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COMPLETE CONVERGENCE FOR U-STATISTICS

BY

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Abstract. The probability inequality for the complete convergence has been extended to the U-statistics. As an application, let $X, X_1, X_2, \cdots$ be independent, identically distributed random variables, $EX = 0$, $T_n = \sum_{j=2}^{n} \sum_{i=1}^{j-1} X_i X_j$, $L = \sup\{n \geq 1; |T_n| \geq n^{2\alpha}\}$ and $M = \sup_{n \geq 1}(|T_n| - n^{2\alpha})$ for some $\alpha > 0$. If $\alpha p > 1$ and $X \in L_p$, then $L \in L_{\alpha p - 1}$ and $M \in L_{(\alpha p-1)/(2\alpha)}$.

1. Introduction. Let $X, X_1, X_2, \cdots$ be iid,

$$S_n = \sum_1^n X_j, T_n = \sum_{j=1}^{n} S_{j-1}\ X_j = \sum_{1 \leq i < j \leq n} X_i X_j, S_0 = X_0 = T_0 = 0,$$

and $EX = 0$. Baum and Katz (1965) proved that for $p \geq 1/\alpha, \alpha > 1/2$ and $E|X|^p < \infty$,

$$\sum_1^\infty n^{\alpha p - 2}\ P(|S_n| > n^\alpha) < \infty. \tag{1}$$

When $\alpha = 1$, (1) is due to Hsu and Robbins (1947) for p=2 and Spitzer (1956) for $p = 1$. In Chow and Lai (1975, 1978), (1) has been improved into the following inequality:

If $1 \leq r \leq 2, \alpha > \beta \geq 0, \alpha p > 1 > \beta p$ and $\alpha r > 1 > \beta r$, then

$$\sum n^{\alpha p - 2} P\Big(\max_{j \leq n} j^{-\beta} S_j \geq n^{\alpha - \beta}\Big) \leq C_{p,r,\alpha,\beta}\{E(X^+)^p + (E|X|^r)^{(\alpha p-1)/(\alpha r-1)}\},$$

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where $C_{p,r,\alpha,\beta}$ is an absolute constant.

In this paper, we shall prove that:

Theorem 1. *If $\alpha > 1/2$ and $E|X|^p < \infty$ for some $p \geq 1/\alpha$, then*

$$\sum_1^\infty n^{\alpha p-2} P(|T_n| > n^{2\alpha}) < \infty. \tag{3}$$

Theorem 2. *For $1 \leq r \leq 2, \alpha > \beta \geq 0, \alpha p > 1 > \beta p$ and $\alpha r > 1 > \beta r$, there exists a constant $C = C_{p,r,\alpha,\beta}$ such that*

$$\begin{aligned} &\sum_1^\infty n^{\alpha p-2} P\Big(\max_{j\leq n} |j^{-2\beta} T_j| \geq n^{2\alpha-2\beta}\Big) \\ &\leq C\{E|X|^p + (E|X|^r)^{(\alpha p-1)/(\alpha r-1)}\}. \end{aligned} \tag{4}$$

Theorems 1 and 2 extend (1) and (2) to U-statistics respectively. Theorem 1 follows from Theorem 2 immediately, except when $\alpha p = 1$, since we can let $r = p$ because $E|X| < \infty$. Their proofs will be given in Section 2 and 3.

As an application of Theorem 2, let

$$M = \sup_{n\geq 0}(|T_n| - n^{2\alpha}), \qquad L = \sup\{n \geq 1; |T_n| > n^{2\alpha}\}. \tag{5}$$

Theorem 3. *For $p \geq 1, \alpha p > 1, 1 \leq r \leq 2$ and $\alpha r > 1$, there exists a constant $C = C_{p,r,\alpha}$ such that*

$$E\{L^{\alpha p-1} + M^{(\alpha p-1)/(2\alpha)}\} \leq C\{E|X|^p + (E|X|^r)^{(\alpha p-1)/(\alpha r-1)}\}. \tag{6}$$

The proof of Theorem 3 follows from Theorem 2 and the following lemma, which is a restatement of Lemma 2 of Chow and Lai (1975), see also Chow and Teicher (1988, Lemma 10.4.1).

Lemma 1. *For $p > 0$ and $\alpha p > 1$, there exists a constant $C = C_{p,\alpha}$ such that*

$$E\{L^{\alpha p-1} + M^{(\alpha p-1)/(2\alpha)}\} \leq C \int_1^\infty v^{\alpha p-2} P\Big(\max_{j\leq v} |T_j| \geq v^{2\alpha}\Big) dv. \tag{7}$$

