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HOLLAND'S METHOD IN 3-DIMENSIONAL COVARIANCE STABILIZING TRANSFORMATIONS

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Abstract. Holland's method is extended to the 3-dimensional case. A necessary and sufficient condition for the existence of solutions is derived. Also, whenever the solutions exist, they may be obtained by solving a system of partial differential equations of the first order.

1. Introduction. Holland (1973) extended the idea of covariance stabilizing transformations from one-dimensional case to multivariate case, and gave a necessary and sufficient condition for the existence of 2-dimensional covariance stabilizing transformations. He also noted that his method does not appear to generalize to the cases more than 2-dimensions. The main reason that we are not likely to generalize Holland's method is that it is difficult to write a general $n \times n$ orthogonal matrix in terms of $n(n-1)/2$ independent parameters. However, this can be done for a 3×3 orthogonal matrix. In this paper, following Holland's idea, we extend his result to the 3-dimensional case by proving a necessary and sufficient condition for the existence of 3-dimensional covariance stabilizing transformations. To get through with this job, some tedious computations more complicated than the 2-dimensional case are inevitable.

2. Review of Holland's method. The question of whether a covariance stabilizing transformation exists is equivalent to solve a system of

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partial differential equations $J_F \cdot \Sigma \cdot J_F^T = I$ for $F = (f_1, f_2, \ldots, f_n)$ where J_F is the Jacobian matrix of F, and Σ is a given positive definite covariance matrix. Notice that the function F, and the matrix Σ are assumed to be functions of independent variables x, y, z and the entries of Σ are usually at least C^2 functions. It is also assumed that the domain of F is an open, connected set Ω. The equation $J_F \cdot \Sigma \cdot J_F^T = I$ can be rewritten as $J_F^T \cdot J_F = (\Sigma^{-\frac{1}{2}})^T \Sigma^{-\frac{1}{2}}$, in which $\Sigma^{-\frac{1}{2}}$ is any square root of Σ^{-1}. Then the existence of a solution to the equation is equivalent to the existence of an orthogonal matrix Γ such that $\Gamma\Sigma^{-\frac{1}{2}}$ is a Jacobian matrix for some function F. Holland's technique for solving the general covariance stabilizing transformations problems is to find out the condition under which such an orthogonal matrix Γ will exist. As noted by Holland, it is often easier to do so by choosing $\Sigma^{-\frac{1}{2}}$ to be a triangular matrix. In the sequel, we will focus on the 3-dimensional case. It is noteworthy that the procedure described in Section 3 presents a way to solving this problem in the multi-dimensional case.

3. Necessary and sufficient condition for the existence of 3-dimensional covariance stabilizing transformations. Let

$$\Sigma^{-\frac{1}{2}} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a^{(1)}, a^{(2)}, a^{(3)}).$$

Suppose that there exists an orthogonal matrix Γ such that $\Gamma\Sigma^{-\frac{1}{2}} = (\Gamma a^{(1)}, \Gamma a^{(2)}, \Gamma a^{(3)})$ is a Jacobian matrix of some function. For the case $\det \Gamma = 1$, Γ can be expressed as a product of three matrices Γ_ψ, Γ_θ and Γ_ϕ, *i.e.*, $\Gamma = \Gamma_\psi \Gamma_\theta \Gamma_\phi$, where

$$\Gamma_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

$$\Gamma_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $0 < \psi, \theta, \phi < \pi$ are assumed to be C^2-functions of x, y and z (for details, see for example, Hsiung (1980), p. 50). When $\det \Gamma = -1$, since $\det(-\Gamma) = 1$ and the relation $(-\Gamma)\Sigma^{-\frac{1}{2}} = J_{-F}$ holds, thus, we only need consider the situation for which $\det \Gamma = 1$.

