BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 21, Number 1, March 1993

ON λ-FIRMLY NONEXPANSIVE MAPPINGS IN NONCONVEX SETS

BY

YEOU-MING HONG (洪有明) AND YOUNG-YE HUANG (黃永裕)

Abstract. It is shown that a weakly commutative family $\{T_i\}_{i\in I}$ of λ-firmly nonexpansive self-maps on a finite union $C=\cup_{k=1}^{n}C_k$ of nonempty, disjoint, bounded, closed convex subsets C_k of a uniformly convex Banach space X has a common fixed point in C whenever their graphs have a nonempty intersection. Moreover, in case $\{T_i\}_{i\in I}$ is a finite commutative family, the result still holds without the assumption of nonempty intersection for graphs.

1. Introduction. Let C be a nonempty subset of a Banach space X. A mapping $T: C \to C$ is said to be *nonexpansive* if

$$\|Tx-Ty\| \le \|x-y\| \tag{1}$$

for all x, y in C. If there exists a $\lambda \in (0,1)$ so that

$$\|Tx-Ty\| \le \|(1-\lambda)(x-y)+\lambda(Tx-Ty)\| \tag{2}$$

for all x, y in C, then T is said to be λ-firmly nonexpansive. It is obvious that every λ-firmly nonexpansive mapping is nonexpansive. The behaviors of nonexpansive mappings and λ-firmly nonexpansive mappings are quite different in nonconnected sets. Recently, R. Smarzewski [6] obtained the following result.

Received by the editors November 8, 1991 and in revised form August 10, 1992.

1991 Mathematics Subject Classification: 47H09, 47H10.

Key words and phrases: λ-firmly nonexpansive mappings, fixed points, weakly commutative.

Let X be a uniformly convex Banach space and $C = \bigcup_{k=1}^{n} C_k$ a finite union of nonempty, bounded, closed convex subsets C_k of X. Then any λ-firmly nonexpansive mapping $T : C \to C$ has a fixed point in C.

If $C_1 = C_2 = \ldots = C_n = C$, then the above theorem is true for nonexpansive mapping, which is the well-known fixed point theorem of Browder [1], Göhde [3] and Kirk [4]. In this note we shall prove the following theorem which is a generalization of the Smarzewski's fixed point theorem.

Thorem. *Let $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty, disjoint, bounded, closed convex subsets C_k of a uniformly convex Banach space X, and let $\{T_i\}_{i \in I}$ be any weakly commutative family of λ-firmly nonexpansive self-maps on C. If there is an x in C such that its orbit under $\{T_i\}_{i \in I}$ is a singleton, then T_i, $i \in I$, have a common fixed point in C.*

2. Common fixed point theorems for λ-firmly nonexpansive mappings.

To begin with we deduce some lemmas which will be used in the proof of the main theorem.

Lemma 2.1. *Suppose $C = \bigcup_{k=1}^{n} C_k$ is a finite union of nonempty, disjoint, bounded, closed convex subsets C_k of a Banach space X, and suppose $T : C \to C$ is nonexpansive. For $k \in \{1, 2, \ldots, n\}$ if there is an $x \in C_k$ such that $Tx \in C_r$ for some $r \in \{1, 2, \ldots, n\}$, then $T(C_k) \subseteq C_r$.*

Proof. For any $s \in \{1, 2, \ldots, n\}$ and for any $z \in C_s$, let $r_p(z) = d(z, C_p)$ for $p \neq s$, and let $t_q(z) = d(Tz, C_q)$ for $q \neq j$, where $Tz \in C_j$ and $d(z, C_p)$ denotes the distance from z to C_p. Since each C_p is closed, $r_p(z)$ and $t_q(z)$ are positive, and so

$$\delta(z) = \min_{\substack{p \neq s \\ q \neq j}} \{r_p(z), t_q(z)\} \quad \text{is positive.}$$

Now, suppose $x \in C_k$ and $Tx \in C_r$. We show that $T(C_k) \subseteq C_r$. For any y in C_k, if $\|x - y\| < \delta(y)$, then we are done. Otherwise, choose $m(x) \in \mathbb{N}$

such that $\|x-y\|/m(x) < \delta(x)$, and put $u_1 = x + ((y-x)/m(x))$. If Tu_1 were not in C_r, we would have $Tu_1 \in C_q$ for some q, and so we would have

$$\|Tu_1 - Tx\| \geq d(Tx, C_q) \geq t_q(x) \geq \delta(x) > \|Tu_1 - Tx\|,$$

which is a contradiction. Therefore Tu_1 must lie in C_r.

