzero centralizing derivation $d$, that is, $xd(x)-d(x)x \in \mathcal{Z}$ for all $x \in R$. In [6], Hirano, Kaya and Tominaga established a similar result for the existence of a nonzero skew-centralizing derivation $d$, that is, $xd(x)+d(x)x \in \mathcal{Z}$ for all $x \in R$. Recently, Brešar [1] generalized these results by showing that $R$ is commutative if there exist derivations $d$ and $\delta$, not both zero, such that $xd(x)-\delta(x)x \in \mathcal{Z}$ for all $x \in R$. The condition arises naturally in the study of centralizing generalized inner derivations which have been extensively investigated on operator algebras. By a generalized inner derivation $f$ on $R$ we mean a mapping $f$ defined by $f(x)=xa+bx$ for some $a$ and $b$. The hypothesis that $f$ is centralizing on $R$ can be written as $x[x,a]-[b,x]x \in \mathcal{Z}$ for all $x \in R$, where $d(x)=[x,a]$ and $\delta(x)=[b,x]$ are derivations. Indeed Brešar obtained a result under the weaker assumption that $xd(x)-\delta(x)x \in \mathcal{Z}$ holds merely for all $x$ in some nonzero left ideal. In the present note, we shall consider the situation when $xd(x)-\delta(x)x \in \mathcal{Z}$ holds for all $x$ in some noncentral Lie ideal. Our result extends the theorems proved previously by Lee and Lee

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[11, Theorem 5] and Lanski [9, Theorem 2].

We begin with a result on generalized polynomial identities (GPIs) which seems not appear in the literature.

**Lemma.** *Let $k$ be an infinite field and $A$ be an algebra over $k$. Let $f(\xi_1,\dots,\xi_n)$ be a polynomial in commuting indeterminates $\xi_1,\dots,\xi_n$ over $A$. Suppose that $f(\alpha_1,\dots,\alpha_n)=0$ for all $\alpha_1,\dots,\alpha_n$ in $k$. Then $f(\xi_1,\dots,\xi_n)=0$.*

*Proof.* We proceed by induction on $n$. Consider first the case when $n=1$. Let $f(\xi)=a_0+a_1\xi+\cdots+a_m\xi^m$ where $a_i\in A$ and $a_m\neq 0$. Choose $m+1$ distinct elements $\alpha_0,\alpha_1,\dots,\alpha_m$ in $k$. Then $f(\alpha_i)=0$ for $i=0,1,\dots,m$, and a van der Monde argument yields $a_0=a_1=\cdots=a_m=0$. Hence $f(\xi)=0$. Assume next that $n>1$ and that the assertion holds for $n-1$. Write $f(\xi_1,\dots,\xi_{n-1},\xi_n)=f_0(\xi_1,\dots,\xi_{n-1})+f_1(\xi_1,\dots,\xi_{n-1})\xi_n+\cdots+f_m(\xi_1,\dots,\xi_{n-1})\xi_n^m$ where $f_i(\xi_1,\dots,\xi_{n-1})$ are polynomials in $\xi_1,\dots,\xi_{n-1}$ over $A$. For any fixed $n-1$ elements $\alpha_1,\dots,\alpha_{n-1}$ in $k$, set $F(\xi)=f(\alpha_1,\dots,\alpha_{n-1},\xi)$. Then $F(\alpha_n)=f(\alpha_1,\dots,\alpha_{n-1},\alpha_n)=0$ for all $\alpha_n\in k$. Thus $F(\xi)=0$ or, equivalently, $f_i(\alpha_1,\dots,\alpha_{n-1})=0$ for $i=0,1,\dots,m$. Since the choice for $\alpha_1,\dots,\alpha_{n-1}$ is arbitrary, by the induction hypothesis $f_i(\xi_1,\dots,\xi_{n-1})=0$ for each $i$ and so $f(\xi_1,\dots,\xi_{n-1},\xi_n)=0$.

