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AN IMPROVED DETERMINATION FOR DECAY AT INFINITY OF SOLUTIONS TO CONVOLUTION EQUATION

BY

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Abstract. Let T be a finite distribution with Fourier transform F (which is then an entire function of finite exponential type), let T satisfy the two conditions stated below and let $N(F)$ denote the set of all $\xi \in R^n$ such that $F(\xi) = 0$. Then the convolution equation $T * u = \Phi$ in R^n has at most one distributional solution u such that at infinity $u(x) = o(|x|^{-d})$ for any $d \geq n - 1 - l/2$ for some positive integer l depending solely on the geometric structure of $N(F)$. Moreover, if for each $\xi \in R^n$ there exists $\eta \in R^n$ such that $|\xi - \eta| \leq a \log(1 + |\xi|)$ and $|F(\eta)| \geq (a + |\eta|)^{-a}$, the only solution of the convolution equation fulfilling the above uniqueness condition is a C_0^∞-function for each C_0^∞-function Φ. Denote by f_j all distinct irreducible factors of F, by $I(\xi)$ the set of all indices j of f_j such that $\xi \in N(f_j)$, and by $|I(\xi)|$ the number of elements in $I(\xi)$. The above finite distribution T must satisfy the two conditions:

(i) the gradient of f_j does not vanish on $N(f_j)$ for each irreducible factor such that $N(f_j) \neq \Phi$;

(ii) the matrix $[\nabla f_j(\xi)]$, $j \in I(\xi)$, is of rank $|I(\xi)|$ for each $\xi \in N(F)$.

Introduction. For a distribution T with compact support (i.e. a finite distribution) satisfying the two conditions (i) and (ii) listed in the abstract, let f be the product of all irreducible factors f_j of F such that $N(f_j) \neq \Phi$. Then $N(f) = N(F)$ is an analytic manifold; that is, for each point ξ of $N(f)$ there exist analytic local coordinates in some neighborhood of ξ. Furthermore, the set J_κ of ξ such that $|I(\xi)|$ is a constant, where κ is an analytic submanifold of $N(f)$ and the union of the disjoint submanifolds J_κ, $\kappa = 1, \cdots, n-1$, related to different individual values of $I(\xi)$ is $N(F)$. Let $k(\xi)$ denote "the number of nonzero principal curvatures at ξ of the submanifold J_κ"

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(defined in the next paragraph) provided the manifold J_κ is of positive dimension, and denote zero provided the manifold J_κ is of dimension zero. Then the positive integer l in the main result asserted in the abstract is the minimum of $k(\xi)$ for all $\xi \in N(f)$. The assertion means actually that the decay rate in the direction $\omega = x/|x|$, $x \neq 0$, of the solution $u(|x|\,\omega)$ depends solely on the dimension of the given Euclidean space R^n, on the number of "nonzero principal curvatures of $\bigcap N(f_j)$ when it is embedded in $N(f_j)$, and on the number $|I(\xi)|$ of distinct manifolds $N(f_j)$ intersecting at $\xi \in N(f)$ at which the normal of one of the manifolds $N(f_j)$ is in the direction ω.

After translation, we can assume that the origin O is contained in J_κ and that $N(f_1),\cdots, N(f_\kappa)$ contain O, say. Then near O the manifold can be described by (ξ', ξ'') with small $\xi'' = (\xi_{\kappa+1},\cdots, \xi_n)$ and $\xi'(\xi'') = (\xi_1(\xi''),\cdots, \xi_\kappa(\xi'')) = 0$ (which is guaranteed by the implicit function theorem). The manifold M_i defined by the equation $\xi_i(\xi'') = 0$ with such ξ'' is $(n-\kappa)$-dimensional, $i = 1,\cdots, \kappa$. Denote by $\xi_i^{(j)}(\xi'')$ the partial derivative of the vector $(\xi_i(\xi''), \xi'')$ with respect to the variable ξ_j, $j = \kappa + 1,\cdots, n$; by $\nu(\xi)$ the unit normal of M_i when it is embedded in $R^{n-\kappa+1}$; and by $L_i^{(jk)}(\xi'')$ the inner product of $\xi_i^{(jk)}(\xi'')$ with $\nu(\xi'')$. The $n-\kappa$ principal curvatures of M_i at $\xi = 0$ are defined to be the $n-\kappa$ eigenvalues of the matrix $[L_i^{(jk)}(0)]$. "The number of nonzero principal curvatures at $\xi = 0$ of J_κ" is defined to be the minimum of the nonzero principal curvatures of M_i at $\xi = 0$ for all $i \in I(\xi)$.

