BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 21, Number 2, June 1993

HARMONIC ANALYSIS FOR OPERATORS IN HOMOGENEOUS CLASSES

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Abstract. Let $\mathcal{H}$ be the Hilbert space of square integrable functions on the circle group. We study the problems of convergence of Fourier series for certain classes of operators on $\mathcal{H}$. Our analysis applies to the Schatten p-class for $1 < p < \infty$. In particular, a Parseval formula is obtained.

1. Introduction. Throughout this paper, $\mathcal{H}$ will denote the Hilbert space $L^2(\mathbf{T})$ of square integrable functions on the circle group $\mathbf{T}$ with inner product

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt, \quad f,g \in \mathcal{H},$$

and norm $\|f\| = (f,f)^{1/2}$. $\mathcal{B}(\mathcal{H})$ will denote the Banach algebra of all bounded linear operators on $\mathcal{H}$ with the usual operator norm $\|\cdot\|$. The Schatten classes $\mathcal{B}_p(\mathcal{H})$, $1 \leq p \leq \infty$, are the $*$-ideals of compact operators T on $\mathcal{H}$ for which the p-norm:

$$\|T\|_p = \begin{cases} (tr|T|^p)^{1/p} & \text{if } 1 \leq p < \infty \\ \|T\| & \text{if } p = \infty \end{cases}$$

is finite, where $|T| = (T^*T)^{1/2}$.

Received by the editors October 4, 1991.

A.M.S. (1985) subject classification: 43A50, 47B10.

Key Words and Phrases: Homogeneous classes, Schatten classes, Almost invariant operators, Fourier transform, Interpolation and duality.

If $T \in \mathcal{B}(\mathcal{H})$, its Fourier transform is defined in [1] to be the $\mathcal{B}(\mathcal{H})$-valued function $\widehat{T}$ defined on $\mathbf{Z}$ by

$$\widehat{T}(n)f = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}R_{-t}TR_t f\,dt, \quad n \in \mathbf{Z}, f \in \mathcal{H}, \tag{1.1}$$

where R_t is the translation operator defined by $(R_t f)(s) = f(s-t)$, $s \in \mathbf{T}$, and the formal series

$$T \sim \sum_{-\infty}^{\infty} \widehat{T}(n) \tag{1.2}$$

is called the Fourier series of T. It is clear from the definition that $\|\widehat{T}(n)\| \leq \|T\|$ for all n. Moreover, each $\widehat{T}(n)$ satisfies the functional equation

$$\widehat{T}(n)R_t = e^{int}R_t\widehat{T}(n), \quad t \in \mathbf{T}. \tag{1.3}$$

We should point out that the definition (1.2) yields an extension of the usual notion of Fourier series. For, if $T \in \mathcal{B}(\mathcal{H})$ is the operator of multiplication by a function $\varphi \in L^\infty(\mathbf{T})$, then $\widehat{T}(n) = \widehat{\varphi}(n)M_n$ for all n, where $\widehat{\varphi}(n) = (\varphi, e^{in\cdot})$ is the nth Fourier coefficient of φ and M_n is the operator of multiplication by $e^{in\cdot}$.

Once the Fourier transform is defined, it is natural to ask if the series (1.2) converges to T in a certain topology. DeLeeuw [1] showed that the Fourier series of any $T \in \mathcal{B}(\mathcal{H})$ is $C-1$ summable to T in the strong operator topology of $\mathcal{B}(\mathcal{H})$. That is, for each $f \in \mathcal{H}$, $\lim_{n\to\infty} \|\sigma_n(T)f - Tf\| = 0$ where $\sigma_n(T) = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)\widehat{T}(j)$. In particular, the following uniqueness result holds:

$$T = 0 \text{ if and only if } \widehat{T}(n) = 0 \text{ for all } n. \tag{1.4}$$

The purpose of this paper is to present a study of harmonic analysis for certain classes of operators in $\mathcal{B}(\mathcal{H})$. We shall be concerned mostly with the problems of convergence of Fourier series for operators in homogeneous classes. Our analysis relies highly on the methods presented in [3], where the general theory of classical Fourier series is given. In Section 2, we begin by introducing the homogeneous classes and several of their important properties connected with the Fourier transform. We show that the Schatten p-class is homogeneous for $1 \leq p \leq \infty$. In Section 3, we use the results of Section 2 and some known facts to establish a Parseval formula. A stronger form of that formula is given in Section 5. We also prove a Plancherel theorem for Hilbert-Schmidt operators. In Section 4, the convergence of Fourier series and the conjugate Fourier series are discussed. We give a characterization of the homogeneous classes which admit convergence in norm. Finally, in Section 5, we apply our results to the Schatten classes.

2. Homogeneous classes and examples.

Definition 2.1. A subspace $\mathcal{F}$ of $\mathcal{B}(\mathcal{H})$ is called a homogeneous class if there is a norm $\|\cdot\|_{\mathcal{F}}$ on $\mathcal{F}$ with the following properties:

(a) $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ forms a Banach space.

(b) $\|T\|_{\mathcal{F}} \geq \|T\|$ for all $T \in \mathcal{F}$.

(c) If $T \in \mathcal{F}$ and $t \in \mathbf{T}$, then $T_t = R_{-t}TR_t \in \mathcal{F}$ and $\|T_t\|_{\mathcal{F}} = \|T\|_{\mathcal{F}}$.

