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ON EXISTENCE OF SIMILARITY SOLUTIONS FOR LAMINAR FLOW IN A CHANNEL WITH ONE POROUS WALL*

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Abstract. We study a two point boundary value problem

$$f''' - R[(f')^2 - ff''] = \beta,\ f(0) = f'(0) = f'(1) = f(1) - 1 = 0,$$

which arises from the study of fluid injection and suction through a porous wall in a vertical channel. Various types of solutions and multiple solutions are found numerically. All possible solutions are classified by studying an equivalent problem. Moreover, we verify that for every real R the problem possesses at least one solution and the solution is unique if $R \geq 0$.

1. Introduction. We study the following two-point boundary-value problem

(1) $$f''' - R[(f')^2 - ff''] = \beta \quad ('= d/d\eta)$$

(2) $$f(0) = f'(0) = f'(1) = f(1) - 1 = 0.$$

The given problem arises from a similarity reduction for the Navier-Stokes system which was applied to describe a fluid injection or suction through one porous wall of a long vertical channel. The fluid is injected or sucked

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with a constant velocity V in the y-direction through the porous wall at $y = d$ (or $\eta = y/d = 1.0$), where d is the thickness of the flow channel, and, strikes another vertical impermeable plate at $y = 0$ (or $\eta = 0.0$). It flows, due to the action of gravity along the z-axis, out through the opening of the plates. Assume that the dimensions of the plates are large so that the edge effects can be ignored. Let (u, v) be the velocity components in the direction (x, y), respectively. Utilizing the equation of continuity, the velocity components may be written in terms of a potential function $f(\eta)$ by $u = Vxf'(\eta)/d$ and $v = -Vf(\eta)$. Then equation (1) is obtained. Here β is an integration constant and $R = Vd/\nu$ is the crossflow Reynolds numbers. Positive (negative) R denotes the fluid injection (suction).

Preliminary studies, Wang and Skalak [1] and C. L. Huang [2], reported only a few data of R's at which (1), (2) possesses a solution. In this paper, we apply the continuation scheme to give a delicate numerical study in Section 2. A family of solutions is obtained and multiple solutions are found at some negative R's. In Section 3, we first classify all possible solutions for the problem (1), (2) and then obtain a connected set in the $R - \beta$ space on which the solutions exist. Our results show that (1), (2) has at least one solution for every real R. Moreover, the uniqueness is also verified if $R > 0$.

2. Numerical results. It is clear that the problem (1), (2) is over-defined since it is a third order system but consisting of four conditions. Then, an additional equation either $R' = 0$ or $\beta' = 0$ is added that the numerical solvability of (1), (2) becomes reasonable. In fact, (1), (2) has a unique solution $f_0(\eta) = 3\eta^2 - 2\eta^3$ when $R = 0$, which yields $\beta = -12$. Therefore, $(R, \beta) = (0, -12)$ is a proper choice of starting point for the continuation scheme.

For numerical computations, the code BVPSOL[3]-[5], with local accuracy EPS=1.0E-8, is chosen and implemented with the adaptive stepsize control scheme on CYBER 170/720 at NCTU. By adding $\beta' = 0$, a family of solutions is found for R varying in $(-14.6169, 291)$. Moreover, replacing $\beta' = 0$ by $R' = 0$, we have successfully obtained a family of solutions for β

varying in an interval $(-765.747, 2310.56)$. The termination of the continuation scheme by varying β is due to the stiffness. Selected data are shown in Table 1 and the corresponding bifurcation diagrams are plotted in Fig. 1(a)-1(c). It is interesting to point out that the curve in Fig. 1(b) exhibits two turning points which indicates the occurrence of multiple solutions. It is also found numerically that (1), (2) possesses two types of solutions with either $f > 0$ or f changes signs once on $(0, 1]$, as shown in Fig. 2(a), 2(c) respectively, when R crosses -13.119. Furthermore, by observing Fig. 1(a), 1(b), we may give the following conjectures:

C-1. *For all real R, there exists at least one β, such that the* problem (1), (2) *has a solution.*

C-2. *The* problem (1), (2) *possesses at least two types of solutions.*

C-3. *There exist constants R_* and R^* with $R_* < R^* < 0$ such that there exist at least three different β's such that the problem* (1), (2) *has a solution for $R \in (R_*, R^*)$. Moreover, β is unique for all $R \notin (R_*, R^*)$.*

Note that the values R_* and R^* are expected to be close to -14.6169 and -14.1, as shown in Table 1, respectively. We shall verify a portion of the conjectures in the next section.