From the Jacobian matrix $\Gamma\Sigma^{-\frac{1}{2}}$, it follows that

$$\frac{\partial}{\partial y}(\Gamma a^{(1)}) = \frac{\partial}{\partial x}(\Gamma a^{(2)}),\ \frac{\partial}{\partial z}(\Gamma a^{(1)}) = \frac{\partial}{\partial x}(\Gamma a^{(3)}),\ \frac{\partial}{\partial z}(\Gamma a^{(2)}) = \frac{\partial}{\partial y}(\Gamma a^{(3)}),$$

or,

$$\begin{aligned} \Gamma^T \cdot \left(\left(\frac{\partial}{\partial x}\Gamma \right) a^{(2)} - \left(\frac{\partial}{\partial y}\Gamma \right) a^{(1)} \right) &= \frac{\partial}{\partial y} a^{(1)} - \frac{\partial}{\partial x} a^{(2)} \\ \Gamma^T \cdot \left(\left(\frac{\partial}{\partial y}\Gamma \right) a^{(3)} - \left(\frac{\partial}{\partial z}\Gamma \right) a^{(2)} \right) &= \frac{\partial}{\partial z} a^{(2)} - \frac{\partial}{\partial y} a^{(3)} \\ \Gamma^T \cdot \left(\left(\frac{\partial}{\partial z}\Gamma \right) a^{(1)} - \left(\frac{\partial}{\partial x}\Gamma \right) a^{(3)} \right) &= \frac{\partial}{\partial x} a^{(3)} - \frac{\partial}{\partial z} a^{(1)}. \end{aligned} \tag{1}$$

By taking partial derivatives on both sides of $\Gamma = \Gamma_\psi \Gamma_\theta \Gamma_\phi$ with respective to x, y and z, we may obtain

$$\begin{aligned} \Gamma^T \cdot \left(\frac{\partial}{\partial x}\Gamma \right) &= \frac{\partial \psi}{\partial x} M_1 + \frac{\partial \phi}{\partial x} M_2 + \frac{\partial \theta}{\partial x} M_3 \\ \Gamma^T \cdot \left(\frac{\partial}{\partial y}\Gamma \right) &= \frac{\partial \psi}{\partial y} M_1 + \frac{\partial \phi}{\partial y} M_2 + \frac{\partial \theta}{\partial y} M_3 \\ \Gamma^T \cdot \left(\frac{\partial}{\partial z}\Gamma \right) &= \frac{\partial \psi}{\partial z} M_1 + \frac{\partial \phi}{\partial z} M_2 + \frac{\partial \theta}{\partial z} M_3 \end{aligned} \tag{2}$$

where

$$M_1 = \begin{pmatrix} 0 & \cos\theta & \cos\phi \sin\theta \\ -\cos\theta & 0 & \sin\phi \sin\theta \\ -\cos\phi \sin\theta & -\sin\phi \sin\theta & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & -\sin\phi \\ 0 & 0 & \cos\phi \\ \sin\phi & -\cos\phi & 0 \end{pmatrix}.$$

Now substitute expressions in (2) for $\Gamma^T \cdot \left(\frac{\partial}{\partial x}\Gamma \right)$, $\Gamma^T \cdot \left(\frac{\partial}{\partial y}\Gamma \right)$ and $\Gamma^T \cdot \left(\frac{\partial}{\partial z}\Gamma \right)$ into (1). Then, after a tedious simplication and rearrangement it yields

$$(3)\qquad \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \left[\frac{\partial\psi}{\partial x}\ \frac{\partial\phi}{\partial x}\ \frac{\partial\theta}{\partial x}\ \frac{\partial\psi}{\partial y}\ \frac{\partial\phi}{\partial y}\ \frac{\partial\theta}{\partial y}\ \frac{\partial\psi}{\partial z}\ \frac{\partial\phi}{\partial z}\ \frac{\partial\theta}{\partial z}\right]^T$$

$$=[B_1\ B_2\ B_3\ B_4\ B_5\ B_6\ B_7\ B_8\ B_9]^T$$

where

$$P = \begin{pmatrix} \cos\theta & 1 & 0 \\ \cos\phi\sin\theta & 0 & -\sin\phi \\ \sin\phi\sin\theta & 0 & \cos\phi \end{pmatrix},$$