(d) $\lim_{t\to 0} \|T_t - T\|_{\mathcal{F}} = 0$ for all $T \in \mathcal{F}$.

Remark 2.2. We note that if $\mathcal{F}$ is a homogeneous class in $\mathcal{B}(\mathcal{H})$, then properties (c) and (d) imply that for any $T \in \mathcal{F}$, the mapping $t \longmapsto T_t$ is continuous from $\mathbf{T}$ into the $\|\cdot\|_{\mathcal{F}}$ norm topology of $\mathcal{F}$. As a consequence, the Fourier transform of T can be defined directly by the $\mathcal{F}$-valued integral

$$\widehat{T}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}R_{-t}TR_t dt, \quad n \in \mathbb{Z}. \tag{2.1}$$

It then follows from elementary properties of the integral and (c) that

$$\|\widehat{T}(n)\|_{\mathcal{F}} \leq \|T\|_{\mathcal{F}} \quad \text{for all } n. \tag{2.2}$$

With these observations, we can pass to the same argument as in the proof of Proposition 3.4 of [1] to get

Proposition 2.3. *Let $\mathcal{F}$ be a homogeneous class in $\mathcal{B}(\mathcal{H})$. If $T \in \mathcal{F}$, then its Fourier series is $C-1$ summable to T in the $\|\cdot\|_{\mathcal{F}}$ norm.*

As a corollary, we obtain the following analogue of the Riemann-Lebesgue lemma.

Corollary 2.4. *Let $\mathcal{F}$ be a homogeneous class in $\mathcal{B}(\mathcal{H})$. If $T \in \mathcal{F}$, then*

$$\lim_{|n|\to\infty} \|\widehat{T}(n)\|_{\mathcal{F}} = 0.$$

Proof. From (1.1) and (1.3), we find that

$$[\widehat{T}(n)]^{\wedge}(k) = \delta_{k,n}\widehat{T}(n) \tag{2.3}$$

where $\delta_{k,n}$ is the Kronecker delta. Thus, for $|k| > m$, we have $[\sigma_m(T) - T]^{\wedge}(k) = -\widehat{T}(k)$. So by (2.2), if $|k| > m$, then $\|\widehat{T}(k)\|_{\mathcal{F}} \leq \|\sigma_m(T) - T\|_{\mathcal{F}}$. The corollary now follows from Proposition 2.3.

Recall that an operator T in $\mathcal{B}(\mathcal{H})$ is said to be almost invariant if $\lim_{t\to 0}\|TR_t - R_tT\| = 0$. The space of almost invariant operators in $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{B}_{\sharp}(\mathcal{H})$. It is easy to check that $\mathcal{B}_{\sharp}(\mathcal{H})$ forms a closed subalgebra of $\mathcal{B}(\mathcal{H})$. Also, with the usual operator norm $\mathcal{B}_{\sharp}(\mathcal{H})$ is a homogeneous class. In order to exhibit some important examples of homogeneous classes, we need the following lemma.

Lemma 2.5. *Every compact operator on $\mathcal{H}$ is almost invariant.*

Proof. Since every compact operator on $\mathcal{H}$ is a norm limit of finite rank operators and $\mathcal{B}_{\sharp}(\mathcal{H})$ forms a closed subalgebra of $\mathcal{B}(\mathcal{H})$, it is sufficient to prove the lemma for finite rank T and thus for a T of rank one. Let $T = (\cdot, \varphi)\psi$, where $\varphi, \psi \in \mathcal{H}$. Then for each $f \in \mathcal{H}$,

$$\begin{aligned}(TR_t - R_tT)f &= (f, R_{-t}\varphi)\psi - (f, \varphi)R_t\psi \\ &= (f, R_{-t}\varphi - \varphi)\psi + (f, \varphi)(\psi - R_t\psi).\end{aligned}$$

Thus, by the Schwarz inequality, we obtain

$$\|TR_t - R_tT\| \leq \|R_{-t}\varphi - \varphi\|\,\|\psi\| + \|\varphi\|\,\|\psi - R_t\psi\|.$$

Since the functions in $\mathcal{H}$ translate continuously, this implies that T is almost invariant.

Example 2.6. Recall that the Schatten classes $\mathcal{B}_p(\mathcal{H})$, $1 \leq p \leq \infty$, are the $*$-ideals of compact operators T on $\mathcal{H}$ for which the p-norm:

$$\|T\|_p = \begin{cases} (tr|T|^p)^{1/p} & \text{if } 1 \leq p < \infty \\ \|T\| & \text{if } p = \infty \end{cases}$$

is finite. We claim that for $1 \leq p \leq \infty$, $(\mathcal{B}_p(\mathcal{H}), \|\cdot\|_p)$ is a homogeneous class in $\mathcal{B}(\mathcal{H})$. It is well-known that $\mathcal{B}_p(\mathcal{H})$ is a Banach space under the p-norm, and that $\|T\| \leq \|T\|_p \leq \|T\|_1$. Also, since the translation operators are unitary on $\mathcal{H}$, it follows that $\|T_t\|_p = \|T\|_p$. It remains to verify

$$\lim_{t\to 0} \|T_t - T\|_p = 0 \quad \text{for all } T \in \mathcal{B}_p(\mathcal{H}). \tag{2.4}$$

Since T is compact, we conclude by Lemma 2.5 that $\lim_{t\to 0} \|T_t - T\| = \lim_{t\to 0} \|T_t^* - T^*\| = 0$. (2.4) now follows from the Grümm convergence theorem [5].