The method employed here is simply the symmetrization of distributions introduced in Chen [1], and again in Chen [2], applied to the manifold $N(f)$. Therefore the results derived here refine those of Chen [1, 2] and Littman [4, 5].

The idea is applied to solve the wave propagation problem which is to appear in another paper.

The organization of the paper is as follows: The assumptions and main results are given in the first section; the determination of the decay at infinity forms the second section; the third section contains the idea of symmetrization, the corresponding representations, and the behavior at infinity of inverse Fourier transforms of symmetrized distributions; in §4, we determine the decay rate which yields the uniqueness condition and we then construct the finite smooth solution satisfying the condition; and finally we give some remarks about the uniqueness condition independent of the geometric structure of $N(F)$ and the references.

1. Assumptions and Main Results. The notation and terminology are based on [1]. Denote by $\mathfrak{C}$ the class of entire functions of finite exponential type satisfying the following two conditions:

($\mathfrak{C}$ i) the null set $N(f)=\{\xi\in R^n : f(\xi)=0\}$ of f is non-empty: $N(f)\neq\Phi$;

($\mathfrak{C}$ ii) the gradient of f at each point $\xi\in N(f)$ does not vanish: $\nabla f(\xi)\neq 0$.

As for the classification of $\mathfrak{C}$, let $\mathfrak{C}(k)$ denote the class of functions $f\in\mathfrak{C}$ such that

($\mathfrak{C}(k)$) $$k=\operatorname{Inf} k(\xi) \qquad (\xi\in N(f)),$$

where $k(\xi)$ is the number of the nonzero principal curvatures of $N(f)$ at ξ.

For an entire function $F(z)$ of finite exponential type, let f be the product of its distinct irreducible factors f_j. Denote by $I(\xi)$, $\xi\in N(f)$, the set of the indices j of all irreducible factors f_j with $\xi\in N(f_j)$ and by $|I(\xi)|$ the number of the elements in $I(\xi)$.

DEFINITION 1.1. A distribution $T\in\mathcal{D}'$ belongs to $\mathfrak{C}(k, l)$ if and only if

($\mathfrak{C}(k, l)$ i) T has compact support;

($\mathfrak{C}(k, l)$ ii) if F denotes the Fourier transform of T, then

$$k=\min\{k_j : f_j\in\mathfrak{C}(k_j) \text{ are irreducible factors of } F\};$$

($\mathfrak{C}(k, l)$ iii) for each $\xi\in N(f)$ the matrix $|\nabla f_j(\xi)|$, $j\in I(\xi)$, is of rank $|I(\xi)|$ (and hence $|I(\xi)|\leq n-1$);

($\mathfrak{C}(k, l)$ iv) $l=\inf\{k(\xi) : \xi\in N(f)\}$ where $k(\xi)$ is the number of the nonzero principal curvatures of $\bigcap N(f_j)$, $j\in I(\xi)$.

Consider the convolution equation

(1.1) $$T*u=\Phi, \qquad \Phi\in\mathcal{D}'.$$

THEOREM 1.1. *For $T \in \mathfrak{C}(k, l)$, the convolution equation has at most one solution satisfying the condition at infinity*:

$$u(x) = o(|x|^{-d}) \qquad \text{for any } \ d \geq n-1-l/2\,. \tag{1.2}$$

Let us recall some ideas from L. Ehrenpreis [3].

DEFINITION 1.2. For $T \in \mathcal{D}'$ with compact support, then T is said to be *invertible* if and only if its Fourier transform F satisfies the property: for each point $\xi \in R^n$ there is an $\eta \in R^n$ such that $|\xi - \eta| \leq a \log(1 + |\xi|)$ and $|F(\eta)| \geq (a + |\eta|)^{-a}$. The function F is called *slowly decreasing.*

THEOREM 1.2. *For an invertible distribution $T \in \mathfrak{C}(k, l)$, $k > 0$, and the inhomogeneous term $\Phi \in C_0^\infty(R^n)$, a continuous solution u of the equation* (1.1) *belongs to the class $C_0^\infty(R^n)$ if u satisfies the uniqueness condition* (1.2).

Let $T \in \mathfrak{C}(k, l)$ and let f be the corresponding function as before. Let $\mu \neq 0$ be a nonnegative function of the class $C_0^\infty(R^n)$ with value 1 at certain points ξ in $N(f)$, where $k(\xi) = l$ in condition ($\mathfrak{C}(k, l)$ iv).

