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ON THE SPECTRUM OF THE HILL DIFFERENCE OPERATOR

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Abstract. We characterize the spectrum of the Hill difference operator

$$(Hu)_n = b_{n-1}u_{n-1} + a_n u_n + b_n u_{n+1}$$

where a_n and b_n are real periodic functions of n with the same period and $b_n > 0$ for all n. We also show that H has purely absolutely continuous spectrum.

1. Introduction. The purpose of this paper is to characterize the spectrum of the linear operator $H : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ defined by

$$(Hu)_n = b_{n-1}u_{n-1} + a_n u_n + b_n u_{n+1} \tag{1}$$

where $a_n, b_n \in \mathbf{R}$ and $b_n > 0$ for all n. In addition, it is assumed that a_n and b_n are periodic functions of n with the same period p so that

$$a_m = a_n, \quad b_m = b_n \quad \text{if } m \equiv n \pmod p. \tag{2}$$

Under these conditions, H is a bounded self-adjoint operator.

The operator H is connected with the recurrence equation

$$b_{n-1}u_{n-1} + (a_n - \lambda)u_n + b_n u_{n+1} = 0 \quad (n \in \mathbf{Z}) \tag{3}$$

where $\lambda \in \mathbf{R}$ is regarded as the eigenvalue parameter. It was shown in Hochstadt [3] that the theory associated with (2) and (3) is completely analogous to the theory of Hill's equation (the second-order linear differential

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equation with real periodic coefficients). For this reason we shall call H a Hill's difference operator. As references to the general theory of Hill's equation we mention Magnus and Winkler [5] and Eastham [2].

Following [3], we define the discriminant associated to (2) and (3) to be

$$\Delta(\lambda) = \text{trace}(A_{p-1}A_{p-2}\dots A_0)$$

where

$$A_k = \begin{pmatrix} 0 & 1 \\ -b_k/b_{k+1} & (\lambda - a_{k+1})/b_{k+1} \end{pmatrix}.$$

The discriminant is a polynomial in λ of precise degree p which plays an important role in the discussion of the solutions of (3).

Let $\sigma(H)$ denote the spectrum of H. In this paper, we shall prove the following

Theorem. (a) *H has no eigenvalues.*
(b) $\sigma(H) = \{\lambda \in \mathbf{R} : |\Delta(\lambda)| \leq 2\}$.
(c) *H has purely absolutely continuous spectrum.*

2. Boundedness and Periodicity of Solutions. In this section, we state some basic facts that we need in the proof of the theorem. The following results concerning boundedness and periodicity of solutions of (3) and the location of stability and instability intervals were established in Hochstadt [3].

(I) If $|\Delta(\lambda)| \neq 2$ and ρ, ρ^{-1} are solutions of $\rho^2 - \Delta(\lambda)\rho + 1 = 0$, then (3) has two linearly independent solutions u_n and v_n satisfying

$$u_{k+np} = \rho^n u_k, \quad v_{k+np} = \rho^{-n} v_k \quad (k, n \in \mathbf{Z}).$$

It follows that all non-trivial solutions of (3) are unbounded if $|\Delta(\lambda)| > 2$ while all solutions of (3) are bounded if $|\Delta(\lambda)| < 2$.

(II) If $\Delta(\lambda) = 2$, then $\rho = 1$, and (3) has at least one solution of period p. A second solution either is also of period p or else grows linearly with n as $n \to \infty$. If $\Delta(\lambda) = -2$, then $\rho = -1$, and (3) has at least one solution of semi-period p. A second solution either is also of semi-period p or else grows linearly with n as $n \to \infty$.

(III) Denote the zeros of $\Delta(\lambda)-2$ by $\lambda_1, \lambda_2, \ldots, \lambda_p$, and those of $\Delta(\lambda)+2$ by $\mu_1, \mu_2, \ldots, \mu_p$. These zeros are real and at most of multiplicity 2. Moreover, they occur in the order:

$$\lambda_p > \mu_p \geq \mu_{p-1} > \lambda_{p-1} \geq \lambda_{p-2} > \mu_{p-2} \geq \cdots.$$

The intervals (μ_p, λ_p), $(\lambda_{p-1}, \mu_{p-1})$, $(\mu_{p-2}, \lambda_{p-2})$, $\ldots$ in which $|\Delta(\lambda)| < 2$ are called the stability intervals of (3), and the intervals (λ_p, ∞), (μ_{p-1}, μ_p), $(\lambda_{p-2}, \lambda_{p-1})$, $\ldots$ in which $|\Delta(\lambda)| > 2$ are called the instability intervals of (3).