3. Parseval's Formula. In this section, we establish and recall (cf. [1, 2, 4, 5]) some results that will be used later. In particular, we prove a Parseval formula for operators in the Schatten classes and a Plancherel theorem for operators in the Hilbert-Schmidt class, $\mathcal{B}_2(\mathcal{H})$.

For each $n \in \mathbb{Z}$, we define

$$\mathcal{B}^n(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : TR_t = e^{int} R_t T \text{ for all } t \in \mathbf{T}\}.$$

Following DeLeeuw [1], we call an operator in $\mathcal{B}(\mathcal{H})$ simple if it is in one of the $\mathcal{B}^n(\mathcal{H})$. It is easy to see that $\mathcal{B}^n(\mathcal{H}) \cap \mathcal{B}^m(\mathcal{H}) = \{0\}$ if $n \neq m$, and that $TS \in \mathcal{B}^{n+m}(\mathcal{H})$ if $T \in \mathcal{B}^n(\mathcal{H})$ and $S \in \mathcal{B}^m(\mathcal{H})$. Moreover, if $\pi_n(T)$ denotes the operator $\widehat{T}(n)$, then π_n is a projection of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}^n(\mathcal{H})$.

Let us call an operator almost simple if it is a finite linear combination of simple operators. An easy computation then shows that an almost simple operator T is of the form

$$T = \sum_{j=-n}^{n} \widehat{T}(j). \tag{3.1}$$

The largest integer n such that $\widehat{T}(n) + \widehat{T}(-n) \neq 0$ will be called the order of T. The space of almost simple operators will be denoted by $\mathcal{B}^\natural(\mathcal{H})$.

Note that Proposition 2.3 implies that the almost simple operators in a homogeneous class $\mathcal{F}$ are dense in the $\|\cdot\|_{\mathcal{F}}$ norm.

We shall need the following lemma.

Lemma 3.1. (a) *An operator T in $\mathcal{B}(\mathcal{H})$ is in $\mathcal{B}^n(\mathcal{H})$ if and only if, for each $m \in \mathbb{Z}$, $T(e^{im\cdot})$ is a constant multiple of $e^{i(n+m)\cdot}$.*
(b) *Let $T \in \mathcal{B}(\mathcal{H})$. Then for each $n, m \in \mathbb{Z}$,*

$$(3.2) \qquad (Te^{im\cdot}, e^{i(n+m)\cdot}) = (\widehat{T}(n)e^{im\cdot}, e^{i(n+m)\cdot}).$$

(c) *If $T \in \mathcal{B}^n(\mathcal{H})$, then $T^* \in \mathcal{B}^{-n}(\mathcal{H})$.*
(d) *Let $T \in \mathcal{B}(\mathcal{H})$. Then $(T^*)^\wedge(n) = [\widehat{T}(-n)]^*$. In particular, T is self-adjoint if and only if $\widehat{T}(-n) = [\widehat{T}(n)]^*$ for all $n \in \mathbb{Z}$.*

Proof. See [1,2].

As usual, we define the nth Fourier coefficient of $f \in \mathcal{H}$ by $\widehat{f}(n) = (f, e^{in\cdot})$. Then (a) and (b) of Lemma 3.1 imply that for any $T \in \mathcal{B}(\mathcal{H})$:

$$(3.3) \qquad (\widehat{T}(n)e^{im\cdot})(t) = (Te^{im\cdot})^\wedge(n+m)e^{i(n+m)t}$$

for all $n, m \in \mathbb{Z}$ and $t \in \mathbf{T}$.

The Hilbert-Schmidt operators are an especially nice class of operators. For example, if $S, T \in \mathcal{B}_2(\mathcal{H})$, then $(S, T) = tr(T^*S)$ defines an inner product so that $\mathcal{B}_2(\mathcal{H})$ becomes a Hilbert space with $\|\cdot\|_2$ norm as its norm. In particular, if $T \in \mathcal{B}_2(\mathcal{H})$, then $\|T\|_2^2 = tr(T^*T) = \sum_n \|Tf_n\|^2$ for any orthonormal basis $\{f_n\}$ of $\mathcal{H}$. Moreover, there is a simple criterion for $T \in \mathcal{B}_2(\mathcal{H})$ in terms of its Fourier transform which is an analogue of the Plancherel theorem for L^2 functions.

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{B}_2(\mathcal{H})$ if and only if, for each n, $\widehat{T}(n) \in \mathcal{B}_2(\mathcal{H})$ and $\sum_{n=-\infty}^{\infty} \|\widehat{T}(n)\|_2^2 < \infty$. In this case, we have*

$$\|T\|_2^2 = \sum_{n=-\infty}^{\infty} \|\widehat{T}(n)\|_2^2.$$

Proof. By (3.3), we have $\|\widehat{T}(n)e^{im\cdot}\| = |(Te^{im\cdot})^\wedge(n+m)|$. Since the exponentials $\{e^{in\cdot} : n \in \mathbb{Z}\}$ form an orthonormal basis for $\mathcal{H}$, it follows that

$$\sum_n \|\widehat{T}(n)\|_2^2 = \sum_n \sum_m \|\widehat{T}(n)e^{im\cdot}\|^2 = \sum_m \sum_n |(Te^{im\cdot})^\wedge(n+m)|^2$$
$$= \sum_m \|Te^{im\cdot}\|^2 = \|T\|_2^2$$

where we have used the usual Plancherel theorem. This completes the proof.