THEOREM 1.3. *The inverse Fourier transform u of μ has the following asymptotic behavior at infinity,*

$$u(x) = O(|x|^{-m}) \qquad \text{uniformly in direction } \ x/|x|\,; \tag{1.3)$_O$ (}$$

$$u(x) \neq o(|x|^{-m}) \qquad \text{for some unit vector } \ x/|x|\,, \tag{1.3)$_o$ (}$$

for some $m \geq l/2$.

2. Fourier Transforms of nonnegative surface-carried measures and proof of Theorem 1.3. As a key point in this and the next sections, let us recall the well-known theorem on inverse functions (e.g. H. Whitney [8, p. 68]).

THEOREM 2.1. *Let F be an s-smooth mapping ($s \geq 1$) of the open set $\Omega \subset R^n$ into R^n, and suppose the Jacobian $J_F(p_0) \neq 0$ at $p_0 \in \Omega$. Then there are neighborhoods U of p_0 and U' of $f(p_0)$ such that F is a one-to-one mapping of U onto U' and $G = F^{-1}$, considered in U' only, is s-smooth, with $J_G(f(p_0)) \neq 0$.*

It suffices to consider the case $|I(\xi)| > 1$ with $\xi = \xi_0$ in supp μ. Because the arguments are similar for the general situation, let us restrict the discussion to the case that $I(\xi_0)=\{1, 2\}$; i. e., $\xi_0 \in N(f_i)$, $i=1, 2$ and $\xi_0 \notin N(f_j)$, $j\neq 1, 2$. Assume further that the normal of $N(f_1)$ at ξ_0 is ω_0, say $\omega_0=(1, 0,\cdots, 0)$; i. e. $n_1(\xi_0)=(1, 0,\cdots, 0)$ and that $|I(\xi)|\leq 1$ outside some small neighborhood U of ξ_0 and $D_1 f_1(\xi)\neq 0$ on $U\cap N(f)$. By condition ($\mathfrak{C}(k, l)$ iii), $n_2(\xi_0)\neq \omega_0$ and then we can assume $D_2 f_2(\xi)\neq 0$ on $U\cap N(f)$. Because of the partition of unity related to the compact set supp μ, we assume supp $\mu \subset U$. Theorem 2.1 implies that

$$\eta=(f_1(\xi), f_2(\xi), \xi'), \qquad \eta'=\xi'=(\xi_3,\cdots, \xi_n)$$

is a C^∞-diffeomorphism on supp μ with the Jacobian $J(\eta, \xi)\neq 0$, and the inverse mapping

$$\xi=(\xi_1, \xi_2, \eta'), \qquad \xi_1=\xi_1(\eta_1, \eta_2, \eta'), \qquad \xi_2=\xi_2(\eta_1, \eta_2, \eta')$$

is C^∞-diffeomorphic and $J(\xi, \eta)\neq 0$.

Let $\Phi_i \in C_0^\infty(U)$ be such that $1\geq \Phi_i \geq 0$, let $\Phi_2=1$ on $\{\xi \in \text{supp}\, \mu : |I(\xi)|=2\}$ and $\Phi_2=0$ outside a small neighborhood of the set, and let $\Phi_1+\Phi_2=1$ on U. Then $|I(\xi)|=1$ on supp $\Phi_1 \cap N(f)$. Then with $x=\rho\omega$,

$$\begin{aligned} u(x) &= \int e^{i\rho\omega\cdot\eta}\, \mu(\eta)\, d_\sigma \eta \\ &= \sum \int e^{i\rho\omega\cdot\eta} (\mu\, \Phi_i)(\eta)\, d_\sigma\eta \equiv u_1(x)+u_2(x)\,. \end{aligned}$$

Since the estimate at infinity in ρ for $u_1(\rho\omega)$ follows directly from the argument for $u_2(\rho\omega)$, we can assume that $u(x)=u_2(x)$ or $\mu=\mu\, \Phi_2$, say $\Phi_2=1$.