Remark. Our Theorem (b) now asserts that $\sigma(H) = [\mu_1, \lambda_1] \cup [\lambda_2, \mu_2] \cup \ldots \cup [\lambda_{p-1}, \mu_{p-1}] \cup [\mu_p, \lambda_p]$ if p is odd while $\sigma(H) = [\lambda_1, \mu_1] \cup [\mu_2, \lambda_2] \cup \ldots \cup [\lambda_{p-1}, \mu_{p-1}] \cup [\mu_p, \lambda_p]$ if p is even, which is formed by the union of the closures of the stability intervals.

3. Proofs of Theorem (a) and (b).

(a) Assume, to reach a contradiction, that some λ is an eigenvalue of H. Then the eigenspace $W = \{u \in \ell^2(\mathbf{Z}) : Hu = \lambda u\} \neq \{0\}$, and W is clearly finite dimensional. Let $S : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ be the shift operator defined by $(Su)_n = u_{n-p}$. Then (1) and (2) imply that $SH = HS$. It follows that W is an invariant subspace of S. Since S is unitary, so is its restriction $S|_W$ on W. As a result, $S|_W$ must have an eigenvalue α with $|\alpha| = 1$. Thus, there exists a non-zero vector $u \in W$ such that $Su = \alpha u$; that is, $u_{n-p} = \alpha u_n$ for all n. It follows that $|u_{n-p}| = |u_n|$ for all n. This contradicts that $u \in \ell^2(\mathbf{Z})$. Therefore, H has no eigenvalues.

(b) Let $\lambda \in \mathbf{R}$ with $|\Delta(\lambda)| \leq 2$. To prove $\lambda \in \sigma(H)$, it suffices to show there exists a sequence of unit vectors $\{v^k : k = 1, 2, \ldots\}$ in $\ell^2(\mathbf{Z})$ such that $\|(H - \lambda)v^k\| \to 0$ as $k \to \infty$. Referring to (I) and (II) of Section 2, the equation (3) has at least one non-trivial solution u_n satisfying

$$u_{n+p} = \rho u_n, \quad \text{where } |\rho| = 1. \tag{4}$$

Put

$$c_k = 1 \Big/ \left(2k \sum_{n=1}^{p} |u_n|^2\right)^{1/2}, \quad d_{k,n} = \begin{cases} 1 & \text{if } 1 - kp \leq n \leq kp \\ 0 & \text{otherwise} \end{cases}$$

and define $(v^k)_n = c_k d_{k,n} u_n \quad (k, n \in \mathbf{Z},\ k \geq 1)$. Then, using (4), one easily computes that $\|v^k\| = 1$ for all k. Also, using (3), one has

$$\begin{aligned}[(H-\lambda)v^k]_n &= c_k[d_{k,n-1}b_{n-1}u_{n-1} + d_{k,n}(a_n-\lambda)u_n + d_{k,n+1}b_n u_{n+1}] \\ &= c_k[(d_{k,n}-d_{k,n-1})(a_n-\lambda)u_n + (d_{k,n+1}-d_{k,n-1})b_n u_{n+1}].\end{aligned}$$