Notice that there are examples of non-compact operators having $\mathcal{B}_2(\mathcal{H})$-valued Fourier transform. For example, let $T \in \mathcal{B}(\mathcal{H})$ be defined by $(Tf)(t) = f(-t)$. It is clear that T is not compact. On the other hand,

$$(Te^{im\cdot})^\wedge(n+m) = (Te^{im\cdot}, e^{i(n+m)\cdot}) = (e^{-im\cdot}, e^{i(n+m)\cdot})$$
$$= \begin{cases} 0 & \text{if } n+2m \neq 0 \\ 1 & \text{if } n+2m = 0. \end{cases}$$

Thus, using (3.3), we find that

$$\|\widehat{T}(n)\|_2^2 = \sum_m \|\widehat{T}(n)e^{im\cdot}\|^2 = \sum_m |(Te^{im\cdot})^\wedge(n+m)|^2$$
$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even}. \end{cases}$$

Of course, we have $\sum_n \|\widehat{T}(n)\|_2^2 = \infty$ in this case.

Finally, we want to give a Parseval formula for operators in the Schatten classes. We recall a Hölder inequality for the $\|\cdot\|_p$ operator norms (see [5], p.31): if $A \in \mathcal{B}_p(\mathcal{H})$, $B \in \mathcal{B}_q(\mathcal{H})$ and $p^{-1} + q^{-1} = r^{-1}$, $1 \le p, q, r \le \infty$, then $AB \in \mathcal{B}_r(\mathcal{H})$ and

$$\|AB\|_r \le \|A\|_p \|B\|_q. \tag{3.4}$$

It is well-known (see [4], p.43) when $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$ that the bounded linear functionals Λ on $\mathcal{B}_p(\mathcal{H})$ are in one-to-one correspondence with the members T of $\mathcal{B}_q(\mathcal{H})$: each $\Lambda \in \mathcal{B}_p(\mathcal{H})^*$ is of the form

$$\Lambda S = (S, T) = tr(T^*S), \quad S \in \mathcal{B}_p(\mathcal{H}),$$

and $\|\Lambda\| = \|T\|_q$. Thus $\mathcal{B}_p(\mathcal{H})^* = \mathcal{B}_q(\mathcal{H})$.

Theorem 3.3. *Suppose* $1 < p,\ q < \infty$ *and* $p^{-1}+q^{-1} = 1$. *If* $S \in \mathcal{B}_p(\mathcal{H})$ *and* $T \in \mathcal{B}_q(\mathcal{H})$, *then*

$$(S,T) = \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right)(\widehat{S}(n), \widehat{T}(n)). \tag{3.5}$$

Proof. For notational simplicity, let $\varphi_j = e^{ij\cdot}$. Then, using (3.3),

$$\begin{aligned}(\widehat{S}(n), \widehat{T}(m)) &= tr(\widehat{T}(m)^*\widehat{S}(n)) = \sum_j (\widehat{S}(n)\varphi_j, \widehat{T}(m)\varphi_j) \\ &= \sum_j ((S\varphi_j)^\wedge(n+j)\varphi_{n+j}, (T\varphi_j)^\wedge(m+j)\varphi_{m+j}) \\ &= \sum_j (S\varphi_j)^\wedge(n+j)\overline{(T\varphi_j)^\wedge(m+j)}\delta_{n+j,m+j}\end{aligned}$$

so

$$(\widehat{S}(n), \widehat{T}(m)) = 0 \text{ if } n \neq m. \tag{3.6}$$

From Proposition 2.3 and Example 2.6, we see that

$$\lim_{N\to\infty} \|\sigma_N(S) - S\|_p = \lim_{N\to\infty} \|\sigma_N(T) - T\|_q = 0.$$

Thus by (3.6), we have for each n,

$$\begin{aligned}(\widehat{S}(n), T) &= \lim_{M\to\infty} \sum_{m=-M}^{M} \left(1 - \frac{|m|}{M+1}\right)(\widehat{S}(n), \widehat{T}(m)) \\ &= \lim_{M\to\infty} \left(1 - \frac{|n|}{M+1}\right)(\widehat{S}(n), \widehat{T}(n)) = (\widehat{S}(n), \widehat{T}(n)).\end{aligned}$$

It follows that

$$\begin{aligned}(S,T) &= \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right)(\widehat{S}(n), T) \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right)(\widehat{S}(n), \widehat{T}(n)).\end{aligned}$$

Remark 3.4. Using a further result (see Section 5), one can improve (3.5) to read:

$$(S,T) = \sum_{n=-\infty}^{\infty} (\widehat{S}(n), \widehat{T}(n)).$$

4. Convergence in Norm and Conjugation. Let $\mathcal{F}$ be a homogeneous class in $\mathcal{B}(\mathcal{H})$. From Remark 2.2, we see that if $T \in \mathcal{F}$, then the nth partial sum of the Fourier series of T:

$$S_n(T) = \sum_{j=-n}^{n} \widehat{T}(j) \in \mathcal{F} \quad \text{for all } n \geq 0.$$

We say that $\mathcal{F}$ admits convergence in norm if for every $T \in \mathcal{F}$,

$$\lim_{n\to\infty} \|S_n(T) - T\|_{\mathcal{F}} = 0.$$

In this section, we wish to characterize the homogeneous classes which have this property. These results are analogous to those for the classical function spaces on the circle group given in Katznelson [3].