From ($\mathfrak{C}(k, l)$ iii) it follows that $N_{12}\equiv N(f_1)\cap N(f_2)$ is $(n-2)$-dimensional at ξ_0 and its local coordinates are

$$\xi=(\xi_1(0, 0, \eta'), \xi_2(0, 0, \eta'), \eta')\,. \tag{2.1}$$

Then $\xi=(\alpha+\xi_1(0, 0, \eta'), \xi_2(0, 0, \eta'), \eta')$ and $\xi=(\xi_1(0, 0, \eta'), \alpha+\xi_2(0, 0, \eta'), \eta')$ describe some neighborhoods U_2, U_1 of N_{12} on $N(f_2)$ and $N(f_1)$, respectively. Let χ_i be $C_0^\infty(U_i)$ functions, $1\geq \chi_i \geq 0$, $\chi_1+\chi_2=1$ on supp μ and let supp χ_1 and supp χ_2 be described by $\xi=(\xi_1(0, 0, \eta'), \alpha+\xi_2(0, 0, \eta'), \eta')$ and $\xi=(\alpha+\xi_1(0, 0, \eta'), \xi_2(0, 0, \eta'), \eta')$, respectively. Then

$$u(x)=\sum \int e^{i\rho\omega\cdot\eta} (\chi_i\, \mu)(\eta)\, d_\sigma\eta\,, \qquad i=1, 2\,.$$

Since the arguments are the same for both cases, we consider only one of them, say the one with χ_1, and denote the corresponding function by $u_1(x)$ and use χ for χ_1.

By Taylor's formula, with $p=\xi_0$,

$$\begin{aligned}\theta(\eta') &= \xi_1(0,\ 0,\ \eta')\\ &= \theta(p)+\nabla'\theta(p)\cdot\eta'+\sum a_{ij}\eta_i\eta_j+O(|\eta'|^3)\,,\end{aligned}$$

where ∇' is the gradient with respect to η'. On N_{12}, $f_i(\theta(\eta'),\ \xi_2(0,0,\eta'),\eta')\equiv 0$, $i=1, 2$, and therefore $D_j\theta(\eta')=-D_jf_1(\xi)/D_1f_1(\xi)$ and

$$\theta(\eta')=f_1(p)-\nabla' f_1(p)\cdot\eta'/D_1f_1(p)+\sum a_{ij}\eta_i\eta_j+O(|\eta|^3)\,. \tag{2.2}$$

Let us set

$$g(\eta')=\theta(\eta')-f_1(p)+\nabla' f_1(p)\cdot\eta'/D_1f_1(p)\,. \tag{2.3}$$

Let s_j be in the direction of the jth nonzero principal curvature of N_{12} at p, $j=1,\cdots,\kappa$, where κ is the number of nonzero principal curvatures λ_j of N_{12} at p. By a Morse lemma [6, p. 172],

$$g(\eta'(s,\ t))=\sum\lambda_j(t)s_j^2 \qquad (1\leq j\leq\kappa) \tag{2.4}$$

with $s=(s_1,\cdots,s_\kappa)$ and with some suitable choice of $t=(t_{\kappa+1},\cdots,t_{n-2})$ such that the Jacobian $J(\eta',\ (s,\ t))$ of the transformation $\eta'\to(s,\ t)$ is 1 at p. On a sufficiently small neighborhood U of p, $J(\xi,\ \eta)>1/2$; say $U\subset\operatorname{supp}\mu$. Without loss of generality, we can assume $\operatorname{supp}\chi\subset\{(s,t)\in U:|t|<\varepsilon\}$ for some $\varepsilon>0$. Since principal curvatures are continuous, we can assume that on $\operatorname{supp}(\chi\mu)$ the nonzero principal curvatures of N_{12} do not change sign. Let d^+ and d^- be the number of positive and negative principal curvatures of $N_{12}\cap\operatorname{supp}(\chi\mu)$, respectively, and let $K_\kappa(s,\ t)$ be the product of the κ nonzero principal curvatures. Then we have the following estimate.

PROPOSITION 2.1. *For $x=\rho\omega$ approaching infinity,*

$$\begin{aligned}u_1(\rho\omega) &= (\pi(2\rho\omega_1)^{-1})^{\kappa/2}\,(1+i)^{d+}\,(1-i)^{d-}\exp\{i\rho\omega_1f_1(p)\}\\ &\quad\cdot\iint\mu_2(t)\exp\Big\{i\rho\omega_2(\sigma+f_2(\eta(0,\ t)))+i\rho\omega'\cdot\eta'(0,\ t)\\ &\quad+i\rho\omega_1\sum D_jf_1(p)\,\eta_j(0,t)/D_1f_1(p)\Big\}d\sigma\,dt+O(\rho^{-(\kappa+1)/2})\,,\end{aligned} \tag{2.5}$$

where the summation runs on $j=3,\cdots,n$ and

(2.6) $$\mu_2(t) = (\chi\mu_1)(\eta'(0, t))J(\eta', (0, t))/|K_\kappa(0, t)|^{1/2}.$$