For a fixed k, since

$$d_{k,n} - d_{k,n-1} = \begin{cases} -1 & \text{if } n = 1+kp \\ 1 & \text{if } n = 1-kp \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_{k,n+1} - d_{k,n-1} = \begin{cases} -1 & \text{if } n = kp,\ n = 1+kp \\ 1 & \text{if } n = -kp,\ n = 1-kp \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$\begin{aligned}&\|(H-\lambda)v^k\|^2 \\ =&c_k^2 \sum_{n=-\infty}^{\infty} |(d_{k,n}-d_{k,n-1})(a_n-\lambda)u_n + (d_{k,n+1}-d_{k,n-1})b_n u_{n+1}|^2 \\ =&c_k^2\{|b_{-kp}u_{1-kp}|^2 + |(a_{1-kp}-\lambda)u_{1-kp} + b_{1-kp}u_{2-kp}|^2 + \\ &\qquad |b_{kp}u_{1+kp}|^2 + |(a_{1+kp}-\lambda)u_{1+kp} + b_{1+kp}u_{2+kp}|^2\} \\ =&2c_k^2\{b_0^2|u_1|^2 + |(a_1-\lambda)u_1 + b_1u_2|^2\}\end{aligned}$$

where we have used (2) and (4) in the last step. Since $c_k \to 0$ as $k \to \infty$, we conclude that $\|(H-\lambda)v^k\| \to 0$, so $\lambda \in \sigma(H)$. This proves that $\{\lambda \in \mathbf{R} : |\Delta(\lambda)| \leq 2\} \subset \sigma(H)$.

To prove the reverse inclusion, we suppose now $|\Delta(\lambda)| > 2$ and prove that $\lambda \notin \sigma(H)$. By the result of (a), it suffices to show that $H - \lambda$, as an operator on $\ell^2(\mathbf{Z})$, is onto. We shall consider only the case that $\Delta(\lambda) > 2$. The proof for $\Delta(\lambda) < -2$ is similar. Referring to (I) of Section 2, the equation (3) has two linearly independent solutions u_n and v_n satisfying

$$u_{k+np} = \rho^n u_k, \quad v_{k+np} = \rho^{-n} v_k \tag{5}$$

where we may assume $\rho > 1$ since $\Delta(\lambda) > 2$. For $w \in \ell^2(\mathbf{Z})$, we define

$$(Gw)_k = \frac{1}{c}\left\{v_k \sum_{n=-\infty}^{k} u_n w_n + u_k \sum_{n=k+1}^{\infty} v_n w_n\right\} \tag{6}$$

where c denotes the constant value of $b_{n-1}(u_{n-1}v_n - v_{n-1}u_n)$. To see the value of c is independent of n, we note that since u_n and v_n are solutions of (3),

$$\begin{aligned}&b_{n-1}(u_{n-1}v_n - v_{n-1}u_n)\\ =&[-b_n u_{n+1} + (\lambda - a_n)u_n]v_n + [b_n v_{n+1} - (\lambda - a_n)v_n]u_n\\ =&b_n(u_n v_{n+1} - v_n u_{n+1}).\end{aligned}$$

Also, since $\rho > 1$, (5) implies that the series in (6) converge absolutely. We now show that $H - \lambda$ is invertible in $\ell^2(\mathbf{Z})$ with inverse G. For this, we compute

$$\begin{aligned}[(H-\lambda)Gw]_n &= b_{n-1}(Gw)_{n-1} + (a_n - \lambda)(Gw)_n + b_n(Gw)_{n+1}\\ &= b_{n-1}\left(v_{n-1}\sum_{j=-\infty}^{n-1} u_j w_j + u_{n-1}\sum_{j=n}^{\infty} v_j w_j\right)\Big/c\\ &\quad + (a_n - \lambda)\left(v_n \sum_{j=-\infty}^{n} u_j w_j + u_n \sum_{j=n+1}^{\infty} v_j w_j\right)\Big/c\\ &\quad + b_n\left(v_{n+1}\sum_{j=-\infty}^{n+1} u_j w_j + u_{n+1}\sum_{j=n+2}^{\infty} v_j w_j\right)\Big/c\\ &= \left(\sum_{j=-\infty}^{n} u_j w_j\right)[b_{n-1}v_{n-1} + (a_n - \lambda)v_n + b_n v_{n+1}]/c\\ &\quad + \left(\sum_{j=n+1}^{\infty} v_j w_j\right)[b_{n-1}u_{n-1} + (a_n - \lambda)u_n + b_n u_{n+1}]/c\\ &\quad + b_{n-1}(u_{n-1}v_n - v_{n-1}u_n)w_n/c\\ &= w_n.\end{aligned}$$