Replacing x by u_1 and arguing in the like manner, if $\|y - u_1\| < \delta(y)$ then we are done. Otherwise, putting $u_2 = u_1 + ((y-u_1)/m(u_1))$ we see that $Tu_2 \in C_r$. Continuing in this way for w steps we finally get $u_w = u_{w-1} + ((y-u_{w-1})/m(u_{w-1}))$, $Tu_w \in C_r$ and $\|u_w - y\| < \delta(y)$. So we see that $Ty \in C_r$.

Lemma 2.2. *Suppose C is any nonempty subset of a strictly convex Banach space X, and suppose $T : C \to C$ is a λ-firmly nonexpansive mapping for some $\lambda \in (0,1)$. If x, y are in C so that $\|Tx - Ty\| = \|x-y\|$, then $Tx - Ty = x - y$.*

Proof. Since

$$\begin{aligned}\|Tx - Ty\| &\leq \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\| \\ &\leq (1-\lambda)\|x-y\| + \lambda\|Tx - Ty\| \\ &\leq \|x - y\| = \|Tx - Ty\|,\end{aligned}$$

we have

$$\|(1-\lambda)(x-y) + \lambda(Tx-Ty)\| = (1-\lambda)\|x-y\| + \lambda\|Tx-Ty\|.$$

It then follows from the strict convexity of the norm that $Tx - Ty = \alpha(x-y)$ for some scalar α, and hence the equalities $\|x-y\| = \|Tx-Ty\| = |\alpha|\,\|x-y\|$ show that α is 1 or -1. But, if $\alpha = -1$, then

$$\begin{aligned}\|x-y\| &= \|Tx - Ty\| \\ &\leq \|(1-\lambda)(x-y) + \lambda(Tx-Ty)\| \\ &= \|(1-\lambda)(x-y) + \lambda(y-x)\| \\ &= |1-2\lambda|\,\|x-y\| \\ &< \|x-y\|, \qquad \text{a contradiction.}\end{aligned}$$

Therefore $\alpha = 1$, i.e. $Tx - Ty = x - y$.

Lemma 2.3. *Suppose $C = \bigcup_{k=1}^{n} C_k$ is a finite union of nonempty, disjoint, bounded, closed convex subsets C_k of a uniformly convex Banach space X, and suppose $T : C \to C$ is λ-firmly nonexpansive for some $\lambda \in (0,1)$. Then there exists some $k \in \{1,2,\ldots,n\}$ such that $T(C_k) \subseteq C_k$.*

Proof. Assume that for any $k \in \{1,2,\ldots,n\}$, $T(C_k) \nsubseteq C_k$. In view of Lemma 2.1, there exist integers $\{n_1,n_2,\ldots,n_m\} \subseteq \{1,2,\ldots,n\}$, $(m \geq 2)$, such that $T(C_{n_j}) \subseteq C_{n_{j+1}}$ for $j = 1,2,\ldots,m$, where $C_{n_{m+1}} = C_{n_1}$. We may rearrange the indices so that $T(C_j) \subseteq C_{j+1}$ for $j = 1,2,\ldots,m$, where $C_{m+1} = C_1$. Notice $T^m : C_1 \to C_1$ and T^m is nonexpansive. So there is ξ in C_1 such that $T^m\xi = \xi$. By the λ-firmly nonexpansiveness we have

$$\begin{aligned}
\|T\xi - \xi\| &= \|T\xi - T(T^{m-1}\xi)\| \\
&\leq \|(1-\lambda)(\xi - T^{m-1}\xi) + \lambda(T\xi - \xi)\| \\
&\leq (1-\lambda)\|\xi - T^{m-1}\xi\| + \lambda\|T\xi - \xi\| \\
&= (1-\lambda)\|T^m\xi - T^{m-1}\xi\| + \lambda\|T\xi - \xi\| \\
&\leq (1-\lambda)\|T^{m-1}\xi - T^{m-2}\xi\| + \lambda\|T\xi - \xi\| \\
&\quad\vdots \\
&\leq (1-\lambda)\|T^2\xi - T\xi\| + \lambda\|T\xi - \xi\| \\
&\leq (1-\lambda)\|T\xi - \xi\| + \lambda\|T\xi - \xi\| \\
&= \|T\xi - \xi\|.
\end{aligned}$$

Consequently, we see recursively that

$$\|T\xi - \xi\| = \|T^2\xi - T\xi\| = \cdots = \|T^{m-1}\xi - T^{m-2}\xi\| = \|T^m\xi - T^{m-1}\xi\|,$$

and hence it follows from Lemma 2.2 that

$$T\xi - \xi = T^2\xi - T\xi = \cdots = T^{m-1}\xi - T^{m-2}\xi = T^m\xi - T^{m-1}\xi.$$

This means that $T^i\xi = (T^{i-1}\xi + T^{i+1}\xi)/2$ for any $i = 1,2,\ldots,m$. Hence we conclude that $T\xi = \xi$, and then Lemma 2.1 shows that $T(C_1) \subseteq C_1$.