We begin with some basic definitions and lemmas.

Definition 4.1. Let $T \in \mathcal{B}(\mathcal{H})$. The conjugate Fourier series of T is defined by

$$-i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\widehat{T}(n) \tag{4.1}$$

where $\operatorname{sgn}(n) = 0$ if $n = 0$ and $\operatorname{sgn}(n) = n/|n|$ otherwise. If (4.1) is the Fourier series of some operator $S \in \mathcal{B}(\mathcal{H})$, we call S the conjugate operator of T and denote it by $\tilde{T}$ so that

$$(\tilde{T})^\wedge(n) = -i \operatorname{sgn}(n)\widehat{T}(n) \quad \text{for all } n \in \mathbf{Z}. \tag{4.2}$$

Note that if $\tilde{T}$ exists, it is unique because of (1.4).

Definition 4.2. We say that a space of operators $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ admits conjugation if for every $T \in \mathcal{S}, \tilde{T}$ exists and belongs to $\mathcal{S}$.

Here is a simple example. Let $\mathcal{A}$ denote the space of all analytic operators on $\mathcal{H}$, i.e., $\mathcal{A} = \{T \in \mathcal{B}(\mathcal{H}) : \widehat{T}(n) = 0 \text{ for all } n < 0\}$. It is easy to see that if $T \in \mathcal{A}$, then $\tilde{T} = -i[T - \widehat{T}(0)]$ and is also in $\mathcal{A}$. Thus, $\mathcal{A}$ admits conjugation.

Lemma 4.3. *Let $\mathcal{F}$ be a homogeneous class in $\mathcal{B}(\mathcal{H})$. If $\mathcal{F}$ admits conjugation, then the mapping $T \mapsto \tilde{T}$ is a bounded linear operator on $\mathcal{F}$.*

Proof. The linearity follows easily from (4.2) and (1.4). To prove the boundedness, we shall apply the closed graph theorem. Suppose $T_n \longrightarrow T$ and $\tilde{T}_n \longrightarrow S$ in $\mathcal{F}$. Since

$$\|(\tilde{T}_n)^\wedge(j) - \widehat{S}(j)\| \le \|\tilde{T}_n - S\| \le \|\tilde{T}_n - S\|_{\mathcal{F}}$$

and

$$\|(\tilde{T}_n)^\wedge(j) - (\tilde{T})^\wedge(j)\| \le \|\widehat{T}_n(j) - \widehat{T}(j)\| \le \|T_n - T\| \le \|T_n - T\|_{\mathcal{F}}$$

it follows that $\widehat{S}(j) = (\tilde{T})^\wedge(j)$ for all j. This proves $S = \tilde{T}$ and thus the lemma.

Given $T \in \mathcal{B}(\mathcal{H})$ we shall denote by T^b (if it exists) the operator whose Fourier transform satisfies

$$(T^b)^\wedge(n) = \begin{cases} \widehat{T}(n) & \text{for } n \ge 0 \\ 0 & \text{for } n < 0. \end{cases} \tag{4.3}$$

Corollary 4.4. *Let $\mathcal{F}$ be a homogeneous class in $\mathcal{B}(\mathcal{H})$. Then $\mathcal{F}$ admits conjugation if and only if the mapping $T \mapsto T^b$ is a well-defined, bounded linear operator on $\mathcal{F}$.*

Proof. From (2.3), (4.2) and (4.3), we see that if one of the $\tilde{T}$ or T^b is defined, so is the other. Moreover, they are related by

$$T^b = \frac{1}{2}[\widehat{T}(0) + T + i\tilde{T}]. \tag{4.4}$$

The corollary now follows from (2.2) and Lemma 4.3.

For each $n \in \mathbb{Z}$, we define the multiplication operator M_n on $\mathcal{H}$ by

$$(M_n f)(t) = e^{int} f(t), \quad f \in \mathcal{H}. \tag{4.5}$$

Clearly, M_n is unitary and belongs to $\mathcal{B}^n(\mathcal{H})$. It is useful to note that

$$(M_n T)^\wedge(m) = M_n \widehat{T}(m - n) \tag{4.6}$$

for all $T \in \mathcal{B}(\mathcal{H})$ and $n, m \in \mathbb{Z}$, which follows by a direct computation.

We are now able to give a characterization of the homogeneous classes which admit convergence in norm.