**Proposition.** *Let $A$ be an algebra over an infinite field $k$ and $K$ be a field extension over $k$. If $A$ satisfies a GPI $p(X_1,\dots,X_m)$, so does $A\otimes_k K$.*

*Proof.* For $b_1,\dots,b_m$ in $A\otimes_k K$, we want to show that $p(b_1,\dots,b_m)=0$. Write $b_i=\sum_{j=1}^n a_{ij}\alpha_{ij}$ with $a_{ij}\in A$ and $\alpha_{ij}\in K$, and we are going to show that $p\big(\sum_{j=1}^n a_{1j}\alpha_{1j},\cdots,\sum_{j=1}^n a_{mj}\alpha_{mj}\big)=0$. Set $f(\xi_{11},\xi_{12},\dots,\xi_{mn})=p\big(\sum_{j=1}^n a_{1j}\xi_{1j},\dots,\sum_{j=1}^n a_{mj}\xi_{mj}\big)$. For all $\gamma_{ij}\in k$ we have $f(\gamma_{11},\gamma_{12},\dots,\gamma_{mn})=0$ because $\sum_{j=1}^n a_{ij}\gamma_{ij}\in A$ for each $i$. By the preceding lemma, $f(\xi_{11},\xi_{12},\dots,\xi_{mn})=0$ and hence $p(b_1,\dots,b_m)=f(\alpha_{11},\alpha_{12},\dots,\alpha_{mn})=0$.

**Corollary.** *If the extended centroid $C$ of the prime ring $R$ is ininite, then $RC$ and $RC\otimes_C F$ satisfy the same GPIs for any field extension $F$*

*over $C$.*

We are now ready to prove our main

**Theorem.** *Let $R$ be a prime ring and $U$ be noncentral Lie ideal of $R$. Suppose that $d$ and $\delta$ are derivations on $R$ such that $ud(u) - \delta(u)u \in \mathcal{Z}$, the center of $R$, for all $u \in U$. Then either $d = \delta = 0$ or $R$ satisfies the standard polynomial $s_4$ in 4 variables.*

*Proof.* Assume that either $d \neq 0$ or $\delta \neq 0$ and proceed to show that $R$ satisfies $s_4$. Consider first that both $d$ and $\delta$ are inner derivations, say, $d(x) = [x, a]$ and $\delta(x) = [x, b]$ for some $a, b$ in $R$ with $a \notin \mathcal{Z}$ or $b \notin \mathcal{Z}$. Then $u[u, a] - [u, b]u \in \mathcal{Z}$ or, equivalently, $u^2a - u(a + b)u + bu^2 \in \mathcal{Z}$ for all $u \in U$. Let $I$ be the ideal of $R$ generated by $[U, U]$. Since $U \not\subseteq \mathcal{Z}$, we have $[I, I] \subseteq U$ unless $R$ satisies $s_4$ [10, Theorems 4 and 13]. Thus, replacing $u$ with $[x, y]$ for $x, y$ in $I$, we may assume that $p(x, y, z) = \big[[x, y]^2a - [x, y](a + b)[x, y] + b[x, y]^2, z\big] = 0$ holds for all $x, y, z$ in $I$. By a theorem due to Chuang [2,Theorem 2], this GPI is also satisfied by $RC$ where $C$ is the extended centroid of $R$. In case $C$ is infinite, we have $p(x, y, z) = 0$ for all $x, y, z$ in $RC \otimes_C \overline{C}$ by the corollary above, where $\overline{C}$ is the algebraic closure of $C$. Since both $RC$ and $RC \otimes_C \overline{C}$ are prime and centrally closed [3, Theorems 2.5 and 3.5] we may replace $R$ with $RC$ or $RC \otimes_C \overline{C}$ according as $C$ is finite or infinite respectively. Thus we may assume further that $R$ is centrally closed over $C$ and that $C$ is either finite or algebraically closed.

By Martindale's theorem [12], $R$ is then a primitive ring having nonzero socle $H$ with $C$ as the associated division ring. In light of Jacobson's theorem [7, p.75], $R$ is isomorphic to a dense ring of linear transformations of some vector space $V$ over $C$, and $H$ consists of the linear transformations in $R$ of finite rank. Assume first that $V$ is finite-dimensional over $C$. Then the density of $R$ on ${}_CV$ implies that $R \cong M_n(C)$ with $n = \dim_C V$. Let $\{e_{ij}\}$ be the usual matrix units and write $a = \sum \alpha_{ij}e_{ij}$ and $b = \sum \beta_{ij}e_{ij}$ with $\alpha_{ij}, \beta_{ij}$ in $C$. For $i \neq j$, we have $p(e_{ji}, e_{ii}, e_{jj}) = (\alpha_{ij} + \beta_{ij})e_{ji} = 0$ and so $\alpha_{ij} + \beta_{ij} = 0$. That is, $a + b$ is diagonal. Let $\sigma$ be the inner automorphism