3. Mathematical Result. In fact, the existence of solutions may be verified by applying the Leray-Schauder fixed point theorem at some R. Instead, we classify all possible solutions by studying an equivalent problem to (1), (2). Then, the existence of solutions is a direct consequence of this classification.

3.1. Classification of solutions. Recall that (1), (2) has a unique solution when $R = 0$. We now assume that $R \neq 0$ in the following study. Let $y = \ell\eta$ and $g(y) = Rf(\eta)/\ell$, for given nonzero R, where ℓ is a positive constant which is to be determined. Then the problem (1), (2) is equivalent to

$$g''' + gg'' - g'^2 = R\beta/\ell^4 \tag{3}$$

$$g(0) = g'(\ell) = g'(0) = g(\ell) - R/\ell = 0. \tag{4}$$

Let $g''(0) = B$ and $g'''(0) = C$. Now, by assuming values of B and C, one can study an initial value problem consisting of equation (3) and the conditions

$$g(0) = g'(0) = g''(0) - B = 0. \tag{5}$$

For the simplicity, we denote the unique solution $g(y; B, C)$ of (3), (5) by $g(y)$. Suppose that $g'(y)$ meets the y-axis at some positive a_*. Then, by seting $\ell = a_*$, (1), (2) has a solution when $R = a_* g(a_*)$ and $\beta = a_*^4 C/R$. Meanwhile, the qualitative behavior of f can be obtained directly from g. In fact, (3), (5) has only the trivial solution if $B = C = 0$, hence, $B^2 + C^2 > 0$ is further assumed. Moreover, let $[0, M)$ be the corresponding maximal interval of $g(y)$ for some $0 < M = M(B, C) \le \infty$. Now, a crucial property of $g^{(4)}$ can be obtained as follows.

Property 1.1. *Let $B^2 + C^2 > 0$ and $g(y)$ be a solution of the problem (3), (5). Then $g^{(4)}(y) > 0$ on $(0, M)$.*

Proof. Differentiating (3) twice, we obtain that

$$g^{(4)} = g'g'' - gg''', \tag{6}$$

$$g^{(5)} = (g'')^2 - gg^{(4)}. \tag{7}$$

If $g''(0) = B \neq 0$, then $g^{(4)}(0) = 0$ and $g^{(4)} > 0$ initially. Suppose d_1 is the first positive zero of $g^{(4)}$. Then $g^{(5)}(d_1) \le 0$. From (7), g'' and $g^{(5)}$ vanish at $y = d_1$. By differentiating (7), we get that

$$g^{(6)} = 2g''g''' - g'g^{(4)} - gg^{(5)}, \tag{8}$$

$$g^{(7)} = 2(g''')^2 + g''g^{(4)} - 2g'g^{(5)} - gg^{(6)}. \tag{9}$$

Then, $g^{(6)} = 0$ and $g^{(7)} \ge 0$ at $y = d_1$. Suppose $g'''(d_1) = 0$. Then $g(y) = g(d_1) + g'(d_1)(y - d_1)$ solves (3), (5) uniquely and this contradicts the assumption $B^2+C^2>0$. Therefore, $g'''(d_1)\neq 0$ and $g^{(7)}(d_1)>0$. But it implies that $g^{(5)}$ is nonnegative in a neighborhood of d_1 and this contradicts to the definition of d_1. Hence $g^{(4)}$ is positive for y in $(0,M)$.

For the case $B=0$, we can apply similar arguments and, therefore, omit the proof.

Note that solutions $g(y)$ can only blow up by tending to $+\infty$ if $M<\infty$. Also, from Property 1.1, g''' is increasing and g'' is convex on $(0,M)$. Therefore, the classification can be given by choosing the pair (B,C) from the sets

$$D_1=\{(B,C),B\geq 0,C\geq 0\,\text{and}\,B,C\neq 0\},$$
$$D_2=\{(B,C),B<0,C\geq 0\},$$
$$D_3=\{(B,C),B\leq 0,C<0\}$$

and

$$D_4=\{(B,C),B>0,C<0\}$$

respectively.

Suppose $(B,C)\in D_1$. Then g'', $g'''\geq 0$ at $y=0$. This implies that $g''>0$ on $(0,M)$ and g' possesses no positive zero. Hence, we have the following theorem.

Theorem 1.2. *For $(B,C)\in D_1$, $g'(y)$ has no positive zero.*

Suppose $(B,C)\in D_2$. It is clear that $g''<0$ initially and, then g'' has exactly one positive zero b_1. Also $g'(y)$ is convex on $(0,M)$. Thus g' has a unique positive zero a_1 with $a_1>b_1$ and we have the next theorem.