$$\begin{aligned} B_1 =& \frac{1}{a_{22}}\frac{\partial a_{11}}{\partial y} - \frac{a_{32}a_{21}}{a_{11}a_{22}a_{33}}\frac{\partial a_{21}}{\partial z} + \frac{a_{21}}{a_{11}a_{22}}\left(\frac{\partial a_{21}}{\partial y} - \frac{\partial a_{22}}{\partial x}\right) \\ &+ \frac{1}{2}\Bigg[-\frac{a_{31}a_{32}}{a_{11}a_{22}a_{33}}\left(\frac{\partial a_{31}}{\partial z} - \frac{\partial a_{33}}{\partial x}\right) + \frac{a_{31}a_{21}}{a_{11}a_{22}a_{33}}\frac{\partial a_{22}}{\partial z} \\ &+ \frac{a_{31}}{a_{11}a_{22}}\left(\frac{\partial a_{31}}{\partial y} - \frac{\partial a_{32}}{\partial x}\right) + \frac{a_{31}}{a_{11}a_{33}}\frac{\partial a_{21}}{\partial z} \\ &+ \frac{a_{31}^2}{a_{11}a_{22}a_{33}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right)\Bigg] - \frac{a_{32}}{a_{22}a_{33}}\frac{\partial a_{11}}{\partial z}, \end{aligned}$$

$$\begin{aligned} B_2 =& \frac{1}{2}\Bigg[-\frac{a_{21}a_{32}}{a_{11}a_{22}a_{33}}\left(\frac{\partial a_{31}}{\partial z} - \frac{\partial a_{33}}{\partial x}\right) - \frac{a_{21}^2}{a_{11}a_{22}a_{33}}\frac{\partial a_{22}}{\partial z} + \\ &\frac{a_{21}}{a_{11}a_{22}}\left(\frac{\partial a_{31}}{\partial y} - \frac{\partial a_{32}}{\partial x}\right) + \frac{a_{21}}{a_{11}a_{33}}\frac{\partial a_{21}}{\partial z} - \frac{a_{21}a_{31}}{a_{11}a_{22}a_{33}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right)\Bigg] \\ &+ \frac{1}{a_{33}}\frac{\partial a_{11}}{\partial z} + \frac{a_{31}}{a_{11}a_{33}}\left(\frac{\partial a_{31}}{\partial z} - \frac{\partial a_{33}}{\partial x}\right), \end{aligned}$$

$$\begin{aligned} B_3 =& \frac{1}{2}\Bigg[\frac{a_{32}}{a_{22}a_{33}}\left(\frac{\partial a_{31}}{\partial z} - \frac{\partial a_{33}}{\partial x}\right) + \frac{a_{21}}{a_{22}a_{33}}\frac{\partial a_{22}}{\partial z} - \frac{1}{a_{22}}\left(\frac{\partial a_{31}}{\partial y} - \frac{\partial a_{32}}{\partial x}\right) \\ &+ \frac{1}{a_{33}}\frac{\partial a_{21}}{\partial z} + \frac{a_{31}}{a_{22}a_{33}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right)\Bigg] \end{aligned}$$

$$\begin{aligned} B_4 =& \frac{1}{a_{11}}\left(\frac{\partial a_{21}}{\partial y} - \frac{\partial a_{22}}{\partial x}\right) + \frac{1}{2}\Bigg[-\frac{a_{32}^2}{a_{11}a_{22}a_{33}}\left(\frac{\partial a_{31}}{\partial z} - \frac{\partial a_{33}}{\partial x}\right) \\ &- \frac{a_{21}a_{32}}{a_{11}a_{22}a_{33}}\frac{\partial a_{22}}{\partial z} + \frac{a_{32}}{a_{11}a_{22}}\left(\frac{\partial a_{31}}{\partial y} - \frac{\partial a_{32}}{\partial x}\right) - \frac{a_{32}}{a_{11}a_{33}}\frac{\partial a_{21}}{\partial z} \\ &+ \frac{a_{31}a_{32}}{a_{11}a_{22}a_{33}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right)\Bigg] + \frac{a_{31}}{a_{11}a_{33}}\frac{\partial a_{22}}{\partial z}, \end{aligned}$$

$$B_5 = \frac{1}{2}\left[\frac{a_{32}}{a_{11}a_{33}}\left(\frac{\partial a_{31}}{\partial z} - \frac{\partial a_{33}}{\partial x}\right) - \frac{a_{21}}{a_{11}a_{33}}\frac{\partial a_{22}}{\partial z} + \frac{1}{a_{11}}\left(\frac{\partial a_{31}}{\partial y} - \frac{\partial a_{32}}{\partial x}\right) + \frac{a_{22}}{a_{11}a_{33}}\frac{\partial a_{21}}{\partial z} + \frac{a_{31}}{a_{11}a_{33}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right)\right] - \frac{a_{21}a_{32}}{a_{11}a_{22}a_{33}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right),$$