2. Proof of Theorem 1. For $t > 0$, let $S_t = S_{[t]}$, $T_t = T_{[t]}$, $S_0 = T_0 = 0$, $S_{m,n} = S_{m+n} - S_m$, $T_{m,n} = \sum_{1\le i<j\le n} X_{m+i}X_{m+j}$, $S_{m,0} = T_{m,0} = 0$, $T^*_{m,n} = \max_{1\le j\le n} |T_{m,j}|$, $S^*_{m,n} = \max_{1\le j\le n} |S_{m,j}|$, $T^*_n = T^*_{0,n}$ and $S^*_n = S^*_{0,n}$. Then for $m, n = 1, 2, \cdots$

$$T_{m+n} - T_n = T_{m,n} + S_m\ S_{m,n}. \tag{8}$$

Proof of Theorem 1. Since $EX = 0$, we can assume that $p \ge 1$. If $\alpha p > 1$, then Theorem 1 follows from Theorem 2. Hence we can assume that $\alpha p = 1$. Since $\alpha > 1/2, p < 2$. By Marcinkiewicz and Zygmund SLLN (see Chow and Teicher (1988), Ch. 5)

$$n^{-1/p}\ S_n \to 0, \qquad \text{a.s.} \tag{9}$$

and by its U-statistics version (Teicher (1992)),

$$n^{-2/p}\ T_n \to 0, \qquad \text{a.s.} \tag{10}$$

By (8), (9) and (10),

$$T^*_{2^n,2^n} \le T^*_{2^{n+1}} + T^*_{2^n} + S^*_{2^n} S^*_{2^n,2^n} = o(4^{n/p}), \quad \text{a.s.}$$

Since $(T^*_{2^n,2^n}, n \ge 1)$ are independent, by Borel-Cantelli lemma, $\sum_1^\infty P(T^*_{2^n,2^n} > 4^{n/p}) < \infty$. Hence $\int_1^\infty P(T^*_{2^t} > 4^{(t+1)/p})dt < \infty$. By change of variables, $\int_1^\infty y^{-1}\ P(T^*_y > 4^{1/p}y^{2/p})dy < \infty$. Replacing $(X, X_1, \ldots)$ by $(4^{1/p}X, 4^{1/p}X_1, \ldots)$, we have

$$\int_1^\infty y^{-1}P(T^*_y > y^{2/p})dt < \infty,$$

yielding (3) when $\alpha p = 1$.

3. Proof of Theorem 2. For the proof of Theorem 2, we need the following lemma.

Lemma 2. *For* $1 \le r \le 2, 0 \le \beta r < 1, x > 0$ *and* $k = 1, 2, \ldots,$

$$I \equiv P\Big(\max_{j\le n}|j^{-2\beta}\,T_j| \ge 5kx\Big)$$
(11)
$$\le kP\Big(\max_{j\le n}|j^{-\beta}\,S_j| \ge x^{1/2}\Big) + P^k\Big(\max_{j\le n}|j^{-2\beta}\,T_j| \ge x\Big),$$

and

(12) $$P\Big(\max_{j\le n}|j^{-2\beta}\,T_j| \ge x\Big) \le C_{r,\beta}\,x^{-2r}n^{2-2\beta r}\,(E|X|^r)^2,$$

where $C_{r,\beta}$ is a constant, depending only on r and β.

Proof. Let $\delta = 2\beta$ and $n = 1,2,\ldots$. Define $\theta = \theta_1 = \inf\{j \ge 1; |T_j| \ge xj^\delta\} \wedge (n+1)$, and for $m = 1,2,\ldots$, $\theta_{m+1} = \inf\{j \ge 1; |T_{\theta_1+\cdots+\theta_m,j}| \ge xj^\delta\} \wedge (n+1)$.

Then (see Chow and Teicher (1988, Ch. 5)) $\theta_1, \theta_2, \ldots$ are iid. Put

(13) $$A = \Big\{\max_{j\le n}|j^{-\beta}\,S_j| < x^{1/2}, \max_{j\le n}|j^{-\delta}\,T_j| \ge 5kx\Big\}.$$

On A, $\theta = \theta_1 \le n$ and

$$x \le \max_{j\le\theta}|j^{-\delta}\,T_j| = |\theta^{-\delta}T_\theta| = |\theta^{-\delta}(T_{\theta-1} + S_{\theta-1}X_\theta)|$$
(14)
$$\le x + |\theta^{-\beta}S^*_{\theta-1}\cdot\theta^{-\beta}(S^*_\theta + S^*_{\theta-1})| \le 3x.$$

By (8) and (14),

$$\max_{1\le m\le n-\theta}(m+\theta)^{-\delta}|T_{\theta,m}| = \max_m (m+\theta)^{-\delta}|T_{\theta+m} - T_\theta - S_\theta S_{\theta,m}|$$
$$\ge \max_{1\le m\le n}|m^{-\delta}T_m| - |\theta^{-\delta}T_\theta| - \max_{1\le m\le n-\theta}\theta^{-\beta}|S_\theta|\cdot(m+\theta)^{-\beta}|S_{m+\theta} - S_\theta|$$
$$\ge 5kx - 3x - 2x = 5(k-1)x.$$