Theorem 1.3. *For $(B,C)\in D_2$, $g'(y)$ has exactly one positive zero.*

By assuming $\ell=a_1$, we have that $g(\ell)<0$ and it leads to a solution f of (1), (2) with the corresponding $R<0$, $\beta<0$. In fact, f satisfies that $f>0$, $f'''<0$ on $(0,1]$, as obtained on branch (i) in Fig. 1, and we denote it be the type I solution.

Now suppose $(B,C)\in D_3$. Then g, g' and $g''<0$ initially.

Theorem 1.4. *For $(B,C) \in D_3$, $g'(y)$ has exactly one positive zero.*

Proof. Suppose g''' never changes signs. Then g, g' and g'' are all negative $(0,\infty)$. Now, from (7), $g^{(5)} > 0$ for $y > 0$. This shows that g''' is convex and increasing. Then, g''' must cross the y-axis and it is a contradiction. Therefore, g''' has a unique positive zero, say c_1, and it implies that g'' has a unique zero b_2 with $b_2 > c_1$. Also, g' is convex and increasing for $y > b_2$. Then g' has a unique zero a_2 with $a_2 > b_2$.

Note that the corresponding (R,β) satisfies that $R < 0$, $\beta > 0$ since $g(a_2) < 0$. Also, the corresponding solution f of (1), (2) satisfies that $f > 0$ on $(0.1]$ but f''' changes signs once as the ones found on branch (ii) in Fig. 1. Moreover, the property of f''' is different from the ones of type I and, therefore, we designate such f to be the type II solution, as shown in Fig 2(b).

Now suppose $(B,C) \in D_4$. Then $g'' > 0$, $g''' < 0$ at $y = 0$ and $g' > 0$ initially.

Theorem 1.5. *For $(B,C) \in D_4$, $g'(y)$ has either no or exactly two positive zeros.*

Proof. We divide the proof into the following cases.

Case 1. g''' never changes sings. Let b_3 be the first zero of g''. Then g' has a unique zero a_3 with $a_3 > b_3$. Again, g has a unique zero y_0 with $y_0 > a_3$. But from (7), g''' is convex for $y > y_0$. Then g''' must have a zero and this is a contradiction. Hence, $g'' > 0$ for $y > 0$ and g' has no positive zero.

Case 2. g''' changes signs. In fact, g''' can only change signs once, say at $y = c_2$. Then g'' reaches its minimum at $y = c_2$. Also, from (6), $g'g'' > 0$ at $y = c_2$. Suppose $g''(c_2) > 0$. Then g'', $g' > 0$ on $(0,M)$. Suppose $g''(c_2) < 0$. Then $g'(c_2) < 0$ and g'' has exactly two zeros at b_4, b_5 with $b_4 < c_2 < b_5$. Obviously, $g'(b_4) > 0$ and this implies that g' has a unique zero a_4 in (b_4, c_2). Also g' is convex for $y > b_5$. Then g' has the second zero a_5 with $a_5 > c_2$.

However, Theorem 1.5 gives no direct evidence for existence of solutions to (1), (2). In fact, there exist regions in D_4 on which g' has no and two zeros respectively.

Theorem 1.6. *Under the hypotheses of* Theorem 1.5, *there exists a negative number N such that*

(i) *if* $(B,C) \in D_4^+ = \{(B,C), C < NB^{4/3}\}$, *then* $g'(y)$ *has exactly two zeros;*

(ii) *if* $(B,C) \in D_4^- = \{(B,C), C \geq NB^{4/3}\}$, *then* $g'(y)$ *has no zero.*

Proof. We first show that D_4^+ is not empty. From Theorem 1.4, we have that $g'(y;0,-1)$ has a unique zero a_2. Let $\epsilon = |g'(a_2/2;0,-1)|/2$. By the continuity on initial data, there is a sufficiently small $\delta > 0$ such that $|g'(y;0,-1) - g'(y;B,-1)| < \epsilon$ on $[0, a_2/2]$ provided that $0 < B < \delta$. Then $g'(a_2/2;B,-1) < 0$ and $g'(y;B,-1)$ has exactly one zero in $(0, a_2/2)$ since $g'(y;B,-1) > 0$ initially. Then, Theorem 1.5 implies that $g'(y;B,-1)$ possesses two zeros.

