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THE GENERALIZED RENEWAL THEORIES OF THE Z.AR(1) PROCESS

BY

HSIAW-CHAN YEH (葉小蓁)

Abstract. A Markov chain $\{A_i\}_0^\infty$ in Z.AR(1) process with Zipf(III) marginal distribution was introduced in Yeh (1990). In this paper, some generalized renewal properties about Z.AR(1) will be discussed. Let $E_m = \{\omega \,|\, k_0 \leq \omega < m\}$ be the mainly concerned set with $k_0 < m$, two fixed integers. The counting process and the waiting time of the visits to E_m in the Markov chain are discussed. The recurrence and the number of visits to E_m are studied. The limiting behavior of the number of visits to E_m during a period of time is also studied in this paper.

1. Introduction. In Yeh (1990), a Markov chain $\{A_i\}_0^\infty$ with Zipf marginal distribution in Z.AR(1) process is defined as

$$A_n = \max\{V_n A_{n-1}, (1 - V_n)Y_n\}, \tag{1.1}$$

where $\{V_n\}$ is a sequence of i.i.d. Bernoulli(ρ) random variables with $P(V_n = 1) = \rho$, $0 \leq \rho \leq 1$, a fixed constant, and $\{Y_n\}$ is a sequence of i.i.d. Zipf(III) (k_0, r, σ) random variables. (Arnold (1983)) for each $n = 1, 2, \ldots$.

The transition probability of such a chain has also been shown to satisfy equation (2.6) in Yeh (1990). That is

$$P_{ij} = P(A_{n+1} = j | A_n = i) = \begin{cases} (1-\rho)\pi(j) & \text{if } j \neq i \\ \rho + (1-\rho)\pi(i) & \text{if } j = i \end{cases}$$

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for any i, j in the state space $E = \{k_0, k_{0+1}, \ldots\}$, where $\pi(i)$ is the marginal probability mass function

$$\pi(i) = P(A_n = i) = \left[1 + \left(\frac{i - k_0}{\sigma}\right)^{1/r}\right]^{-1} - \left[1 + \left(\frac{i + 1 - k_0}{\sigma}\right)^{1/r}\right]^{-1}.$$

The set of states which is mainly considered in this paper is $E_m = \{\omega | k_0 \leq \omega < m\}$. The corresponding counting process is defined as

$$N_m^* = \inf\{i : A_i \in E_m\}. \tag{1.2}$$

In other words, N_m^* is the first time the Markov chain $\{A_i\}_0^\infty$ hits the set E_m. In the classical renewal theory, the counting process of any renewal process is defined in terms of summation of i.i.d. components. However, the counting process of the Markov chain $\{A_i\}_0^\infty$ in Z.AR(1) process is defined in terms of the minimum order statistic in the observed sequence $\{A_i\}_0^\infty$ (refer to equations (1.2) and (2.3) in this paper). Therefore, the classic renewal theory can not be directly applied to Z.AR(1) process. This causes me to study the generalized renewal theories of the Z.AR(1) process.

The probability mass function, the mean and variance of N_m^* are calculated in section 2. The waiting time T_m^r is the time of the r-th visit to E_m for $r = 1, 2, \ldots$. The interarrival times $\{\omega_m^i\}$ are defined as

$$\omega_m^i = \begin{cases} T_m^1, & \text{if } i = 1 \\ T_m^i - T_m^{i-1}, & \text{if } i = 2, 3, \ldots. \end{cases}$$

Some interesting properties about $\{\omega_m^i\}_1^\infty$ and T_m^r are studied in section 3. The recurrency and the number of visits to E_m are discussed in section 4, the limiting behavior of the number of visits to E_m during a period of time in section 5.