Theorem 4.5. *Let $\mathcal{F}$ be a homogeneous class in $\mathcal{B}(\mathcal{H})$, and suppose for all $T \in \mathcal{F}$ and $n \in \mathbb{Z}$, $M_nT \in \mathcal{F}$ and $\|M_nT\|_{\mathcal{F}} = \|T\|_{\mathcal{F}}$. Then the following are equivalent:*

(a) *$\mathcal{F}$ admits convergence in norm.*

(b) *$\mathcal{F}$ admits conjugation.*

(c) *There exists a constant c such that $\|S_n(T)\|_{\mathcal{F}} \leq c\|T\|_{\mathcal{F}}$ for all $T \in \mathcal{F}$ and $n \geq 0$.*

Before giving the proof we note that this theorem applies to $\mathcal{F} = \mathcal{B}_p(\mathcal{H})$, $1 \leq p \leq \infty$, since the Schatten classes are ideals of $\mathcal{B}(\mathcal{H})$ and the operators M_n are unitary on $\mathcal{H}$.

Proof.. If (a) holds, then for each $T \in \mathcal{F}$, $\{\|S_n(T)\|_{\mathcal{F}} : n \geq 0\}$ is bounded. So by the principle of uniform boundedness, (c) follows.

We next prove that (c) implies (a). Fix $T \in \mathcal{F}$. By Proposition 2.3, for any $\epsilon > 0$, there is an almost simple operator $A \in \mathcal{F}$ with $\|T - A\|_{\mathcal{F}} < \epsilon$. If n is greater than the order of A, then we have $S_n(A) = A$ and thus

$$\|S_n(T) - T\|_{\mathcal{F}} \leq \|S_n(T) - S_n(A)\|_{\mathcal{F}} + \|A - T\|_{\mathcal{F}} < (c+1)\epsilon.$$

This proves (a).

Now suppose (b) holds, we will prove (c). By Corollary 4.4, the mapping $T \mapsto T^b$ is a well-defined, bounded linear operator on $\mathcal{F}$. Thus, if $T \in \mathcal{F}$, then $(M_nT)^b \in \mathcal{F}$ for all n, and we have

$$[M_{-n}(M_nT)^b]^\wedge(j) = \begin{cases} \widehat{T}(j) & \text{for } j+n \geq 0 \\ 0 & \text{for } j+n < 0 \end{cases}$$

by (4.3) and (4.6). A simple computation then shows that

$$S_n(T) = M_{-n}(M_nT)^b - M_{n+1}(M_{-n-1}T)^b.$$

From this and the hypothesis, we conclude that $\|S_n(T)\|_{\mathcal{F}} \leq 2c\|T\|_{\mathcal{F}}$, where c is the operator norm of the mapping $T \mapsto T^b$.

Finally, we will show that (c) implies (b). Define

$$S_n^b(T) = \sum_{j=0}^{2n} \widehat{T}(j) = M_n S_n(M_{-n}T)$$

where in the last step, we use (4.6). Then by hypothesis, we see that $\|S_n^b(T)\|_{\mathcal{F}} \leq c\|T\|_{\mathcal{F}}$ for all $T \in \mathcal{F}$ and $n \geq 0$. Now fix $T \in \mathcal{F}$ and let $\epsilon > 0$ be given. As above, we can find an almost simple operator $A \in \mathcal{F}$ so that $\|T - A\|_{\mathcal{F}} < \epsilon$. It follows that

$$\|S_n^b(T) - S_n^b(A)\|_{\mathcal{F}} = \|S_n^b(T - A)\|_{\mathcal{F}} < c\epsilon$$

for all $n \geq 0$. If n and m are greater than the order of A, say k, we have $S_n^b(A) = S_m^b(A)$. Thus, using the triangle inequality, we get

$$\|S_n^b(T) - S_m^b(T)\|_{\mathcal{F}} < 2c\epsilon$$

whenever $n, m > k$. This shows that $\{S_n^b(T)\}$ is a Cauchy sequence, and hence converges to an element B of $\mathcal{F}$. It then follows by (2.2) and (2.3) that the operator B has the same Fourier transform as T^b. Consequently, $T^b = B \in \mathcal{F}$ and so, by (4.4), we see that $\tilde{T}$ exists and belongs to $\mathcal{F}$. This proves (b).

This theorem is important in that it reduces the convergence question of Fourier series to the existence question of conjugation. We shall prove in the next section that for $1 < p < \infty$, $\mathcal{B}_p(\mathcal{H})$ admits conjugation. The case $p = 2$ is particularly easy. Indeed, let $T \in \mathcal{B}_2(\mathcal{H})$. Then by Theorem 3.2, we have $\|T\|_2^2 = \sum_{k=-\infty}^{\infty} \|\widehat{T}(k)\|_2^2$. Thus, using (2.3),

$$\|S_n(T)\|_2^2 = \sum_{k=-\infty}^{\infty} \|[S_n(T)]^\wedge(k)\|_2^2 = \sum_{k=-n}^{n} \|\widehat{T}(k)\|_2^2 \leq \|T\|_2^2.$$

The desired result now follows from Theorem 4.5.

5. Applications. Our purpose in this section is to show for $1 <$

$p < \infty$ that $\mathcal{B}_p(\mathcal{H})$ admits conjugation. This, combined with Theorem 4.5, immediately yields

Theorem 5.1. *For $1 < p < \infty$, the Fourier series of every $T \in \mathcal{B}_p(\mathcal{H})$ converges to T in the Schatten p-norm.*

With this fact, one can strengthen the result of Theorem 3.3:

Theorem 5.2. (Parseval's formula) *Suppose $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$. If $S \in \mathcal{B}_p(\mathcal{H})$ and $T \in \mathcal{B}_q(\mathcal{H})$, then*

$$(S,T) = \sum_{n=-\infty}^{\infty} (\widehat{S}(n), \widehat{T}(n)). \text{ Equivalently, } tr(TS) = \sum_{n=-\infty}^{\infty} tr(\widehat{T}(-n)\widehat{S}(n)).$$

We shall first show that if $p \geq 2$ is an even integer, then $\mathcal{B}_p(\mathcal{H})$ admits conjugation. The result for $1 < p < \infty$ will be based on use of the interpolation and duality theory for the Schatten classes.