To prove that D_4^- is not empty, we apply the fact that $g''(y;1,0) > 1$ and $g'''(y;1,0) > 0$ for $y > 0$. By fixing a $\bar{y} > 0$, one can choose a $\delta > 0$ such that $g''(y;1,0) > 0$ on $(0,\bar{y})$ and $g'''(\bar{y};1,c) > 0$ provided that $-\delta < c < 0$. This implies that $g'' > 0$ at the first zero of g'''. By applying similar arguments as in the proof of Theorem 1.5, $g'(y;1,c)$ has no positive zero.

To obtain the desired result, we must explore some properties of g first. It is clear that $\lambda g(\lambda y; B/\lambda^3, C/\lambda^4)$ also solves (3), (5) with the pair (B,C). This yields the "homogeneity" property of g by

$$g(y;B,C) = \lambda g(\lambda y; B/\lambda^3, C/\lambda^4), \tag{10}$$

for all $\lambda > 0$. Then $g(B^{1/3}y;1,r) = B^{-1/3}g(y;B,C)$ if $r = CB^{-4/3}$. Let

$$S = \{c; g'(y;1,c) \text{ has exactly two zeros}\}.$$

The desired result follows immediately if $S = (-\infty, N)$ for some $N < 0$. In fact, there exists a $\bar{c} \in S$ such that $g'(y;1,\bar{c})$ has two zeros at $\bar{a}, \tilde{a}$ with $\bar{a} < \tilde{a}$ since S is not empty. Let $\epsilon = \min\{|g'(\bar{a};1,\bar{c})|, |g'((\bar{a}+\tilde{a})/2, 1, \bar{c})|,$

$|g'(\tilde{a}+\epsilon;1,\bar{c})|\}/2$ where $\tilde{a} < \tilde{a}+\epsilon < M$. By the continuity on initial data, there is a sufficiently small $\delta > 0$ such that $|g(y;1,c) - g'(y;1,\bar{c})| < \epsilon$ on $[0,\tilde{a}+\epsilon]$ provided that $|c-\bar{c}| < \delta$. This implies that $g'(\bar{a};1,c) > 0$, $g'((\bar{a}+\tilde{a})/2;1,c) < 0$ and $g'(\tilde{a}+\epsilon;1,c) > 0$. Hence $g'(y;1,c)$ has exactly two positive zeros and S is open.

Therefore, there is a $\tilde{c}$ such that $g'(y;c,-1)$ has two zeros provided $0 < c < \tilde{c}$. By homogeneity (10), $g'(y;c,-1) = c^{2/3}g'(c^{1/3}y;1,-c^{-4/3})$ for $c > 0$. Then $-c^{-4/3}$ tends to $-\infty$ as c tends to 0^+. It is clear that S must contain an interval $(-\infty,c^*)$ for some $c^* < 0$. Suppose S is not connected. Then there is a component $(\tilde{c}_1,\tilde{c}_2) \subset S$, $\tilde{c}_1 > -\infty$. In fact, $g(y;1,\tilde{c}_1)$ and $g(y;1,\tilde{c}_2)$ must be defined on $(0,\infty)$. Suppose either $M(1,\tilde{c}_1)$ or $M(1,\tilde{c}_2)$ is finite. Then, g''' will blow up at M. From the arguments in the proof of Theorem 1.5 and the continuous dependence on initial data, g' have no zero in the neighborhood of $\tilde{c}_1$ or $\tilde{c}_2$. This is a contradiction. Hence, $g''(y;1,\tilde{c}_1)$ and $g''(y;1,\tilde{c}_2)$ are positive on $(0,\infty)$. By the continuity on initial data, we get that

$$\lim_{c\to\tilde{c}_1^+} b_3(1,c) = \infty \text{ and } \lim_{c\to\tilde{c}_2^-} b_3(1,c) = \infty \tag{11}$$

where b_3 is the first zero of $g''(y;1,c)$. We shall show that this never occur.

Consider the variational equation of $g''(b_3(1,c);1,c) = 0$, $c \in S$,

$$\varphi''(b_3(1,c);1,c) + g'''(b_3(1,c);1,c)\partial b_3(1,c)/\partial c = 0. \tag{12}$$

where $\varphi^{(i)}(y;1,c) = \partial g^{(i)}(y;1,c)/\partial c$, $i = 0,\dots 4$. Some qualitative properties of φ can be obtained from the next lemma.