2. Synchronous counting process. Two sets E and E_m are defined in section one, the corresponding counting process N_m^* for $\{A_i\}_0^\infty$ is also defined as

$$N_m^* = \inf\{i : A_i \in E_m\}. \tag{2.1}$$

In this section, the probability mass function (pmf), the mean and variance of N_m^* are calculated as follows. First,

$$\begin{aligned} P(N_m^* = 1) &= P(A_1 < m) = 1 - P(A_1 \geq m) \\ &= 1 - \frac{1}{1 + (\frac{m-k_0}{\sigma})^{1/r}}. \end{aligned} \tag{2.2}$$

$N_m^* = n$, when $n = 2, 3, \ldots$, the pmf of N_m^* is derived from the survival function of the minimum $m_n = \min_{1 \leq i \leq n} A_i$, calculated in Yeh (1990)'s equation (3.1) as

$$\bar{F}_{m_n}(i) = \frac{1}{1 + (\frac{i-k_0}{\sigma})^{1/r}} \left\{ \frac{1 + \rho(\frac{i-k_0}{\sigma})^{1/r}}{1 + (\frac{i-k_0}{\sigma})^{1/r}} \right\}^{n-1}$$

for any $i \in E$ and any integer $n \geq 1$. Then given $n = 2, 3, \ldots$

$$\begin{aligned} P(N_m^* = n) &= P(\min_{1 \leq i \leq n-1} A_i \geq m,\ A_n < m) \\ &= P(m_{n-1} \geq m) - P(m_n \geq m) \\ &= (1-\rho) \frac{(\frac{m-k_0}{\sigma})^{1/r}}{[1 + (\frac{m-k_0}{\sigma})^{1/r}]^2} \left\{ \frac{1 + \rho(\frac{m-k_0}{\sigma})^{1/r}}{1 + (\frac{m-k_0}{\sigma})^{1/r}} \right\}^{n-2}. \end{aligned} \tag{2.3}$$

It is straightforward to check that $\sum_{n=1}^{\infty} P(N_m^* = n) = 1$, and conclude that N_m^* is a non-defective random variable. The survival function of N_m^* is directly calculated from equation (2.3) as

$$P(N_m^* \geq n) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{1}{1+(\frac{m-k_0}{\sigma})^{1/r}} \left\{ \frac{1+\rho(\frac{m-k_0}{\sigma})^{1/r}}{1+(\frac{m-k_0}{\sigma})^{1/r}} \right\}^{n-2} & \text{if } n = 2, 3, \ldots. \end{cases} \tag{2.4}$$

The mean of N_m^* is

$$E(N_m^*) = \sum_{n=1}^{\infty} P(N_m^* \geq n) = 1 + (1-\rho)^{-1} \left(\frac{m-k_0}{\sigma} \right)^{-1/r}. \tag{2.5}$$

The variance of N_m^* is

$$\begin{aligned} \mathrm{Var}(N_m^*) &= E(N_m^{*2}) - [E(N_m^*)]^2 \\ &= (1-\rho)^{-2} \left(\frac{m-k_0}{\sigma} \right)^{-2/r} \left\{ 1 + 2 \left(\frac{m-k_0}{\sigma} \right)^{1/r} \right\} \\ &\quad - (1-\rho)^{-1} \left(\frac{m-k_0}{\sigma} \right)^{-1/r}. \end{aligned} \tag{2.6}$$

3. Waiting times of the set E_m in Z.AR(1) process. Supposing the Markov chain $\{A_i\}_0^\infty$ in Z.AR(1) process starts from $A_0 \in E_m$. In this section, two types of waiting times of the set E_m are introduced. One is the T_m^r which is the waiting time of the r-th visit to the set E_m, the other one is ω_m^r, the interarrival time between the $(r-1)$-th visit to E_m and the r-th visit to E_m for any fixed $r = 1, 2, \ldots$.