We begin with two preliminary lemmas.

Lemma 5.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that its conjugate operator $\tilde{T}$ exists. Then T is self-adjoint if and only if $\tilde{T}$ and $\widehat{T}(0)$ are both self-adjoint.*

Proof. The lemma follows directly from part (d) of Lemma 3.1 and (4.2).

Lemma 5.4. (a) $\|S\|_1 = \||S|^\wedge(0)\|_1$ *for all $S \in \mathcal{B}_1(\mathcal{H})$.*

(b) *Let m be a positive integer. If $S \in \mathcal{B}_{2m}(\mathcal{H})$ and is self-adjoint, then $(S^{2m})^\wedge(0) \in \mathcal{B}_1(\mathcal{H})$ and*

$$\|S\|_{2m}^{2m} = \|(S^{2m})^\wedge(0)\|_1.$$

Proof. (a) $\|S\|_1 = tr|S| = \sum_n (|S|e^{in\cdot}, e^{in\cdot}) = \sum_n (|S|^\wedge(0)e^{in\cdot}, e^{in\cdot}) = \||S|^\wedge(0)\|_1$, where we have used (3.2) and the fact that $|S|^\wedge(0)$ is positive.

(b) Since S is self-adjoint, $|S|^{2m} = S^{2m}$. Thus, $\|S\|_{2m}^{2m} = tr(|S|^{2m}) = \|S^{2m}\|_1 = \|(S^{2m})^\wedge(0)\|_1$ by part (a).

Proposition 5.5. *If $p \geq 2$ is an even integer, then $\mathcal{B}_p(\mathcal{H})$ admits conjugation.*

Proof. Let $T \in \mathcal{B}^\sharp(\mathcal{H}) \cap \mathcal{B}_p(\mathcal{H})$. By (3.1), T can be expressed as

$$T = \sum_{j=-n}^{n} \widehat{T}(j) \tag{5.1}$$

for some $n \geq 0$. It follows that $\tilde{T} = -i\sum_{j=-n}^{n} \operatorname{sgn}(j)\widehat{T}(j) \in \mathcal{B}_p(\mathcal{H})$.

Since $\mathcal{B}^\sharp(\mathcal{H}) \cap \mathcal{B}_p(\mathcal{H})$ is dense in $\mathcal{B}_p(\mathcal{H})$, the proposition will be proven if we can show that there exists a constant c such that

$$\|\tilde{T}\|_p \leq c\|T\|_p \tag{5.2}$$

for all $T \in \mathcal{B}^\sharp(\mathcal{H}) \cap \mathcal{B}_p(\mathcal{H})$. To prove (5.2), we may assume $\widehat{T}(0) = 0$. For, suppose $\widehat{T}(0) \neq 0$. Then the operator $S = T - \widehat{T}(0)$ satisfies $\widehat{S}(0) = 0$ and $\tilde{S} = \tilde{T}$ so that $\|\tilde{T}\|_p = \|\tilde{S}\|_p \leq c\|S\|_p \leq 2c\|T\|_p$.

Now, let T be as in (5.1) with $\widehat{T}(0) = 0$. Then $T + i\tilde{T} = 2\sum_{j=1}^{n} \widehat{T}(j)$ so that

$$[(T + i\tilde{T})^p]^\wedge(0) = 0. \tag{5.3}$$

Write $p = 2m$. We have

$$\begin{aligned}(T + i\tilde{T})^{2m} = (i\tilde{T})^{2m} &+ i^{2m-1} \sum_{\substack{k_1+k_2=2m-1\\ k_j\geq 0}} (\tilde{T}^{k_1}T\tilde{T}^{k_2}) \\ &+ i^{2m-2} \sum_{\substack{k_1+k_3+k_5=2m-2\\ k_2+k_4=2\\ k_j\geq 0}} (\tilde{T}^{k_1}T^{k_2}\tilde{T}^{k_3}T^{k_4}\tilde{T}^{k_5}) + \cdots + T^{2m}.\end{aligned}$$

Taking the Fourier transform and using (5.3), we get

$$\begin{aligned}-(\tilde{T}^{2m})^\wedge(0) = i^{-1} \sum(\tilde{T}^{k_1}T\tilde{T}^{k_2})^\wedge(0) + i^{-2} \sum(\tilde{T}^{k_1}T^{k_2}\tilde{T}^{k_3}T^{k_4}\tilde{T}^{k_5})^\wedge(0) \\ + \cdots + i^{-2m}(T^{2m})^\wedge(0).\end{aligned} \tag{5.4}$$