Lemma 1.7. *Let a and b be the first zero of g' and g'' respectively. Then $\varphi''' > 0$ on $(0,b)$ and $\varphi'' > 0$ on $(0,a)$.*

Proof of Lemma 1.7. Differentiating (3) with respect to c, we obtain the following variational equation

$$\varphi''' + \varphi g'' + \varphi'' g - 2g'\varphi' = 1 \tag{13}$$

subjects to $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$. It is clear that $\varphi^{(i)} > 0$ initially, $i = 0, \ldots, 3$ since $\varphi'''(0) = 1$. Suppose ξ_3 is the first zero of φ''' and $\xi_3 < b$. By differentiating (13) with respect to y, we get that

$$\varphi^{(4)} + g\varphi''' + \varphi g''' - g''\varphi' - g'\varphi'' = 0. \tag{14}$$

Then $\varphi^{(4)}(\xi_3) > 0$ since $g''' < 0$ and g'', $g' > 0$ at $y = \xi_3$. This contradicts the definition of ξ_3. Hence $\varphi''' > 0$ on $[0, b]$. Suppose ξ_2 is the first zero of φ'' and $b < \xi_3 < \xi_2 < a$. Now from (13), we have $\varphi'''(\xi_2) > 0$. This yields a contradiction again and completes the proof of Lemma 1.7.

Now, from (12), we get that $b_3(1, c)$ is increasing on $(\tilde{c}_1, \tilde{c}_2)$ and this contradicts assertion (11). Hence S is connected and there must exist a negative N with $S = (-\infty, N)$. This completes the proof of Theorem 1.6.

Note that $f(\eta)$, which corresponds to the first zero of g', must satisfy that $R > 0$, $\beta < 0$ and $f > 0$, $f''' < 0$ on $(0, 1]$. We denote such f as the type III solution and it is obtained from branch (iii) in Fig. 1(a). Moveover, the second zero of g' leads to a type IV solution f, as obtained from branch (iv) in Fig. 1(a) and shown in Fig. 2(c), which satisfies $R < 0$ and $\beta > 0$ and f changes signs once in $(0, 1]$. In fact, the classifications yield the first main result:

Theorem A.

(i) *For every* $R \geq 0$, $\beta \geq 0$, *the* problem (1), (2) *possesses no solution.*

(ii) *The problem* (1), (2) *can only possess three different types of solutions which consist of either* (a) $f > 0$, $f''' < 0$, (b) $f > 0$, f''' *changes signs once, or* (c) f *changes signs once in* $(0, 1]$.

3.2. Existence of solutions. In fact, $a(B, C)$, $R(B, C)$ and $\beta(B, C)$ are C^1 functions in (B, C) since g'' never vanishes at positive zeros of $g'(y; B, C)$. Let $\vec{x}(B, C) = (R(B, C), \beta(B, C))$ for $(B, C) \in D_2$, D_3 and $\vec{x}^{\pm}(B, C) = (R^{\pm}(B, C), \beta^{\pm}(B, C))$ where the $(R^{\pm}, \beta^{\pm})$'s are obtained from two positive zeros $a^+ < a^-$ of $g'(y; B, C)$ for $(B, C) \in D_4^+$. Then, (1), (2) may possess type I, II, III or IV solutions by choosing (R, β) from the following sets, respectively,

$$\Gamma_1 = \{\vec{x}(B,C); (B,C) \in D_2\},$$
$$\Gamma_2 = \{\vec{x}(B,C); (B,C) \in D_3\},$$
$$\Gamma_3 = \{\vec{x}^+(B,C); (B,C) \in D_4^+\}$$

and

$$\Gamma_4 = \{\vec{x}^-(B,C); (B,C) \in D_4^+\}.$$

Furthermore, by the homogeneity (10), it is clear that

$$\lambda a(B,C) = a(B/\lambda^3, C/\lambda^4) \tag{15}$$

and, consequently,

$$R(B,C) = R(B/\lambda^3, C/\lambda^4), \tag{16}$$

$$\beta(B,C) = \beta(B/\lambda^3, C/\lambda^4) \tag{17}$$

for $(B,C) \in D_2, D_3$ and D_4^+. This enables us to obtain existence of solutions in the following section.

3.2.1. Type I solutions. By the homogeneities (16), (17), Γ_1 can be written as $\Gamma_1 = \{\vec{x} = (-1,r) : 0 \leq r < \infty\}$ or equivalently $\Gamma_1 = \{\vec{x}(-1,r) : 0 \leq r \leq 1\} \cup \{\vec{x}(\omega,1) : -1 \leq \omega < 0\}$, where $r = C/B^{4/3}$ and $\omega = B/C^{3/4}$ respectively. Then the connectness of Γ_1 is clear. Also, the necessary and sufficient condition for existence of type I solutions can be obtained by the next theorem.

Theorem 2.1.1. *The* problem (1), (2) *has a type* I *solution if and only if* $(R,\beta) \in \Gamma_1$.