Clearly, the relation between T_m^r and ω_m^1 is

$$T_m^r = \omega_m^1 + \omega_m^2 + \cdots + \omega_m^r, \tag{3.1}$$

and

$$\omega_m^i = \begin{cases} T_m^i & \text{for } i = 1 \\ T_m^i - T_m^{i-1} & \text{for } i = 2, 3, \ldots. \end{cases} \tag{3.2}$$

Theorem 1. *Given $r \geq 1$, the interarrival times $\{\omega_m^1, \omega_m^2, \ldots, \omega_m^r, \omega_m^{r+1}\}$ satisfy*

(i) $P(\omega_m^{r+1} = k_{r+1} | A_0 < m, \omega_m^1 = k_1, \ldots, \omega_m^r = k_r) = P(\omega_m^1 = k_{r+1} | A_0 < m)$

(ii) $P(\omega_m^1 = k_1, \ldots, \omega_m^r = k_r | A_0 < m) = P(\omega_m^1 = k_1 | A_0 < m) \cdots P(\omega_m^r = k_r | A_0 < m)$,

for any positive integer set $\{k_1, k_2, \ldots, k_r, k_{r+1}\}$. In other words, the random variables $\{\omega_m^1, \omega_m^2, \cdots\}$ are identically and independently distributed if conditioned on that $A_0 \in E_m$.

Proof. To prove (i)

$$\begin{aligned} &P(\omega_m^{r+1} = k_{r+1} | A_0 < m, \omega_m^1 = k_1, \ldots, \omega_m^r = k_r) \\ &= \frac{P(A_0 < m, \omega_m^1 = k_1, \ldots, \omega_m^r = k_r, \omega_m^{r+1} = k_{r+1})}{P(A_0 < m, \omega_m^1 = k_1, \ldots, \omega_m^r = k_r)} = \frac{\text{I}}{\text{II}} \end{aligned} \tag{3.3}$$

where I $= P(A_0 < m, A_1 \geq m, \ldots, A_{k_1 - 1} \geq m, A_{k_1} < m, A_{k_1+1} \geq m, \ldots, A_{k_1+k_2-1} \geq m, A_{k_1+k_2} < m, A_{k_1+k_2+1} \geq m, \ldots, A_{\Sigma_{i=1}^r k_i - 1} \geq m, A_{\Sigma_{i=1}^r k_i} < m, A_{\Sigma_{i=1}^r k_i + 1} \geq m, \ldots, A_{\Sigma_{i=1}^{r+1} k_i - 1} \geq m, A_{\Sigma_{i=1}^{r+1} k_i} < m)$, II $= P(A_0 < m, A_1 \geq m, \ldots, A_{k_1-1} \geq m, A_{k_1} < m, \ldots, A_{\Sigma_{i=1}^r k_i - 1} \geq m, A_{\Sigma_{i=1}^r k_i} < m)$. Denote

$$A = (A_0 < m, A_1 \geq m, \ldots, A_{\Sigma_{i=1}^r k_i} < m)$$

$$B = (A_{\Sigma_{i=1}^r k_i + 1} \geq m, \ldots, A_{\Sigma_{i=1}^{r+1} k_i - 1} \geq m, A_{\Sigma_{i=1}^{r+1} k_i} < m),$$

then equation (3.3) is simplified as

$$\frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B|A)}{P(A)} = P(B|A).$$

By the Markov property of the Z.AR(1) process

$$\begin{aligned} P(B|A) &= P(A_{\Sigma_{i=1}^{r} k_i+1} \geq m, \ldots, A_{\Sigma_{i=1}^{r+1} k_i-1} \geq m, \\ &\qquad A_{\Sigma_{i=1}^{r+1} k_i} < m | A_0 < m, A_1 \geq m, \ldots, A_{\Sigma_{i=1}^{r} k_i} < m) \qquad (3.4) \\ &= P(A_{\Sigma_{i=1}^{r} k_i+1} \geq m, \ldots, A_{\Sigma_{i=1}^{r+1} k_i-1} \geq m, A_{\Sigma_{i=1}^{r+1} k_i} < m | A_{\Sigma_{i=1}^{r} k_i} < m). \end{aligned}$$

By the stationary property of this process, equation (3.4) becomes $P(A_1 \geq m, \ldots, A_{k_{r+1}-1} \geq m, A_{k_{r+1}} < m | A_0 < m)$ by the definition of ω_m^1, $= P(\omega_m^1 = k_{r+1} | A_0 < m)$. Hence (i) follows.