Proof. From (2.2), (2.3) and (2.4),

$$u_1(\rho\omega) = \exp\{i\rho f_1(p)\}\iint \Psi(\rho, t, \sigma)\, d\sigma\, dt, \qquad |t| \leq \varepsilon,$$

where

$$\Psi(\rho, t, \sigma) = \int \exp\Big\{i\rho\omega_2(\sigma + f_2(\eta(s, t))) + i\rho\omega' \cdot \eta'(s, t) + i\rho\omega_1 \sum \lambda_j(t)\, s_j^2\Big\}\psi(s, t)\, ds$$

with

$$\psi(s, t) = (\chi\mu_1)(\eta'(s, t))J(\eta', (s, t)) \cdot \exp\Big\{-i\rho\sum D_j f(p)\,\eta_j(s, t)/D_1 f_1(p)\Big\}.$$

For each fixed t, by the arguments in W. Littman [4, p. 768] and [5, p. 454] for the case of nonzero Gaussian curvature,

$$\Psi(\rho, t, \sigma) = (\pi(2\rho\omega_1)^{-1})^{\kappa/2}(\exp\{i\rho\omega_2(\sigma + f_2(\eta(0, t))) + i\rho\omega'\eta'(0, t)\})(1+i)^{d+}(1-i)^{d-}\psi(0, t)/|K_\kappa(0, t)|^{1/2} + O(\rho^{-(\kappa+1)/2}),$$

when $\rho \to \infty$. This is the assertion of the proposition.

COROLLARY 2.1. *As* $\omega = \omega_0$,

(2.7) $$u(\rho\omega) = c(p)\,\rho^{-\kappa/2}\exp\{i\rho\omega \cdot p\} + O(\rho^{-(\kappa+1)/2})$$

when $\rho \to \infty$ *with* $c(p) \neq 0$.

Proof. $\omega_0 = (1, 0, \cdots, 0)$ yields $\omega_1 = 1$, $\omega_2 = 0$, and $\omega' = 0$. Hence from (2.5)

(2.8) $$c(p) = (\pi/2)^{\kappa/2}(1+i)^{d+}(1-i)^{d-}\int \mu_2(t)\, dt, \qquad (|t| \leq \varepsilon).$$

Since $K_\kappa(0, t) \neq 0$, we choose U so small that $|K_\kappa(0, t)| > \varepsilon_0$ for some ε_0. Since $J(\eta'(0, t)) > 1/2$ and μ is strictly positive on U, (2.6) implies that $c(p) \neq 0$ if U is sufficiently small. This completes the proof of the corollary.

COROLLARY 2.2. *If* ω *is far away from* ω_0, *then when* $\rho \to \infty$

(2.9) $$u(\rho\omega_0) = O(\rho^{-d}) \qquad \text{for any } d > 0.$$

Indeed, the assertion follows from the estimate (2.5) and integration by parts.

Proof of Theorem 1.3. Since $\operatorname{supp}\mu$ is compact, the partition of unity and the assertions in Corollaries 2.1 and 2.2 yield the assertion $(1.3)_O$. Let $x=\rho\omega_0$ with ω_0 in the direction of normal of $\bigcap N(f_j)$, $j\in I(\xi)$ at ξ, where the number of the nonzero principal curvatures of $\bigcap N(f_j)$ is l. Since $\operatorname{supp}\mu$ is compact, the set of such points ξ is bounded. The partition of unity related to the closure of the set implies that there is a finite number of constants $c(p_j)\neq 0$ in (2.8) corresponding to the points $p=p_j$ in the proof of Proposition 2.1. Because $c(p_j)\exp\{i\rho\omega\cdot p_j\}$ is almost periodic, we have the assertion $(1.3)_o$. This completes the proof of the theorem.

REMARK. If $|I(\xi)|=1$ on $N(f)$, the problem is considered by K. Chen [1] and [2]. In this case, $l=k$. Due to the assertion in $(1.3)_o$, the assertions in [2] are true if we replace the number $n-1$ by k. In particular, Theorems 3.2 and 4.4 hold. On the other hand, each term in the summation of the second estimate of Lemma 2.3 in [1, p. 461] should be in the form (2.8).