Hence on A,

(15) $$\max_{j\le n}|j^{-\delta}T_{\theta,j}| \ge 5(k-1)x.$$

Since $(T_{\theta,j}, j \ge 1)$ is independent of θ and identically distributed with $(T_j, j \ge 1)$ (see Chow and Teicher (1988, Ch. 5)), by (13) and (15),

$$P(A) \le P\Big\{\theta \le n, \quad \max_{j\le n}|j^{-\delta}T_{\theta,j}| \ge 5(k-1)x\Big\}$$
(16)
$$= P(\theta \le n)P\Big\{\max_{j\le n}|j^{-\delta}T_j| \ge 5(k-1)x\Big\}.$$

Now, put $B = \{\max_{j\le n} |j^{-\beta}S_j| \ge x^{1/2}\}$ and $C_k = \{\max_{j\le n} |j^{-\delta}T_j| \ge 5kx\}$. Then $I \equiv P(C_k) \le P(B) + P(A) \le P(B) + P(\theta \le n)P(C_{k-1})$. By induction, $I \le kP(B) + P^k(\theta \le n)$, yielding (11). For (12), by the Hajek-Renyi-Chow inequality (cf. Chow, Robbins and Siegmund (1972, p. 25))

$$P\Big(\max_{j\le n} |j^{-\delta}T_j| \ge x\Big)$$
$$\le x^{-r}\Big\{\sum_{j=1}^{n-1} (j^{-\delta r} - (j+1)^{-\delta r})E|T_j|^r + n^{-\delta r}E|T_n|^r\Big\}. \tag{17}$$

Since $1 \le r \le 2$, by Burkholder inequality (1973) there exists a constant $C = C_{r,\beta}$ such that

$$E|T_n|^r \le C\cdot\sum_1^n E|X_j|^r\cdot E|S_{j-1}|^r \le C\cdot n^2E^2|X|^r. \tag{18}$$

Hence $P\Big(\max_{j\le n} |j^{-\delta}T_j| \ge x\Big) \le C\cdot x^{-r}n^{2-\delta r}E^2|X|^r$, yielding (12), since $\delta r < 2$.

Proof of Theorem 2. We can assume that $E|X|^p < \infty$ and $0 < E|X|^r = B < \infty$. Put

$$k = [1 + (\alpha p - 1)/(\alpha r - 1)], \quad a_n = n^{\alpha p-2}P^k\Big(\max_{j\le n} |j^{-2\beta}T_j| > n^{2\alpha-2\beta}\Big).$$

Then by Lemma 1,

$$\begin{aligned} a_n &\le C_{r,\beta}n^{\alpha p-2+2(1-\beta r)k-2(\alpha-\beta)kr}B^{2k} \\ &= C_{r,\beta}n^{\alpha p-2+2(1-\alpha r)k}B^{2k}. \end{aligned} \tag{19}$$

Since $(\alpha r - 1)k > 1$,

$$\begin{aligned} \sum_{n^{\alpha r-1}>B} a_n &\le C_{p,r,\alpha,\beta}\, B^{2k+\{\alpha p-1+2(1-\alpha r)k\}/(\alpha r-1)} \\ &= C_{p,r,\alpha,\beta}B^{(\alpha p-1)/(\alpha r-1)}. \end{aligned} \tag{20}$$

If $B > 1$,

$$\sum_{n^{\alpha r-1}\le B} a_n \le \sum n^{\alpha p-2} \le C_{p,\alpha}B^{(\alpha p-1)/(\alpha r-1)}. \tag{21}$$

Hence

$$(22) \qquad \sum_{n=1}^{\infty} a_n \le C_{p,r,\alpha,\beta} B^{(\alpha p-1)/(\alpha r-1)}.$$

Since by Chow and Lai (1978)

$$(23) \qquad \begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{j\le n} j^{-\beta}|S_j| > n^{\alpha-\beta}\Big) \\ &\le C_{p,r,\alpha,\beta}\{E|X|^p + B^{(\alpha p-1)/(\alpha r-1)}\}, \end{aligned}$$

(4) follows from (22), (23) and lemma 2.

From Theorem 2, we immediately have by letting $r = p$:

Corollary 1. *If $E|X|^p < \infty$ some $1 \le p \le 2$ and $\alpha p > 1 > \beta p \ge 0$, then for some constant $C = C_{p,\alpha,\beta}$,*

$$(24) \qquad \sum_{1}^{\infty} n^{\alpha p-2} P\Big(\max_{j\le n}\ |j^{-2\beta}T_j| \ge n^{2\alpha-2\beta}\Big) \le C\ E|X|^p.$$

Corollary 2. *If $E|X|^p < \infty$ some $p \ge 1$ and $\alpha p > 1 > \beta p \ge 0$, then as $\epsilon \to 0$ or $\epsilon \to \infty$,*

$$(25) \qquad \epsilon^{p+(p-2)^+/(2\alpha-1)} \sum_{1}^{\infty} n^{\alpha p-2} P\Big(\max_{j\le n} |j^{-2\beta}T_j| \ge \epsilon n^{2\alpha-2\beta}\Big) = 0(1),$$

where $a^+ = \max(a, 0)$.

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