We now come to the main result of this paper. But at first recall that a family $\{T_i\}_{i\in I}$ of self-mappings on C is said to be a *weakly commutative family* if it satisfies

$$\|T_iT_jx - T_jT_ix\| \le \|T_jx - T_ix\| \tag{3}$$

for any i, j in I and for any x in C.

Theorem 2.4. *Let $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty, disjoint, bounded, closed convex subsets C_k of a uniformly convex Banach space X, and let $\{T_i\}_{i\in I}$ be any weakly commutative family of λ-firmly nonexpansive self-maps on C. If there is an x in C such that its orbit under $\{T_i\}_{i\in I}$ is a singleton, then T_i, $i \in I$, have a common fixed point in C.*

Proof. Let $A = \{x \in C : T_ix = T_jx \quad \forall\, i,j \text{ in } I\}$. By assumption $A \neq \emptyset$. For any x in A and for any T_i, T_j, since T_i, T_j are weakly commutative, we have $\|T_iT_jx - T_jT_ix\| \le \|T_jx - T_ix\| = 0$, and therefore T_i and T_j are commutative on A. Consequently, $T_i^2x = T_i(T_jx) = T_j(T_ix) = T_j^2x$. By induction, we obtain that $T_i^qx = T_j^qx$ for any i, j in I and for any q in $\mathbb{N}$.

We now fix any T_j. Lemma 2.3 shows that there is C_k such that $T_j(C_k) \subseteq C_k$. We claim that among those T_j-invariant subsets there is one C_k such that $A \cap C_k \neq \emptyset$. If not, let $\{C_1, \ldots, C_p\}$ be the collection of all T_j-invariant subsets and $\{C_{p+1}, \ldots, C_n\}$ the collection of all those that are not T_j-invariant. Note that $p \neq n$. Otherwise $\emptyset \neq A = \bigcup_{i=1}^{n}(A \cap C_i) = \emptyset$. Furthermore, note that we have either

(i) $T_j^s(C_v) \not\subseteq C_u$ for any $s \in \mathbb{N}$ and for any $u \in \{1, 2, \ldots, p\}$ and for any $v \in \{p+1, \ldots, n\}$ or

(ii) there exist $s \in \mathbb{N}$, $u \in \{1, 2, \ldots, p\}$ and $v \in \{p+1, \ldots, n\}$ so that $T_j^s(C_v) \subseteq C_u$.

To proceed, we assume that $T_j(C_{p+r}) \subseteq C_{p+r+1}$ for $r = 1, 2, \ldots, n-p$, where C_{n+1} denotes C_{p+1}. We at first show that case (i) cannot happen. Since $T_j : \bigcup_{i=p+1}^{n} C_i \to \bigcup_{i=p+1}^{n} C_i$, it follows from Lemma 2.3 and the Browder-Göhde-Kirk fixed point theorem that there is y in $\bigcup_{i=p+1}^{n} C_i$ so that $T_jy = y$. Choose $q \in \{p+1, \ldots, n\}$ so that $y \in C_q$. Then Lemma 2.1

gives us that $T_j(C_q) \subseteq C_q$, which is impossible. So we can only have case (ii). Choose $x \in A \cap C_v$. By the first paragraph we have $T_i^q x = T_j^q x$ for any $i \in I$ and $q \in \mathbb{N}$. In particular, the point $z = T_i^s x = T_j^s x \in C_u$. Moreover, $T_j(z) = T_j(T_i^s x) = T_j(T_j^s x) = T_j^{s+1}(x) = T_i^{s+1}(x) = T_i(T_i^s x) = T_i(T_j^s x) = T_i(z)$. So we see that $z \in A \cap C_u$, which again contradicts the assumption $A \cap C_u = \emptyset$. Therefore we have shown that there is a T_j-invariant subset C_k such that $A \cap C_k \neq \emptyset$.

For the C_k above, choose $x \in A \cap C_k$. For any $i \in I$, since $T_i(x) = T_j(x) \in C_k$, we see that $T_i : C_k \to C_k$. Let $x_q = T_j^q(x)$ $(= T_i^q x \ \ \forall\, i \in I)$ $(q \in \mathbb{N})$. By [2] the sequence $\{x_q\}$ has a unique asymptotic center ξ in C_k, i.e. $f(\xi) = \inf_{y \in C_k} f(y)$, where $f(y) = \limsup_{q \to \infty} \|y - x_q\|$. But then, since T_i is nonexpansive, we have

$$\begin{aligned} \|T_i\xi - x_{q+1}\| &= \|T_i\xi - T_i^{q+1}x\| \\ &\leq \|\xi - T_i^q x\| \\ &\leq \|\xi - x_q\|, \qquad \forall\, q \in \mathbb{N}. \end{aligned}$$

Thus, $f(T_i\xi) \leq f(\xi)$. The uniqueness of asymptotic center yields $T_i\xi = \xi$ $\forall\, i \in I$. This completes the proof.