$x^\sigma = (1+e_{ij})x(1-e_{ij})$ for $i \neq j$. Then $p^\sigma(x,y,z) = \big[[x,y]^2 a^\sigma - [x,y](a^\sigma + b^\sigma)[x,y] + b^\sigma[x,y]^2, z\big] = 0$ is also a GPI for $R$. As we have shown, $a^\sigma + b^\sigma = (a+b)^\sigma$ must be diagonal. Now $(a+b)^\sigma = (a+b) + \big((\alpha_{jj}+\beta_{jj}) - (\alpha_{ii}+\beta_{ii})\big)e_{ij}$, so $\alpha_{jj} + \beta_{jj} = \alpha_{ii} + \beta_{ii}$. In other words, $a+b$ is central in $R$. Hence, $p(x,y,z) = \big[[x,y]^2 a - a[x,y]^2, z\big]$. If $n \geq 3$, for distinct $i,j,k$, we have $p(e_{ik} - e_{kj}, e_{kk}, e_{ii}) = (\alpha_{ii} - \alpha_{jj})e_{ij} - \sum_{h \neq i,j} \alpha_{jh} e_{ih} = 0$, and so $\alpha_{ii} = \alpha_{jj}$ and $\alpha_{jk} = 0$. That is, $a \in \mathcal{Z}$. However, $a + b \in \mathcal{Z}$ and so it follows that both $a$ and $b$ are in $\mathcal{Z}$, a contradiction. Hence $n = 2$ and $R \cong M_2(C)$ satisfies $s_4$.

Assume next that $V$ is infinite-dimensional over $C$. For any $e = e^2 \in H$, we have $eRe \cong M_n(C)$ with $n = \dim_C Ve$. Since $R$ satisfies $ep(exe, eye, eze)e = 0$, $eRe$ satisfies the GPI $p_e(x,y,z) = \big[[x,y]^2 eae - [x,y](eae + ebe)[x,y] + ebe[x,y]^2, z\big] = 0$. As we have seen above, $eae$ and $ebe$ are both central in $eRe$ if $n \geq 3$. Given any $h \in H$, by Litoff's theorem [9, p.280], there is an idempotent $e \in H$ so that $h, ha, ah, hb$ and $bh$ are all in $eRe$. Since $V$ is infinite-dimensional over $C$, we may choose $e$ so that $n = \dim_C Ve \geq 3$. Then both $eae$ and $ebe$ are central in $eRe$. Hence, $ah = eah = eaeh = heae = hae = ha$ and similarly $bh = hb$. Thus, both $a$ and $b$ centralize the nonzero ideal $H$ of the prime ring $R$ and hence lie in $\mathcal{Z}$. This contradiction shows that $R$ satisfies $s_4$ in case both $d$ and $\delta$ are inner.

Now we consider the general case. Linearize the relation $ud(u) - \delta(u)u \in \mathcal{Z}$ to obtain $ud(v) - \delta(v)u + vd(u) - \delta(u)v \in \mathcal{Z}$ for all $u, v$ in $U$. Replacing $v$ with $[u,r]$ for $r \in R$ and using $ud(u) - \delta(u)u \in \mathcal{Z}$, we have $u[u, d(r)] - [u, \delta(r)]u \in \mathcal{Z}$ for all $u \in U$ and $r \in R$. Thus, for each $r \in R$, the inner derivations $D_r(x) = [x, d(r)]$ and $\Delta_r(x) = [x, \delta(r)]$ satisfy $uD_r(u) - \Delta_r(u)u \in \mathcal{Z}$ for all $u \in U$. Hence, either $R$ satisfies $s_4$ or $D_r = \Delta_r = 0$. In the later case, $d(r)$ and $\delta(r)$ are all central for each $r \in R$. Recall that either $d \neq 0$ or $\delta \neq 0$. Suppose that $d \neq 0$ and $d(r) \in \mathcal{Z}$ for all $r \in R$. By a theorem due to Herstein [5], $R$ is commutative if char $R \neq 2$, and $x^2 \in \mathcal{Z}$ for all $x \in R$. In the later case, $R$ satisfies a polynomial identity (PI) of degree 3 and hence is commutative by the PI results due to Kaplansky, Posner and Rowen [8]. The commutativity of $R$ contradicts to the existence of the noncentral Lie ideal $U$. Therefore $R$ must satisfy $s_4$.

We conclude with a remark that the condition $ud(u) - \delta(u)u \in \mathcal{Z}$ need not imply the commutativity of $R$. For instance, if $R = M_2(F)$ for $F$ a field of characteristic 2 and if $U = [R, R]$, then $u^2 \in \mathcal{Z}$ for all $u \in U$. For a nonzero inner derivation $d$ defined by $d(x) = [x, a]$ we have $ud(u) + d(u)u = u^2a + au^2 = 0$ for all $u \in U$. In this case, $R$ satisfies $s_4$ but is not commutative.

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