So, $(H - \lambda)Gw = w$ for all $w \in \ell^2(\mathbf{Z})$. The proof will be complete if we can show that $Gw \in \ell^2(\mathbf{Z})$. Choose $m > 0$ so that $\rho = e^{mp}$, and define

$$\alpha_k = e^{-mk}u_k, \quad \beta_k = e^{mk}v_k. \tag{7}$$

Then, by (5), α_k and β_k have period p. Let $M = \sup_k\{|\alpha_k|, |\beta_k|\}$. By (6) and (7), we have $|(Gw)_k| \leq M^2[G_1(k)+G_2(k)]/|c|$, where

$$G_1(k) = e^{-mk} \sum_{j=-\infty}^{k} e^{mj}|w_j|, \quad G_2(k) = e^{mk} \sum_{j=k+1}^{\infty} e^{-mj}|w_j|.$$

It follows by Young's inequality that

$$\sum_{k=-\infty}^{\infty} |(Gw)_k|^2 \leq 2M^4 \sum_{k=-\infty}^{\infty} [G_1(k)^2 + G_2(k)^2]/|c|^2$$

$$\leq 2M^4\left[\left(\sum_{j=0}^{\infty} e^{-mj}\right)^2 + \left(\sum_{j=-\infty}^{-1} e^{mj}\right)^2\right]\|w\|^2/|c|^2 < \infty.$$

Thus $Gw \in \ell^2(\mathbf{Z})$, which completes the proof.

4. Proof of Theorem (c). We first note that a simple analysis shows that the following are equivalent:

(i) The equation (3) has a non-trivial solution u_n satisfying the boundary conditions $u_p = \rho u_0$ and $u_{p+1} = \rho u_1$.

(ii) $\rho^2 - \Delta(\lambda)\rho + 1 = 0$.

(iii) $\det(\lambda I - L_\rho) = 0$, where

$$L_\rho = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 & \rho^{-1}b_p \\ b_1 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & b_2 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{p-1} & b_{p-1} \\ \rho b_p & 0 & 0 & \dots & b_{p-1} & a_p \end{pmatrix}.$$

For $\rho = e^{i\theta} (\theta \in \mathbf{R})$, L_ρ is self-adjoint, and (ii) is equivalent to

$$(8) \qquad \Delta(\lambda) = 2\cos\theta.$$

It follows that (8) must have p real solutions for λ. We denote these solutions by $\lambda_1(\theta) \leq \lambda_2(\theta) \leq \ldots \leq \lambda_p(\theta)$. Evidently, no $\lambda_j(\cdot)$ is constant. The $\lambda_j(\theta)$ are the eigenvalues of the matrix $L_{e^{i\theta}}$. Also, we can choose the associated eigenvectors $\xi_1(\theta), \xi_2(\theta), \ldots, \xi_p(\theta)$ so that they form an orthonormal basis of $\mathbf{C}^p$ for each θ. It follows from the analytic perturbation theory that all the $\lambda_j(\theta)$ and $\xi_j(\theta)$ are analytic functions of θ (see e.g. Kato [4], p.71, Theorem 1.10 and Reed-Simon[6], Theorem XII.4).

We write $H_\theta = L_{e^{i\theta}}$. The key idea in the proof will be to realize H as a direct integral of the H_θ over $[0, 2\pi]$. To do this, we prove the following

Lemma. *The map $U : \ell^2(\mathbf{Z}) \to L^2([0, 2\pi], d\theta; \mathbf{C}^p)$ given by*

$$[(Uu)(\theta)]_k = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-in\theta} u_{k+np}$$

for $\theta \in [0, 2\pi]$, $k = 1, 2, \ldots, p$, is well defined for $u \in \ell^2(\mathbf{Z})$ with finite support (only a finite number of terms different from zero), and uniquely extendable to a unitary operator. Moreover,

$$\tilde{H} \equiv UHU^{-1} = \int_{[0,2\pi]}^{\oplus} H_\theta d\theta \tag{9}$$

that is, $(\tilde{H}\varphi)(\theta) = H_\theta \varphi(\theta)$ for all $\varphi \in L^2([0, 2\pi], d\theta; \mathbf{C}^p)$.