$$B_6 = \frac{1}{a_{33}}\frac{\partial a_{22}}{\partial z} + \frac{a_{32}}{a_{22}a_{33}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right),$$

$$B_7 = \frac{1}{2}\left[-\frac{a_{32}}{a_{11}a_{22}}\left(\frac{\partial a_{31}}{\partial z} - \frac{\partial a_{33}}{\partial x}\right) - \frac{a_{21}}{a_{11}a_{22}}\frac{\partial a_{22}}{\partial z} + \frac{a_{33}}{a_{11}a_{22}}\left(\frac{\partial a_{31}}{\partial y} - \frac{\partial a_{32}}{\partial x}\right) + \frac{1}{a_{11}}\frac{\partial a_{21}}{\partial z} + \frac{a_{31}}{a_{11}a_{22}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right)\right],$$

$$B_8 = \frac{1}{a_{11}}\left(\frac{\partial a_{31}}{\partial z} - \frac{\partial a_{33}}{\partial x}\right) - \frac{a_{21}}{a_{11}a_{22}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right),$$

$$B_9 = \frac{1}{a_{22}}\left(\frac{\partial a_{32}}{\partial z} - \frac{\partial a_{33}}{\partial y}\right),$$

From (3) it follows immediately that

$$(4)\qquad P\left[\frac{\partial\psi}{\partial x}\ \frac{\partial\phi}{\partial x}\ \frac{\partial\theta}{\partial x}\right]^T = [B_1\ B_2\ B_3]^T \quad P\left[\frac{\partial\psi}{\partial y}\ \frac{\partial\phi}{\partial y}\ \frac{\partial\theta}{\partial y}\right]^T = [B_4\ B_5\ B_6]^T$$
$$P\left[\frac{\partial\psi}{\partial z}\ \frac{\partial\phi}{\partial z}\ \frac{\partial\theta}{\partial z}\right]^T = [B_7\ B_8\ B_9]^T$$

or

$$(5)\qquad \left[\frac{\partial\psi}{\partial x}\ \frac{\partial\phi}{\partial x}\ \frac{\partial\theta}{\partial x}\right]^T = Q[B_1\ B_2\ B_3]^T \quad \left[\frac{\partial\psi}{\partial y}\ \frac{\partial\phi}{\partial y}\ \frac{\partial\theta}{\partial y}\right]^T = Q[B_4\ B_5\ B_6]^T$$
$$\left[\frac{\partial\psi}{\partial z}\ \frac{\partial\phi}{\partial z}\ \frac{\partial\theta}{\partial z}\right]^T = Q[B_7\ B_8\ B_9]^T$$

in which

$$Q = P^{-1} = \begin{pmatrix} 0 & \dfrac{\cos\phi}{\sin\theta} & \dfrac{\sin\phi}{\sin\theta} \\ 1 & -\dfrac{\cos\theta\cos\phi}{\sin\theta} & -\dfrac{\cos\theta\sin\phi}{\sin\theta} \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

Now differentiate both sides of the first equation in (4) with respect to y, and the second equation with respect to x, we may have

$$(6)\quad \left(\frac{\partial\phi}{\partial y}R+\frac{\partial\theta}{\partial y}S\right)\left[\frac{\partial\psi}{\partial x}\ \frac{\partial\phi}{\partial x}\ \frac{\partial\theta}{\partial x}\right]^T+P\left[\frac{\partial^2\psi}{\partial y\partial x}\ \frac{\partial^2\phi}{\partial y\partial x}\ \frac{\partial^2\theta}{\partial y\partial x}\right]^T = \left[\frac{\partial B_1}{\partial y}\ \frac{\partial B_2}{\partial y}\ \frac{\partial B_3}{\partial y}\right]^T$$