3. **Symmetrization of distributions corresponding to a manifold.** For a function $f\in\mathfrak{C}$ we proved in [2] that $N(f-\varepsilon)$ forms an $(n-1)$-dimensional C^∞-manifold embedded in R^n for each q, $|q|<\varepsilon$, for some $\varepsilon>0$. By the same arguments, applying the preceding theorem, we see that the same assertion holds for an entire function f of finite exponential type such that each irreducible factor $f_j\in\mathfrak{C}$ is of multiplicity one and satisfies condition ($\mathfrak{C}(k, l)$ iii) with $f-\varepsilon$ replaced by $f_q=\prod(f_j-q)$, $|q|<\varepsilon$ with small $\varepsilon>0$. In particular, it is true for $f\in\mathfrak{C}(k, l)$ with multiplicity one for each factor; denote such a class by $\mathfrak{C}_1(k, l)$.

For $b=(b_1,\cdots, b_n)$, $0<b_i<\infty$, with respect to the rectangular set

$$R_b=\{\xi\in R^n : |\xi_i|\leq b_i,\ i=1,\cdots, n\},$$

let $W_{b,\varepsilon}$, $V_{b,\varepsilon}$ and $U_{b,\varepsilon}$ be open sets such that $N_b(f_q)=N(f_q)\cap\mathrm{Cl}(V_{b,\varepsilon})$ is a C^∞-manifold for each $q\in[-\varepsilon, \varepsilon]$.

$$\mathrm{Cl}(W_{b,\varepsilon})\subset V_{b,\varepsilon},\ \mathrm{Cl}(V_{b,\varepsilon})\subset U_{b,\varepsilon},\ \mathrm{Cl}(U_{b,\varepsilon})\subset R_b .$$

Let χ_s, χ_l be two $C_0^\infty(R^n)$ functions such that $0\leq\chi_s\leq 1$, $\chi_s=1$ on $W_{b,\varepsilon}$, $\operatorname{supp}\chi_s\subset V_{b,\varepsilon}$, and such that $0\leq\chi_l\leq 1$, $\chi_l=1$ on $\mathrm{Cl}(V_{b,\varepsilon})$ and supp $\chi_l \subset U_{b,\varepsilon}$. For each $\Phi \in L_{\mathrm{loc}}(R^n)$, let $\Phi^{\#}(q)$ denote the function in q equal to the integral of $\chi_s \Phi$ along the C^∞-manifold $N_b(f_q)$ if $|q| \leq \varepsilon$ and equal to 0 if $|q| > \varepsilon$ and let $\Phi^{\natural}(\xi)$ be equal to $\chi_l(\xi)\Phi^{\#}(f_q(\xi))$ if $\xi \in U_{b,\varepsilon}$ and equal to 0 otherwise. Under suitable choice of $W_{b,\varepsilon}$, $V_{b,\varepsilon}$ and $U_{b,\varepsilon}$, if follows from the theorems on inverse functions and the Heine-Borel theorem, that $\Phi^{\#} \in C_0^\infty(R^1)$ and $\Phi^{\natural} \in C_0^\infty(R^n)$ if $\Phi \in C_0^\infty(R^n)$, with sufficiently large b (see [2]). We say that $(\Phi^{\#})\Phi^{\natural}$ is the (1-*dimensional*) *symmetrization of* Φ *corresponding to the pair* (f, b). For any distribution $\mu \in \mathcal{D}'$ with supp $\mu \subset \mathrm{Cl}(V_{b,\varepsilon})$, we define the *symmetrization* of μ as follows

$$(\mu^{\natural}, \Phi) = (\mu, \Phi^{\natural}) .$$

The corresponding 1-*dimensional symmetrization* is denoted by $\mu^{\#}$.

For the purpose of application to convolution equations, we shall consider the representations of $\mu^{\natural}$ and of the inverse Fourier transform $u^{\natural}$ of $\mu^{\natural}$ from which we can determine the decay at infinity of $u^{\natural}$ and compare it with that of u, the inverse Fourier transform of μ. All these representations have been considered in [1] or [2] for the restricted case that $\nabla f(\xi) \neq 0$ on $N(f)$. Therefore we outline the results and the corresponding proofs, and in particular point out the related changes.