Proof. For $u \in \ell^2(\mathbf{Z})$ with finite support, we have

$$\|(Uu)(\theta)\|^2 = \sum_{k=1}^{p} |[(Uu)(\theta)]_k|^2 = \frac{1}{2\pi} \sum_{k=1}^{p} \left| \sum_{n=-\infty}^{\infty} e^{-in\theta} u_{k+np} \right|^2$$

so that

$$\begin{aligned} \|Uu\|^2 &= \int_0^{2\pi} \|(Uu)(\theta)\|^2 d\theta \\ &= \frac{1}{2\pi} \sum_{k=1}^{p} \int_0^{2\pi} \left| \sum_{n=-\infty}^{\infty} e^{-in\theta} u_{k+np} \right|^2 d\theta \\ &= \sum_{k=1}^{p} \sum_{n=-\infty}^{\infty} |u_{k+np}|^2 = \|u\|^2. \end{aligned}$$

Thus, U has a unique extension to an isometry. To see that U is onto, we define for $\varphi \in L^2([0, 2\pi], d\theta; \mathbf{C}^p)$

$$(U^*\varphi)_{k-mp} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-im\theta} [\varphi(\theta)]_k d\theta \quad (m \in \mathbf{Z}, k = 1, 2, \ldots, p).$$

Then

$$
\begin{aligned}
\|U^*\varphi\|^2 &= \sum_{k=1}^{p} \sum_{m=-\infty}^{\infty} |(U^*\varphi)_{k-mp}|^2 \\
&= \sum_{k=1}^{p} \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \left| \int_0^{2\pi} e^{-im\theta} [\varphi(\theta)]_k d\theta \right|^2 \\
&= \sum_{k=1}^{p} \int_0^{2\pi} |[\varphi(\theta)]_k|^2 d\theta = \|\varphi\|^2
\end{aligned}
$$

where we have used the Parseval formula. Thus, U^* is an isometry. Moreover, it is easy to verify that $U^* = U^{-1}$.

To prove (9), it suffices to show that $H_\theta(Uu)(\theta) = (UHu)(\theta)$ for $u \in \ell^2(\mathbf{Z})$ with finite support. On using the periodicity of a_n and b_n, we have for $k = 2, \ldots, p-1$,

$$
\begin{aligned}
&[H_\theta(Uu)(\theta)]_k \\
=&b_{k-1}[(Uu)(\theta)]_{k-1} + a_k[(Uu)(\theta)]_k + b_k[(Uu)(\theta)]_{k+1} \\
=&\frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-in\theta} [b_{k+np-1}u_{k+np-1} + a_{k+np}u_{k+np} + b_{k+np}u_{k+np+1}] \\
=&\frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-in\theta} (Hu)_{k+np} = [(UHu)(\theta)]_k.
\end{aligned}
$$

Since $e^{-i\theta}[(Uu)(\theta)]_p = [(Uu)(\theta)]_0, \quad e^{i\theta}[(Uu)(\theta)]_1 = [(Uu)(\theta)]_{p+1}$, we also have

$$
\begin{aligned}
[H_\theta(Uu)(\theta)]_1 &= e^{-i\theta} b_p[(Uu)(\theta)]_p + a_1[(Uu)(\theta)]_1 + b_1[(Uu)(\theta)]_2 \\
&= [(UHu)(\theta)]_1
\end{aligned}
$$

and

$$
\begin{aligned}
[H_\theta(Uu)(\theta)]_p &= b_{p-1}[(Uu)(\theta)]_{p-1} + a_p[(Uu)(\theta)]_p + e^{i\theta} b_p[(Uu)(\theta)]_1 \\
&= [(UHu)(\theta)]_p
\end{aligned}
$$

as required. This completes the proof of the lemma.

With this lemma, we can now apply [6] Theorem XIII.86 and conclude that $\tilde{H}$, and thus H, has purely absolutely continuous spectrum (see also [1] Lemma 10.14).

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