and

$$(7)\quad \left(\frac{\partial\phi}{\partial x}R+\frac{\partial\theta}{\partial x}S\right)\left[\frac{\partial\psi}{\partial y}\ \frac{\partial\phi}{\partial y}\ \frac{\partial\theta}{\partial y}\right]^T+P\left[\frac{\partial^2\psi}{\partial x\partial y}\ \frac{\partial^2\phi}{\partial x\partial y}\ \frac{\partial^2\theta}{\partial x\partial y}\right]^T = \left[\frac{\partial B_4}{\partial x}\ \frac{\partial B_5}{\partial x}\ \frac{\partial B_6}{\partial x}\right]^T$$

where

$$R=\begin{pmatrix}0 & 0 & 0\\ -\sin\phi\sin\theta & 0 & -\cos\phi\\ \cos\phi\sin\theta & 0 & -\sin\phi\end{pmatrix},\quad S=\begin{pmatrix}-\sin\theta & 0 & 0\\ \cos\phi\cos\theta & 0 & 0\\ \sin\phi\cos\theta & 0 & 0\end{pmatrix}.$$

Substituting, respectively, $\left[\frac{\partial\psi}{\partial x},\frac{\partial\phi}{\partial x},\frac{\partial\theta}{\partial x}\right]^T$ and $\left[\frac{\partial\psi}{\partial y},\frac{\partial\phi}{\partial y},\frac{\partial\theta}{\partial y}\right]^T$ in (5) into (6) and (7), then, substracting the resulting equation of (7) from that of (6), it gives that

$$\frac{\partial^2\psi}{\partial y\partial x}=\frac{\partial^2\psi}{\partial x\partial y},\quad \frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\phi}{\partial x\partial y},\quad \frac{\partial^2\theta}{\partial y\partial x}=\frac{\partial^2\theta}{\partial x\partial y},$$

if and only if

$$(8)\quad \left(\frac{\partial\phi}{\partial y}R+\frac{\partial\theta}{\partial y}S\right)Q[B_1\ B_2\ B_3]^T-\left(\frac{\partial\phi}{\partial x}R+\frac{\partial\theta}{\partial x}S\right)Q[B_4\ B_5\ B_6]^T = \left[\frac{\partial B_1}{\partial y}-\frac{\partial B_4}{\partial x}\ \ \frac{\partial B_2}{\partial y}-\frac{\partial B_5}{\partial x}\ \ \frac{\partial B_3}{\partial y}-\frac{\partial B_6}{\partial x}\right]^T.$$

Now, first we obtain $\frac{\partial\phi}{\partial y},\frac{\partial\theta}{\partial y},\frac{\partial\phi}{\partial x}$ and $\frac{\partial\theta}{\partial x}$ from (5). Then, substitute these into (8), and, finally, we are given

$$(9)\quad \frac{\partial}{\partial y}[B_1\ B_2\ B_3]^T-\frac{\partial}{\partial x}[B_4\ B_5\ B_6]^T=[B_4\ B_5\ B_6]^T\times[B_1\ B_2\ B_3]^T$$

where $[B_4,B_5,B_6]^T\times[B_1,B_2,B_3]^T=[B_3B_5-B_2B_6,B_1B_6-B_3B_4,B_2B_4-B_1B_5]^T$ is the usual cross product of two vectors. Consequently,

$$\frac{\partial^2\psi}{\partial y\partial x}=\frac{\partial^2\psi}{\partial x\partial y},\quad \frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\phi}{\partial x\partial y},\quad \frac{\partial^2\theta}{\partial y\partial x}=\frac{\partial^2\theta}{\partial x\partial y}$$

if, and only if, (9) holds. By the analogous argument, we may also drive

$$(10)\quad \frac{\partial}{\partial z}[B_4\ B_5\ B_6]^T - \frac{\partial}{\partial y}[B_7\ B_8\ B_9]^T = [B_7\ B_8\ B_9]^T \times [B_4\ B_5\ B_6]^T$$

and

$$(11)\quad \frac{\partial}{\partial z}[B_1\ B_2\ B_3]^T - \frac{\partial}{\partial x}[B_7\ B_8\ B_9]^T = [B_7\ B_8\ B_9]^T \times [B_1\ B_2\ B_3]^T$$

are, respectively, necessary and sufficient conditions for

$$\frac{\partial^2\psi}{\partial z\partial y} = \frac{\partial^2\psi}{\partial y\partial z},\ \frac{\partial^2\phi}{\partial z\partial y} = \frac{\partial^2\phi}{\partial y\partial z},\ \frac{\partial^2\theta}{\partial z\partial y} = \frac{\partial^2\theta}{\partial y\partial z},$$

and

$$\frac{\partial^2\psi}{\partial z\partial x} = \frac{\partial^2\psi}{\partial x\partial z},\ \frac{\partial^2\phi}{\partial z\partial x} = \frac{\partial^2\phi}{\partial x\partial z},\ \frac{\partial^2\theta}{\partial z\partial x} = \frac{\partial^2\theta}{\partial x\partial z}.$$

Hence, we have finished the proof of the "only if" part of the following theorem.