Proof. The verification of the sufficient condition is easy since, for each $r \in [0,\infty)$, $f(\eta) = a_*(-1,r)g(a_*(-1,r)\eta; -1,r)/R$ is a solution of (1), (2) with $R < 0$.

Conversely, let f be the desired solution, i.e. $R < 0$, $f(\eta) > 0$ and $f'''(\eta) < 0$. Let $g(\xi) = Rf(\xi)$. Then $g(\xi)$ solves (3), (5) with $B = Rf''(0)$

and $C = R\beta$. Therefore $(B,C) \in D_2$ since $\beta = f'''(0) < 0$ and $g'(1;B,C) = 0$. Hence, $R(B,C)$ and $\beta(B,C)$ are well-defined. Moreover, we have $a_*(B,C) = 1$, and $g(1;B,C) = R$. This yields the desired results.

Furthermore, the limiting behavior of Γ_1 can be obtained by the following two corollaries.

Corollary 2.1.2. (i) $\lim_{r\to 0^+} a(-1,r) = a(-1,0)$, *for some positive* $a(-1,0)$; (ii) $\lim_{\omega\to 0^-} a(\omega,1) = 0$.

Proof. The assertion (i) is trivial. It is clear that $g'(y;0,1) > 0$ for $y > 0$ whenever $g(y)$ is defined. By the continuity on initial data, for any $\mu > 0$, there exists a $\delta > 0$ with $\omega > -\delta$ such that $g'(\mu;\omega,1) > 0$. However, $g'(y;\omega,1) < 0$ initially. Then $a(\omega,1) < \mu$ and the desired result is obtained.

Corollary 2.1.3. (i) $\lim_{r\to 0^+} R(-1,r) = R(-1,0)$, *for some negative* $R(-1,0)$, *and* $\lim_{r\to 0^+} \beta(-1,r) = 0$; (ii) $\lim_{\omega\to 0^-} R(\omega,1) = 0$ *and* $\lim_{\omega\to 0^-} \beta(\omega,1) = -12$.

Proof. The assertion (i) is clear since (R,β) is continuous and the first part of assertion (ii) is a direct consequence of preceding corollary. To complete the proof, it is required to obtain a priori estimate of β.

Integrating (3) on $(0,a)$, we get that

$$Ba + Ca^2/2 + \int_0^a (a-t)F(t)dt = 0 \tag{18}$$

and

$$g(a) = Ba^2/2 + Ca^3/6 + \int_0^a (a-t)^2 F(t)dt/2, \tag{19}$$

where $F(t) = g'(t)^2 - g(t)g''(t)$ and $a = a(B,C)$ is the first positive zero of $g'(y;B,C)$. Multiplying (18) by $a/2$ and subtracted from (19), it yields that

$$g(a) + Ca^3/12 = \int_0^a t(t-a)F(t)dt/2. \tag{20}$$

Since g''' is increasing, $F(t)$ is positive and bounded by $-g(a)g'''(a)$. Therefore, multiplying (20) by $1/g(a)$ and using $\beta = Ca^3/g(a)$, we obtain that

$$0 \le |12+\beta| \le a^3 g''(a). \tag{21}$$

Now the desired limit of β can be obtained as ω tends to 0^- if $g''(a(\omega,1);\omega,1)$ is bounded as ω small in magnitude. In fact, there is a $K > 0$ such that $a(\omega,1) < K$ for $-1 \le \omega < 0$. Then, by the continuity on initial data again, we get that

$$|g''(a(\omega,1);\omega,1)| \le 1 + \sup_{\xi\in[0,K]} |g''(\xi;0,1)|$$

for sufficiently small $|\omega|$. This completes the proof.

Corollary 2.1.3 implies that Γ_1 is a connected subset of the quadrant $R < 0$, $\beta \le 0$ which connects the limit point $(0,-12)$ and the endpoint $(R(-1,0),0))$. Note that, integrating (3), (5) when $B = -1$, $C = 0$ with the subroutine code SDRIV2 [6], the zero $a(-1,0) \approx 2.71$ is obtained and the value $R(-1,0) \approx -6.304$ is consistent with the one in Table 1.

3.2.2. Type II solutions. From (16), (17), write $\Gamma_2 = \{\vec{x}(-1,r) : -1 \le r < 0\} \cup \{\vec{x}(\omega,-1) : -1 \le \omega \le 0\}$. By using similar arguments as in the preceding section, we obtain the following results.

Theorem 2.2.1. *The* problem (1), (2) *has a type* II *solution if and only if* $(R,\beta) \in \Gamma_2$.