To prove (ii)

$$\begin{aligned} &P(\omega_m^1 = k_1, \ldots, \omega_m^r = k_r | A_0 < m) \\ &\quad = P(A_1 \geq m, \ldots, A_{k_1-1} \geq m, A_{k_1} < m, A_{k_1+1} \geq m, \\ &\qquad \ldots, A_{k_1+k_2-1} \geq m, A_{k_1+k_2} < m, \qquad (3.5) \\ &\qquad \ldots, A_{\Sigma_{i=1}^{r} k_i-1} \geq m, A_{\Sigma_{i=1}^{r} k_i} < m | A_0 < m). \end{aligned}$$

Let

$$\begin{aligned} S_1 &= (A_1 \geq m, \ldots, A_{k_1-1} \geq m, A_{k_1} < m) \\ S_2 &= (A_{k_1+1} \geq m, \ldots, A_{k_1+k_2-1} \geq m, A_{k_1+k_2} < m) \\ &\ \vdots \\ S_r &= (A_{\Sigma_{i=1}^{r-1} k_i+1} \geq m, \ldots, A_{\Sigma_{i=1}^{r} k_i-1} \geq m, A_{\Sigma_{i=1}^{r} k_i} < m). \end{aligned}$$

Then equation (3.5) is simplified as

$$\begin{aligned} &= P(S_1 \cap S_2 \cap \cdots \cap S_r | A_0 < m) \\ &= \frac{P((A_0 < m) \cap S_1 \cap S_2 \cap \cdots \cap S_r)}{P(A_0 < m)} = \frac{\mathrm{I}}{P(A_0 < m)}, \end{aligned}$$

where $\mathrm{I} = P(A_0 < m)P(S_1 | A_0 < m)P(S_2 | (A_0 < m) \cap S_1) \cdots P(S_r | (A_0 < m) \cap S_1 \cap \cdots \cap S_{r-1})$. The following equation results from applying the Markov property twice and the definition of ω_m^1,

$$P(A_1 \geq m, \ldots, A_{k_1-1} \geq m, A_{k_1} < m | A_0 < m)$$
$$P(A_{k_1+1} \geq m, \ldots, A_{k_1+k_2-1} \geq m, A_{k_1+k_2} < m | A_{k_1} < m)$$
$$\ldots P(A_{\Sigma_{i=1}^{r-1} k_i+1} \geq m, \ldots, A_{\Sigma_{i=1}^{r} k_i-1} \geq m, A_{\Sigma_{i=1}^{r} k_i} < m | A_{\Sigma_{i=1}^{r-1} k_i} < m)$$
$$= P(\omega_m^1 = k_1 | A_0 < m) P(\omega_m^1 = k_2 | A_0 < m) \ldots P(\omega_m^1 = k_r | A_0 < m).$$

Thus, the property (ii) follows.

The above theorem says that the random variables $\{\omega_m^1, \omega_m^2, \ldots\}$ are i.i.d. if conditioned on that $A_0 \in E_m$. Let $F_m(\cdot)$ be the common conditional probability function of ω_m^i for $i = 1, 2, \ldots$. Calculation of $F_m(k) = P(\omega_m^1 = k | A_0 < m)$, $k = 1, 2, \ldots$ is easy. For $k = 1$,

$$F_m(1) = P(\omega_m^1 = 1 | A_0 < m) = P(A_1 < m | A_0 < m) = \frac{\rho + (\frac{m-k_0}{\sigma})^{1/r}}{1 + (\frac{m-k_0}{\sigma})^{1/r}}. \tag{3.6}$$