Consider first the case where T is self-adjoint. Then by Lemma 5.3, $\tilde{T}$ is self-adjoint, so by Lemma 5.4 (b), $\|(\tilde{T}^{2m})^\wedge(0)\|_1 = \|\tilde{T}\|_{2m}^{2m}$. It follows from (5.4) and (2.2) that

$$\|\tilde{T}\|_{2m}^{2m} \leq \sum \|\tilde{T}^{k_1}T\tilde{T}^{k_2}\|_1 + \sum \|\tilde{T}^{k_1}T^{k_2}\tilde{T}^{k_3}T^{k_4}\tilde{T}^{k_5}\|_1 + \cdots + \|T^{2m}\|_1. \tag{5.5}$$

Using the Hölder inequality (3.4) inductively, we have

$$\begin{aligned}\|\tilde{T}^{k_1}T\tilde{T}^{k_2}\|_1 &\le \|\tilde{T}^{k_1}T\tilde{T}^{k_2-1}\|_{2m/(2m-1)}\|\tilde{T}\|_{2m}\\ &\le \|\tilde{T}^{k_1}T\tilde{T}^{k_2-2}\|_{2m/(2m-2)}\|\tilde{T}\|_{2m}^2\\ &\le \cdots \le \|\tilde{T}^{k_1}T\|_{2m/(2m-k_2)}\|\tilde{T}\|_{2m}^{k_2}\\ &\le \|\tilde{T}^{k_1}\|_{2m/(2m-k_2-1)}\|\tilde{T}\|_{2m}^{k_2}\|T\|_{2m}\\ &\le \cdots \le \|\tilde{T}\|_{2m/(2m-k_2-k_1)}\|\tilde{T}\|_{2m}^{k_2+k_1-1}\|T\|_{2m}\\ &= \|\tilde{T}\|_{2m}^{2m-1}\|T\|_{2m}.\end{aligned}$$

By a similar argument, we find that

$$\|\tilde{T}^{k_1}T^{k_2}\tilde{T}^{k_3}T^{k_4}\tilde{T}^{k_5}\|_1 \le \|\tilde{T}\|_{2m}^{2m-2}\|T\|_{2m}^2, \cdots, \|T^{2m}\|_1 \le \|T\|_{2m}^{2m}.$$

Since the jth summation on the right-hand side of (5.5) consists of $\binom{2m}{j}$ terms, it follows that

$$\|\tilde{T}\|_{2m}^{2m} \le \sum_{j=1}^{2m}\binom{2m}{j}\|\tilde{T}\|_{2m}^{2m-j}\|T\|_{2m}^{j}.$$

Thus, if we set $Q = \|\tilde{T}\|_{2m}\|T\|_{2m}^{-1}$, the inequality

$$Q^{2m} \le \sum_{j=1}^{2m}\binom{2m}{j}Q^{2m-j}$$

shows that $\|\tilde{T}\|_p \le c\|T\|_p$ for some constant c.

In the general case, T can be written as $T = A + iB$ where $A = (T + T^*)/2$ and $B = (T - T^*)/2i$ are self-adjoint. Note that $A, B \in \mathcal{B}^{\natural}(\mathcal{H}) \cap \mathcal{B}_p(\mathcal{H})$ and satisfy $\widehat{A}(0) = \widehat{B}(0) = 0$. Thus,

$$\|\tilde{T}\|_p = \|\tilde{A} + i\tilde{B}\|_p \le \|\tilde{A}\|_p + \|\tilde{B}\|_p \le c(\|A\|_p + \|B\|_p) \le 2c\|T\|_p$$

and the proof is complete.

We are now in a position to establish the result announced at the beginning of this section, namely:

Theorem 5.6. *For $1 < p < \infty$, $\mathcal{B}_p(\mathcal{H})$ admits conjugation.*

Proof. We have proven above that for each $m \in \mathbb{Z}^+$, the mapping $S \mapsto \tilde{S}$ is a bounded linear operator on $\mathcal{B}_{2m}(\mathcal{H})$. By an interpolation argument for the Schatten classes (see e.g. Reed-Simon [4], Theorem IX. 22), this remains true if $\mathcal{B}_{2m}(\mathcal{H})$ is replaced by $\mathcal{B}_p(\mathcal{H})$ for any $p \in [2, \infty)$.

Now fix $1 < q \leq 2$ and let $p = q/(q-1)$. Then $2 \leq p < \infty$. Using (3.6) and the fact that $\mathcal{B}_p(\mathcal{H})$ admits convergence in norm, we see that

$$(S, S_n(T)) = (S_n(S), T) \longrightarrow (S, T) \text{ as } n \longrightarrow \infty$$

for all $S \in \mathcal{B}_p(\mathcal{H})$ and $T \in \mathcal{B}_q(\mathcal{H}) = \mathcal{B}_p(\mathcal{H})^*$. Thus, by the principle of uniform boundedness, there is a constant c such that $\|S_n(T)\|_q \leq c\|T\|_q$ for all $T \in \mathcal{B}_q(\mathcal{H})$ and $n \geq 0$. It follows from Theorem 4.5 that $\mathcal{B}_q(\mathcal{H})$ admits conjugation.

Final Remark. Professor E. Berkson has informed me that Theorem 5.1 is contained in [6] (Theorem 4.2), and the fact that the Schatten p-class is UMD (i.e., to possess the unconditionality property for martingale differences) for $1 < p < \infty$ stems from the Gohberg-Krein proof of Macaev's theorem. However, the proof given here is quite different from the proof in the Berkson-Gillespie paper.

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