Corollary 2.2.2. (i) $\lim_{r\to 0^-} a(-1,r) = a(-1,0)$, *where* $a(-1,0)$ *is defined in* Corollary 2.1.2; (ii) $\lim_{\omega\to 0^-} a(\omega,-1) = a(0,-1)$ *for some positive* $a(0,-1)$.

Corollary 2.2.3. (i) $\lim_{r\to 0^-} R(-1,r) = R(-1,0)$ *and* $\lim_{r\to 0^-} \beta(-1,r) = 0$ *where* $R(-1,0)$ *is defined in* Corollary 2.1.3; (ii) $\lim_{\omega\to 0^-} R(\omega,-1) = R(0,-1)$ *and* $\lim_{\omega\to 0^-} \beta(\omega,-1) = \beta(0,-1)$ *for some* $R(0,-1) < 0$, $\beta(0,-1) > 0$.

Hence, Γ_2 is connected subset of the quadrant $R \le 0$, $\beta > 0$ which connects the endpoint $(R(0,-1),\beta(0,-1))$ and the limit point $(R(-1,0),0)$. In fact, by applying SDRIV2[6], it is found that $R(0,-1) \approx -13.119$ and $\beta(0,-1) \approx 9.389$ as reported in Table 1.

3.2.3. Type III solutions. For convenience, we rewrite $\Gamma_3 = \{\vec{x}^+(\omega, -1) : 0 < \omega < P\}$ where $P = (-1/N)^{3/4}$ and N is defined in Theorem 1.6. We also have the following theorem and corollaries

Theorem 2.3.1. *The problem* (1), (2) *has a type* III *solution if and only if* $(R, \beta) \in \Gamma_3$.

Furthermore, the following corollaries are required for obtaining the asymptotic behaviors of Γ_3.

Corollary 2.3.2. (i) $\lim_{\omega \to 0^+} a^+(\omega, -1) = 0$; (ii) $\lim_{\omega \to P^-} a^+(\omega, -1) = \infty$.

Proof. It is clear that $g'(y; 0, -1) < 0$ on $(0, a(0, -1))$. By the continuity on initial data, we have that for all $\bar{y}$ in $(0, a(0, -1))$, there exists a $\delta > 0$, $g'(\bar{y}; s, -1) < 0$ if $0 < s < \delta$. This implies $a^+ < \bar{y}$ since $g'(y; s, -1) > 0$ initially and then assertion (i) is obtained.

From the proof of Theorem 1.6, $g''(y; P, -1) > 0$ on $(0, \infty)$. Then, given any $\bar{y} > 0$ and let $m = \min\{g''(y; P, -1), y \in [0, \bar{y}]\}$. By the continuity on initial data again, there exists a $\delta > 0$, such that $g''(y; P, -1) > m/2 > 0$ for all $y \in (0, \bar{y})$ provided $0 < P - \omega < \delta$. It implies $g'(y; \omega, -1) > 0$ on $(0, \bar{y})$ and $a^\pm(\omega, -1) > \bar{y}$. This completes the proof.

Now the coreesponding limits of (R, β) can be obtained from the next corollary.

Corollary 2.3.3. (i) $\lim_{\omega \to 0^+} R^+(\omega, -1) = 0$ *and* $\lim_{\omega \to 0^+} \beta^+(\omega, -1) = -12$; (ii) $\lim_{\omega \to P^-} R^+(\omega, -1) = +\infty$ *and* $\lim_{\omega \to P^-} \beta^+(\omega, -1) = -\infty$.

Proof. The first part of assertion (i) is easy since $R^+ = a^+ \cdot g(a^+)$. Also, by applying similar arguments as in Corollary 2.1.3, the second part is clear.

To verify the first part of assertion (ii), it is required to show that $g(y; P, -1)$ tends to ∞ since $g''(y; P - 1) > 0$ for $y > 0$. By the continuity on initial data, there exists $\delta > 0$ and $\bar{y}$ such that $g''(y; c, -1) > 0$ on $(0, \bar{y})$

and $g(\bar{y}; c, -1) > 1$ provided that $|P - c| < \delta$. Then, by Corollary 2.3.2, the desired limit of R^+ is obtained.

To prove the remaining part of (ii), the following fact is required. Let $a^+ < a^-$ be two zeros of g', $b^+ < b^-$ be two zeros of g'' and c be the zero of g''' for $(B, C) \in D_4^+$. Then we get that $b^+ < a^+ < c < b^- < a^-$ and

$$\begin{aligned} g(a^+; \omega, -1) &= \int_0^{a^+} g'(\zeta) d\zeta \leq g'(b^+(\omega, -1); \omega, -1) a^+(\omega, -1) \\ &= a^+(\omega, -1) \int_0^{b^+(\omega,-1)} g''(\zeta) d\zeta \leq \omega (a^+(\omega, -1))^2. \end{aligned}$$

Hence $\beta^+(\omega, -1) \leq -\omega (a^+)^2$ and then $\lim_{\omega \to P^-} \beta^+(\omega, -1) = -\infty$.