Theorem 1. *There exists an orthogonal matrix* Γ *such that* $\Gamma\Sigma^{-\frac{1}{2}}$ *is a Jacobian matrix if, and only if,* (9), (10) *and* (11) *hold.*

Notice that we may still obtain the same resulting equations (9), (10), (11) by directly working with (5). The reason we did not do so is, beside products, quotients of cosine and sine functions also get involved in Q, and when taking partial derivatives, the computations will certainly becomes much more complicated.

To complete the proof of "if" part, we need to use the following well-known theorem, namely, the existence and uniqueness theorem for system of partial differential equations of the first order (see Hsiung (1980), p. 309). In fact we only have to use this theorem to guarantee the existence of solutions for our problem.

Theorem 2. *Consider a system of partial differential equations of the form*

$$(12)\quad \frac{\partial y^k}{\partial u^\alpha} = f_\alpha^k(u^1, u^2, \ldots, u^m; y^1, y^2, \ldots, y^n),\ \ k = 1, \ldots, n; \alpha = 1, \ldots, m,$$

where the function f_α^k are of class C^2 and satisfy the integrability conditions

$$\frac{\partial^2 y^k}{\partial u^\alpha \partial u^\beta} = \frac{\partial^2 y^k}{\partial u^\beta \partial u^\alpha}, \quad k = 1, \ldots, n; \ \alpha, \beta = 1, \ldots, m,$$

in some neighborhood of $(u_0^1, \ldots, u_0^m; 0, \ldots, 0) \in E^m \times E^n$. Then there exist neighborhoods U of the origin in E^n and V of $(u_0^1, \ldots, u_0^m)$ in E^m such that for any $(y_0^1, \ldots, y_0^n) \in u$ and all $(u^1, \ldots, u^m) \in V$ there exists a unique solution $y^i(u^1, \ldots, u^m)$, $i = 1, \ldots, n$, of (12), satisfying $y^i(u_0^1, \ldots, u_0^m) = y_0^i$, $i = 1, \ldots, n$.

Now suppose (9), (10), and (11) hold. Obviously, (5) forms a system of partial differential equations of the first order, and satisfies the integrability conditions (9), (10), (11). Therefore, by Theorem 2, there exists a local solution (ψ, ϕ, θ) for (5). Then it follows that locally there exists an orthogonal matrix Γ such that $\Gamma\Sigma^{-\frac{1}{2}}$ is a Jacobian matrix. Hence for each point (x, y, z) in Ω we have a local solution F satisfying $J_F \cdot \Sigma \cdot J_F^T = I$.

Next we shall describe how to construct a global covariance stabilizing transformation by connecting the local solutions together. First choose and fix an arbitrary point $p_0 = (x_0, y_0, z_0)$ in Ω. Assume at $p_0 = (x_0, y_0, z_0)$ we have a local solution F_0 in the neighborhood $B(p_0)$, an open ball with center at p_0, of p_0. Let $p = (x, y, z)$ be any other point in Ω. To obtain a global solution we shall define a local solution at $p = (x, y, z)$ in the following manner. Since Ω is open and connected, we can join $p_0 = (x_0, y_0, z_0)$ to $p = (x, y, z)$ by a path in Ω. Cover the path by a finite nmber of open balls $B(p_0), B(p_1), \ldots, B(p_n)$ satisfying the following requirements.

(i) $p = (x, y, z) \in B(p_n)$.

(ii) $B(p_i) \cap B(p_{i-1}) \neq \emptyset$, $\forall i = 1, \ldots, n$.

(iii) At p_i there exists a local solution F_i on $B(p_i)$, $\forall i = 1, \ldots, n$.