LEMMA 3.2. *Let* μ *be a distribution with support contained in the closure* $\mathrm{Cl}(N_b(f))$ *of* $N_b(f)$. *Then its symmetrization* $\mu^{\natural}$ *is a linear combination of Dirac-delta measures on* $N_b(f)$,

$$\mu^{\natural} = \sum_{0 \leq h \leq n} C_h (D^h \delta)_b (f) \quad \text{with} \quad C_h = \frac{(-1)^h}{h!} (\mu, \chi_l f^h) ,$$

where Dirac-delta measures are defined by

$$((D^h \delta)_b (f), \Phi) = (D^h \delta, \Phi^{\#}) , \qquad \Phi \in C_0^\infty(R^n) .$$

The proof is the same as for Theorem 1.2 in [1] or [2].

For convenience, denote by $\mathcal{F}\mathfrak{C}_1(k, l)$ the class of the Fourier transforms of all elements in $\mathfrak{C}_1(k, l)$.

With the same arguments as in [2, Theorem 2.3], the representation in the previous theorem yields the following crucial property of the decay at infinity of the inverse Fourier transform of a symmetrized distribution.

LEMMA 3.3. *If the Fourier transform* μ *of a distribution* u *satisfies the condition in Lemma* 3.2, *then the inverse Fourier transform* $u^\natural$ *of the symmetrization* $\mu^\natural$ *of* μ *has the following behavior at infinity*

$$u^\natural(x) = O(|x|^{-m}) \qquad \textit{uniformly in directions } x/|x|;$$

$$u^\natural(x) \neq o(|x|^{-m}) \qquad \textit{for some direction } \omega = x/|x|,$$

with some $m > l/2$, *provided* $u^\natural \neq 0$.

For $f \in \mathfrak{F}\mathfrak{C}_1(k, l)$ and for $\xi_0 \in N(f)$, condition $(\mathfrak{C}(k, l)$ iii) implies that for each small q there is a $\xi \in N(f_q)$ in a neighborhood of ξ_0, $|I(\xi)| = |I(\xi_0)|$. Combining this with the arguments in [1, Lemma 2.2], we have

LEMMA 3.4. *For a function* $f \in \mathfrak{F}\mathfrak{C}_1(k, l)$, *there is a positive number* $\varepsilon > 0$ *such that* $f_q \in \mathfrak{F}\mathfrak{C}_1(k, l)$ *for each* q, $|q| < \varepsilon$.

Let us set $E_b(x, \xi) = \chi_s(\xi)\, e^{ix\cdot\xi}$. Then for each $x \in R^n$, $E_b(x, \cdot) \in C_0^\infty(R^n)$ and $E_b^\#(x, \cdot)$, $E_b^\natural(x, \cdot)$ are defined. With $W_b(x, y)$ as the inverse Fourier transform of $E_b^\natural(x, \cdot)$, due to the change in the definition of the symmetrization and the corresponding modification in the proof of Lemma 2.1 [1], we have

$$W_b(x, y) = \int_{-\varepsilon}^{\varepsilon} E_b^\#(x, q)\, E_b^\#(y, q)\, dq = W_b(y, x).$$

Applying the results of Theorem 1.2 to $E_b^\#(\cdot, q)$ through the representation in Lemma 3.2, the arguments in the proof of Theorem 2.1 [1] with the condition $(\mathfrak{C}(k, l)$ iii), in particular Lemmas 2.3 and 2.4, prove the property at infinity of W_b function as follows:

LEMMA 3.5. *With* $f \in \mathfrak{F}\mathfrak{C}_1(k, l)$, $k > 0$, *for any integer* $p \geq 0$,

$$W_b(x, y) = O(|x|^{-p-l/2}\, |y|^{p-l/2}\, [|x| + |y|]^{-1}),$$

when (x, y) *approaches infinity.*

Using this result with the argument in [1, Theorems 3.1 and 4.1], although we can prove another representation theorem for a wider class of the symmetrized distributions, we state the restricted result which is suitable for our purpose here.

THEOREM 3.6. *For a function* $f \in \mathfrak{F}\mathfrak{C}_1(k, l)$ *with* $k > 0$, *if the support of the Fourier transform* μ *of a distribution* u *is contained* *in* $N_b(f)$, *the inverse Fourier transform* $u^\natural$ *of the symmetrization* $\mu^\natural$ *of* μ *with respect to* $N_b(f)$ *has the representation*

$$u^\natural(x) = \int_{R^n} u(y)\, W_b(x, y)\, dy\,;$$

further, when $x \to \infty$,

$$u^\natural(x) = o(|x|^{n-1-l-m}),$$

provided $u(x) = o(|x|^{-m})$ *for some* $m \geq 0$.