In general, for $k = 2, 3, \ldots$, by the Markov and stationary properties of $\{A_i\}_0^\infty$, then

$$\begin{aligned} F_m(k) &= P(\omega_m^1 = k | A_0 < m) \\ &= P(A_1 \geq m, A_2 \geq m, \ldots A_{k-1} \geq m, A_k < m | A_0 < m) \\ &= \frac{(1-\rho)^2 (\frac{m-k_0}{\sigma})^{1/r}}{[1 + (\frac{m-k_0}{\sigma})^{1/r}]^2} \left\{ \frac{1 + \rho (\frac{m-k_0}{\sigma})^{1/r}}{1 + (\frac{m-k_0}{\sigma})^{1/r}} \right\}^{k-2}. \end{aligned} \tag{3.7}$$

The survival function of ω_m^1 is got directly from equation (3.6) and (3.7).

$$\bar{F}_m(k) = \begin{cases} 1 & \text{if } k = 1 \\ \frac{(1-\rho)}{1 + (\frac{m-k_0}{\sigma})^{1/r}} \left\{ \frac{1 + \rho (\frac{m-k_0}{\sigma})^{1/r}}{1 + (\frac{m-k_0}{\sigma})^{1/r}} \right\}^{k-2} & \text{if } k = 2, 3, \ldots. \end{cases} \tag{3.8}$$

The common conditional mean of ω_m^i is

$$E_m(\omega_m^1) = E(\omega_m^1 | A_0 < m) = \sum_{k=1}^{\infty} \bar{F}_m(k) = 1 + \left(\frac{m - k_0}{\sigma} \right)^{-1/r}. \tag{3.9}$$

Apparently, $E_m(\omega_m^1)$ is finite for any fixed integer $m \in E = \{k_0, k_0 + 1, \ldots\}$. Therefore, the following theorem about the waiting time of the r-th visit to E_m is easily discerned.

Theorem 2. *In the Markov chain,* Z.AR(1) *process, if* $\{A_i\}_0^\infty$ *starts at* $A_0 \in E_m$*, then asymptotically,*

$$\lim_{r\to\infty} \frac{T_m^r}{r} = E_m(\omega_m^1) = 1 + \left(\frac{m-k_0}{\sigma}\right)^{-1/r} \tag{3.10}$$

with probability one.

Proof. From Theorem 1, $\{\omega_m^1, \omega_m^2, \ldots\}$ are i.i.d. on condition that $A_0 \in E_m$, and by equation (3.1) $T_m^r = \omega_m^1 + \omega_m^2 + \cdots + \omega_m^r$, also from (3.9), $E_m(\omega_m^1) < \infty$, so the strong law of large numbers (Chow & Teicher (1978)) can be applied. Thus,

$$\lim_{r\to\infty} \frac{\omega_m^1 + \omega_m^2 + \cdots + \omega_m^r}{r} = E_m(\omega_m^1)$$

with probability one. Therefore, the theorem is proved.

Theorem 2 says that the average waiting time of the r-th visit to the set E_m converges almost surely to the expected interarrival time to E_m as $r \to \infty$.

4. The recurrency and the number of visits to the set E_m. Suppose that the Markov chain $\{A_i\}_0^\infty$ starts from an $A_0 \in E_m$, and let F_m^* be the probability that the Markov chain $\{A_i\}$ ever visits E_m , and N_m^* the total number of visits to E_m in Z.AR(1) process. Then

$$\begin{aligned} F_m^* &= P\{\omega_m^1 < \infty | A_0 \in E_m\} \\ &= \sum_{k=1}^{\infty} P\{\omega_m^1 = k | A_0 < m\}, \qquad \text{from equation (3.8)}, \\ &= \bar{F}_m(1) = 1. \end{aligned} \tag{4.1}$$