Here we like to mention that the Smarzewski's fixed point theorem stated in the introduction comes from the theorem above. As a matter of fact, for fixed $z \in C$, let x_i be the unique asymptotic center of the sequence $\{T^m z\}$ with respect to C_i, $1 \leq i \leq n$. Let C' be the collection of all asymptotic centers of the same sequence $\{T^m z\}$ with respect to C. Obviously, C' is a nonempty subset of $\{x_1, \ldots, x_n\}$ and is T-invariant. For definiteness, let $C' = \{x_1, \ldots, x_p\}$, where $p \leq n$. Putting $C_i' = \{x_i\}$, $1 \leq i \leq p$, and applying Theorem 2.4 to $C' = \bigcup_{i=1}^p C_i'$, we get the conclusion.

If the family $\{T_i\}_{i \in I}$ is finite and commutative, the theorem above can be improved.

Theorem 2.5. *Let $C = \bigcup_{k=1}^n C_k$ be a finite union of nonempty, disjoint, bounded, closed convex subsets C_k of a uniformly convex Banach space X, and let $\{T_i\}_{i=1}^m$ be any finite commutative family of λ-firmly nonexpan-*

sive self-maps on C. Then T_i, $1 \le i \le m$, have a common fixed point in C.

Proof. For any $i \in \{1,2,\dots,m\}$ and $k \in \{1,2,\dots,n\}$ let $F_k(T_i)$ be the set of all fixed points of T_i in C_k. It is well-known that if $F_k(T_i) \neq \emptyset$, then it is a closed convex subset of C_k. For any $i \in \{1,2,\dots,m\}$ let $M_i = \{k : F_k(T_i) \neq \emptyset\}$. Then each M_i is nonempty. Let $F_i = \bigcup_{k \in M_i} F_k(T_i)$. By the commutativity of $\{T_i\}_{i=1}^m$, each F_i is T_j-invariant for any $j \in \{1,2,\dots,m\}$. In particular, $T_2 : F_1 \to F_1$ and T_2 has fixed points in F_1 by the Smarzewski's fixed point theorem. Hence $F_1 \cap F_2$ is a union of nonempty disjoint closed subsets of X. Repeating the argument in the same way, we see that, for any $j \in \{1,2,\dots,m-1\}$, $F_1 \cap \cdots \cap F_j$ is T_{j+1}-invariant and $F_1 \cap \cdots \cap F_{j+1} \neq \emptyset$. After $m-1$ steps, we conclude that $\bigcap_{i=1}^m F_i \neq \emptyset$.

Suppose $T : C \to C$ is a contraction mapping with contractive constant $k \in (0,1)$. Putting $\lambda = (1-k)/(1+k)$, we deduce that T is λ-firmly nonexpansive. Using this remark we can easily check the mappings T in the following examples are λ-firmly nonexpansive.

Example 2.6. Let $C = [-2,2]$ and $T_1 : C \to C : T_1x = (1+x)/3$ and $T_2 : C \to C : T_2x = x/2$. Then $T_1(2) = T_2(2) = 1$, while T_1, T_2 have no common fixed points in C. This is due to $\{T_1, T_2\}$ is not a weakly commutative family.

Example 2.7. Let $C = [0,1] \cup \left\{\frac{5}{2}\right\}$ and define

$$T_1 : C \to C : T_1x = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1] \\ \frac{2}{3} & \text{if } x = \frac{5}{2}, \end{cases}$$

and

$$T_2 : C \to C : T_2x = \begin{cases} \frac{x}{3} & \text{if } x \in [0,1] \\ \frac{1}{2} & \text{if } x = \frac{5}{2}. \end{cases}$$

Then $\{T_1, T_2\}$ is a weakly commutative but not commutative family of λ-firmly nonexpansive mappings, where λ can be chosen to be any number in $(0,1)$. In this case, 0 is the only common fixed point for T_1 and T_2.

References

1. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci., **54** (1965), 1041–1044.

2. K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.

3. D. Göhde, *Zum Prinzip der Kontraktiven Abbildung*, Math. Nach., **30** (1965), 251–258.

4. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.

5. S. Reich and I. Shafrir, *The asymptotic behavior of firmly nonexpansive mappings*, Proc. Amer. Math. Soc., **101** (1987), 246–250.

6. R. Smarzewski, *On firmly nonexpansive mappings*, Proc. Amer. Math. Soc., **113** (1991), 723–725.

Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan, 70101, R.O.C.