4. **The Liouville type problem in convolution equations—proofs of Theorems 1.1 and 1.2.** To derive the uniqueness condition (1.2) it suffices to consider the case that the solutions to equation (1.1) are entire functions of finite exponential type and $T \in \mathfrak{C}_1(k, l)$. Indeed, we are considering the homogeneous equation

$$T * u = 0\,, \tag{4.1}$$

with u fulfilling the condition (1.2). For a large finite positive vector b, let function $\kappa_b \in C_0^\infty(R^n)$ satisfy the conditions: $0 \leq \kappa_b \leq 1$ and $\operatorname{supp} \kappa_b \subset W_{b,\varepsilon}$ as constructed in §3. With χ_b and u_b as the inverse Fourier transforms of κ_b and of the product of κ_b with the Fourier transform μ of u, respectively, there are relations:

$$T * u_b = 0$$

with u_b satisfying condition (1.2) and

$$u_b = u * \chi_b\,.$$

Denote by F the Fourier transform of T, by f the product of all distinct irreducible factors of F, and by T_1 and S the inverse Fourier transforms of f and $g = F/f$, respectively. Set $v = S * u_b$. Then we have

$$T_1 * v = 0$$

with $T_1 \in \mathfrak{C}_1(k, l)$, since the Fourier transform of v is $\kappa_b g \mu$ with an entire function g of finite exponential type implying $\kappa_b g \in C_0^\infty(R^n)$, where v is an entire function of finite exponential type satisfying the condition (1.2). Repeatedly using the uniqueness condition on the general equation (4.1), we have the desired assertion.

Assume now that $T \in \mathfrak{C}_1(k, l)$ and that u is an entire function of finite exponential type $\leq b$ satisfying condition (1.2).

Let $u^\natural$ be the inverse Fourier transform of the symmetrization $\mu^\natural$ of μ, the Fourier transform of u, corresponding to the pair (f, b). Then the assertions in Lemma 3.3 and Theorem 3.6 with property (1.2) of u imply $u^\natural = 0$, and so in particular $u^\natural(0) = 0$, because $d \geq n-1-l/2$ implies $n-1-l-d \leq -l/2 \leq -m$. As in the proof in [1, Lemma 6.1], we have $u^\natural(0) = u(0)$; and therefore $u(0) = 0$.

Due to the translation property of the convolution of distributions, the translation u_η of u by $\eta \in R^n$ is a solution of (4.1) fulfilling the property (1.2) for each η. Hence the preceding result for u implies that $u_\eta(0) = 0$; i.e. $u(\eta) = 0$. We have $u = 0$. This is the end of the proof of Theorem 1.1.

The proved result reduces the problem to deriving a solution $u \in C_0^\infty(R^n)$ of the equation (1.1) if $\Phi \in C_0^\infty(R^n)$. But the construction of the solution u is already given in [1; Theorem 6.1], if we replace Lemma 5.1 there by its extension in Lemma 4.1 here, with the help of the equivalent condition of invertibility stated at the end of the section from Leon Ehrenpreis [3].

With the same proof as in [7, p. 107], we have

LEMMA 4.1. *Let g be an irreducible entire function in the class $\mathfrak{C}$ and let V be the set of zeros of $g(z)$ in C^n. Let f be an entire function on C^n.*

Assume that the function f/g defined in $C^n - V$ can be extended, as an holomorphic function, to an open set intersecting V; then f/g can be extended to C^n as an entire function.

THEOREM 4.2 (EHRENPREIS). *A necessary and sufficient condition for a distribution T with compact support to be invertible is that, for any entire function g, the inverse Fourier transform of fg belonging to $C_0^\infty(R^n)$ implies that the inverse Fourier transform of g is in $C_0^\infty(R^n)$, where f is the Fourier transform of T.*

5. **Nongeometric condition.** The facts we wish to mention here are a consequence of our main results or of results in [1] and [2]

which we did not mention; but they are important in applications, due to the simple conditions for the equation.

By $\mathfrak{K}$ we mean the class of all distributions $T \in \mathcal{D}'$ with compact support such that each irreducible factor belongs to the class $\mathfrak{C}$ constructed in the first section.

THEOREM 5.1. *For* $T \in \mathfrak{K}$, *the convolution equation*

$$T * u = \Phi, \qquad \Phi \in C_0^\infty(R^n), \tag{5.1}$$

has only the $C_0^\infty(R^n)$ *solution satisfying the condition at infinity*

$$u(x) = o(|x|^{-(n-1)/2}); \tag{5.2}$$

further $u = 0$ *provided* $\Phi = 0$.

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