Also, it is clearly that

$$N_m^* = r \text{ if and only if } \{\omega_m^1 < \infty, \omega_m^2 < \infty, \ldots, \omega_m^{r-1} < \infty, \omega_m^r = \infty\},$$

for any integer $r \geq 1$. Thus, by Theorem 1 of the previous section and equation (4.1),

$$
\begin{aligned}
&P(N_m^* = r|A_0 < m) \\
&= P(\omega_m^1 < \infty|A_0 < m)\cdots \\
&\qquad P(\omega_m^{r-1} < \infty|A_0 < m)P(\omega_m^r = \infty|A_0 < m) \\
&= (F_m^*)^{r-1}(1 - F_m^*).
\end{aligned} \tag{4.2}
$$

So the random variable N_m^* given the event $A_0 \in E_m$ follows a geometric distribution with parameter F_m^* as the probability of success. Hence the conditional mean is

$$E_m(N_m^*) = E(N_m^*|A_0 < m) = \frac{1}{1 - F_m^*}. \tag{4.3}$$

Note from (4.1), $F_m^* = 1$, so equation (4.2) is 0 for each $r = 1, 2, \ldots$. Then

$$P(N_m^* < \infty|A_0 < m) = \sum_{r=1}^{\infty} P(N_m^* = r|A_0 < m) = 0.$$

Thus

$$P(N_m^* = \infty|A_0 < m) = 1 \tag{4.4}$$

i.e., $N_m^* = \infty$ with probability one. Also equation (4.3), $E_m(N_m^*) = \infty$, as $F_m^* = 1$. Therefore, the following proposition is obtained.

Proposition. *Suppose the Markov chain $\{A_i\}_0^\infty$ in* Z.AR(1) *process starts from $A_0 \in E_m$, then $\{A_i\}_0^\infty$ will visit E_m infinite times with probability one, i.e. $N_m^* = \infty$ almost surely, $E(N_m^*) = \infty$ and $F_m^* = 1$.*

The above Proposition is of great significance. It shows that the set

$$E_m = \{\omega | k_0 \leq \omega < m\}$$

will be visited infinitely. Such a set of states called "recurrent set" according to Çinlar (1975)'s definition.

5. Limiting behavior of the number of visits to E_m during a period of times. Given any fixed integer n, $n \geq 1$, let $N_n(m)$ be the number of visits of the Markov chain $\{A_i\}_0^\infty$ to E_m during the period of discrete times $t = 1, 2, \ldots, n$.

Note that $N_n(m)$ can be expressed as the number of successes in a Bernoulli process, i.e.

$$N_n(m) = \sum_{i=1}^{n} I_m(A_i), \tag{5.1}$$

where $I_m(A_i)$ is the indicator function

$$I_m(A_i) = \begin{cases} 1 & \text{if } A_i \in E_m \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Then the expected number of such visits of this chain starting at $A_0 \in E_m$ is given as

$$\begin{aligned} E(N_m(m)) &= \sum_{i=1}^{n} E(I_m(A_i)) = \sum_{i=1}^{n} P(A_i \in E_m | A_0 \in E_m) \\ &= \sum_{i=1}^{n} P(A_i < m | A_0 < m). \end{aligned} \tag{5.3}$$

Each term of equation (5.3) can be calculated from the representation (1.1) and by induction, the following formula can be got,

$$P(A_i < m | A_0 < m) = \frac{\rho^i + (\frac{m-k_0}{\sigma})^{1/r}}{1 + (\frac{m-k_0}{\sigma})^{1/r}}. \tag{5.4}$$

Then equation (5.3) becomes

$$\begin{aligned} E(N_n(m)) &= \sum_{i=1}^{n} \left\{ \frac{\rho^i + (\frac{m-k_0}{\sigma})^{1/r}}{1 + (\frac{m-k_0}{\sigma})^{1/r}} \right\} \\ &= \frac{1}{1 + (\frac{m-k_0}{\sigma})^{1/r}} \left(\frac{1-\rho^{n+1}}{1-\rho} \right) + \frac{(\frac{m-k_0}{\sigma})^{1/r}}{1 + (\frac{m-k_0}{\sigma})^{1/r}} n. \end{aligned} \tag{5.5}$$

It is obvious that

$$\lim_{n \to \infty} E(N_n(m)) = \infty = E_m(N_m^*) \tag{5.6}$$

where N_m^* is defined in section 4 which is the total number of visits to E_m in Z.AR(1) process.