(iv) $F_{i-1} = F_i$ on $B(p_i) \cap B(p_{i-1})$, $\forall i = 1, \ldots, n$.

Notice that since covariance stabilizing transformations are unique up to an arbitrary additive constant and an arbitrary constant orthogonal transformation, we may modify $F_1, F_2, \ldots, F_n$ to meet the requirement (iv).

Then we define the local solution at p to be F_n. Proceed in this manner, starting from a fix point $p_0 = (x_0, y_0, z_0)$, we may define a global solution, say F, with value $F_n(x,y,z)$ at the point $p = (x,y,z)$, *i.e.*, $F(x,y,z) = F_n(x,y,z)$.

Still we must show that the global solution, described and obtained as before, is independent of the choice of the path connecting $p_0 = (x_0, y_0, z_0)$ to $p = (x,y,z)$ and the open covering. Suppose $B(p_0) = B(p'_0)$, $B(p'_1), \ldots, B(p'_m)$, and $F_0, F'_1, \ldots, F'_m$ are alternative open covering and local solutions satisfying the requirements (i), (ii), (iii), and (iv).

Now define functions F, F' on $\cup_{i=0}^{n} B(p_i)$ and $\cup_{i=0}^{m} B(p'_i)$, respectively, as follows: $F(r,s,t) = F_i(r,s,t)$ whenever $(r,s,t) \in B(p_i)$,and $F'(r,s,t) = F'_i(r,s,t)$ whenever $(r,s,t) \in B(p'_i)$. It follows that F and F' are solutions on $\cup_{i=0}^{n} B(p_i)$ and $\cup_{i=0}^{m} B(p'_i)$. Hence $F' = \Gamma F + c$ on $(\cup_{i=0}^{n} B(p_i)) \cap (\cup_{i=0}^{m} B(p'_i))$ where Γ is a constant orthogonal matrix and c is a constant vector. Note that $B(p_0) \subseteq (\cup_{i=0}^{n} B(p_i)) \cap (\cup_{i=0}^{m} B(p'_i))$. Since $|J_F| \neq 0$, then F is an open mapping on $B(p_0)$, and there exists an open ball $D \subseteq F(B(p_0))$ such that $q = \Gamma q + c$ for all $q \in D$. It is not difficult to show that $\Gamma = I$, the identity matrix, and $c = 0$. Therefore $F = F'$ on $(\cup_{i=0}^{n} B(p_i)) \cap (\cup_{i=0}^{m} B(p'_i))$, and this gives $F_n(x,y,z) = F'_m(x,y,z)$. We not only have finished the proof of Theorem 1, but also shown that under conditions (9), (10), and (11) a global solution exists.

4. Example. Now we give the following simple example to illustrate the existence of covariance stabilizing transformation. Let

$$\Sigma^{-1} = \begin{bmatrix} a_{11}(x) & a_{21}(x) & a_{31}(x) \\ 0 & a_{22}(y) & a_{32}(y) \\ 0 & 0 & a_{33}(z) \end{bmatrix} \begin{bmatrix} a_{11}(x) & 0 & 0 \\ a_{21}(x) & a_{22}(y) & 0 \\ a_{31}(x) & a_{32}(y) & a_{33}(z) \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}^2(x) + a_{21}^2(x) + a_{31}^2(x) & a_{21}(x)a_{22}(y) + a_{31}(x)a_{32}(y) & a_{31}(x)a_{33}(z) \\ a_{21}(x)a_{22}(y) + a_{31}(x)a_{32}(y) & a_{22}^2(y) + a_{32}^2(y) & a_{32}(y)a_{33}(z) \\ a_{31}(x)a_{33}(z) & a_{32}(y)a_{33}(z) & a_{33}^2(z) \end{bmatrix}$$

where $a_{11}, a_{21}, a_{31}, a_{22}, a_{32}, a_{33}$ are C^2-functions, and (x,y,z) is restricted so that $a_{11}(x) > 0$, $a_{22}(y) > 0$, $a_{33}(z) > 0$. Then Σ is positive definite. In this case, $B_i = 0$, $\forall\ i = 1, 2, \ldots, 9$, and obviously conditions (9), (10), and (11) hold. Thus, by Theorem 1, covariance stabilizing transformations exist for this example.

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