The proposition in section 4 tells us that the set E_m is a recurrent set, then the random variable $N_n(m)$ satisfies:

Property 1. *With probability one, then*

$$(5.7)\qquad \begin{aligned} &\text{(i)} \quad \lim_{n\to\infty} N_n(m) \overset{d}{=} \infty \\ &\text{(ii)} \quad \lim_{n\to\infty} \frac{N_n(m)}{n} = \frac{1}{E_m(\omega_m^r)} \end{aligned}$$

it where $E_m(\omega_m^1)$ *is given in* (3.9) *as* $E_m(\omega_m^1) = 1 + \left(\frac{m-k_0}{\sigma}\right)^{-1/r}$.

Proof. (i) is followed straightforward from equations (5.5) and (5.6).

To check (ii): In section 3, T_m^i is defined as the waiting time of the i-th visit to E_m, it is clearly observed that

$$(5.8)\qquad N_n(m) = \max\{i : T_m^i \leq n\}.$$

Set $r = N_n(m)$, the meaning of (5.8) is that by time n, the Markov chain, $\{A_i\}_0^\infty$, has made exactly r visits to E_m. Thus the r-th visit to E_m occurs on or before the discrete time n and the $(r+1)$-th visit to E_m occurs after time n; that is $T_m^{N_n(m)} \leq n \leq T_m^{N_n(m)+1}$, hence

$$(5.9)\qquad \frac{T_m^{N_n(m)}}{N_n(m)} \leq \frac{n}{N_n(m)} \leq \frac{T_m^{N_n(m)+1}}{N_n(m)}.$$

Assuming equation (5.9) hold for n large enough so that $N_n(m) \geq 1$. From the result (i) of this property and equation (3.10) both imply that

$$\lim_{n\to\infty} \frac{n}{N_n(m)} = E_m(\omega_m^1),$$

with probability one, or equivalently that (5.7) holds, i.e., that

$$\lim_{n\to\infty} \frac{N_n(m)}{n} = \frac{1}{E_m(\omega_m^1)}$$

with probability one. Hence the Property (ii) is followed.

Note that by definition $0 \leq N_n(m) \leq n$ and hence $0 \leq \frac{N_n(m)}{n} \leq 1$, apply the dominated convergence theorem in measure theory, conclusion is

Property 2.

$$(5.10)\qquad \lim_{n\to\infty} \frac{E(N_n(m))}{n} = \frac{1}{E_m(\omega_m^1)}.$$

Observe (5.10), it says that the expect proportion for the first n units (n is large) of time that the Markov chain, $\{A_i\}_0^\infty$, hits the set E_m is asymptotically equal to the reciprocal of the mean return time to E_m.

6. Conclusion. In this paper, the Markov chain $\{A_i\}_0^\infty$ in Z.AR(1) process has the counting process whose property is analogous to that of the Pareto Process P.AR(1) (Yeh, Arnold and Robertson (1988)). The properties of the waiting times and interarrival times about Z.AR(1) are similar to those of a classic renewal process. (Hoel, Port, and Stone (1972)). The set $E_m = \{\omega | k_0 \leq \omega < m\}$ is found to be a recurrent set. In general, the set E_m can be taken any of the forms, such as $E_m^1 = \{\omega|\omega > m\}$ or any interval form $\underset{m_1, m_2}{E} = \{\omega|m_1 < \omega < m_2\}$ with $m_1 < m_2$ as two fixed integers, and the counting process can be defined in terms of the maxmum order statistics, then the renewal theories corresponding to either form of E_m can be studied. However, their properties are expected to be very analogous to what discussed in this paper. Therefore, no more discussion regarding this topic.

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Department of Finance, National Taiwan University, Taipei, Taiwan, R.O.C.