Note that Γ_3 is a connected subset of the quadrant $R > 0, \beta < 0$ which connects the limit points $(0, -12)$ and $(+\infty, -\infty)$.

3.2.4. Type IV Solutions. Write $\Gamma_4 = \{\vec{x}^-(\omega, -1) : 0 < \omega < P\}$. By applying preceding arguments, existence of type IV solutions is clear.

Theorem 2.4.1. *The* problem (1), (2) *has a type* IV *solution if and only if* $(R, \beta) \in \Gamma_4$.

Furthermore, the following corollaries yield the asymptotic behavior of the connected set Γ_4.

Corollary 2.4.2. (i) $\lim_{\omega \to 0^+} a^-(\omega, -1) = a(0, -1)$ *for some positive* $a(0, -1)$; (ii) $\lim_{\omega \to P^-} a^-(\omega, -1) = \infty$.

Proof. We omit the verification of assertion (ii) since it is trivial. Recall that $a(0, -1)$ is the zero of $g'(y; 0, -1)$. Let $0 < \epsilon < a(0, -1)/2$ and $y^\pm = a(0, -1) \pm \epsilon$ where $y^+ < M$. Again by the continuity on initial data, there exists a $\delta > 0$ such that

$$\begin{aligned} &|g'(y^\pm; s, -1) - g'(y^\pm; 0, -1)| \\ &\qquad < \min\{|g^+(a(0, -1); 0, -1)|, |g^-(a(0, -1); 0, -1)|\}/2 \end{aligned}$$

for $0 < s < \delta$. Hence $g'(y^+; s, -1) > 0 > g'(y^-; s, -1)$ and $|a^-(s, -1) - a(0, -1)| < \epsilon$ since $a^+(s, -1) < y^-$. Therefore, the assertion (b) follows.

Also the asymptotic behavior of (R,β) can be obtained as follows.

Corollary 2.4.3. *(i)* $\lim_{\omega\to 0^+} R^-(\omega,-1) = R(0,-1)$ *and* $\lim_{\omega\to 0^+}\beta(\omega,-1) = \beta(0,-1)$ *where* $R(0,-1)$ *and* $\beta(0,-1)$ *are defined in* Corollary 2.2.3*;* (ii) $\lim_{\omega\to P^-} R^-(\omega,-1) = -\infty$ *and* $\lim_{\omega\to P^-}\beta^-(\omega,-1) = \infty$.

Proof. By the continuity, assertion (i) is clear. To prove assertion (ii), it is required to estimate $g(a^-)$ properly. Since $g'''(b^-) > 0$ and $g'(b^-) < 0$, (3) yields that $g'(b^-) < -1$ and hence there is a $\eta_* \in (b^-,a^-)$ such that $g'(\eta_*) = -1$ and $g'''(\eta_*) = -g(\eta_*)g''(\eta_*)$. Now from (6), we get that $g^{(4)}(\eta_*) = g''(\eta_*)(g(\eta_*)^2-1) > 0$. Then, $g(\eta_*) < -1$ and, then, $g(a^-) < -1$. Hence the desired limit of R^- is obtained. Again, we have

$$\begin{aligned} g(a(\omega,-1) >&g'(b^-(\omega,-1))a^-(\omega,-1)\\ =&a^-(\omega,-1)\int_0^{b^-(\omega,-1)} g''(\zeta)d\zeta\\ \geq&(a^-(\omega,-1))^2 g''(c(\omega,-1))\\ =&(a^-)^2\left(\int_0^{c(\omega,-1)} g'''(\zeta)d\zeta+\omega\right)\\ \geq&(a^-)^2(\omega-a^-). \end{aligned}$$

Hence $\beta^-(\omega,-1) \geq -a^-(\omega,-1)^2/(\omega-a^-(\omega,-1))$. This completes the proof.

Again, Γ_4 is a connected subset of the quadrant $R < 0$, $\beta > 0$ with the limit points $(R(0,-1),\beta(0,-1))$ and $(-\infty,\infty)$. Also, by SDRIV2[6], the values $R(0,-1) \approx -13.119$, $\beta(0,-1) \approx 9.389$ are consistent with the ones reported in Table 1.

3.2.5. Uniqueness of type III solution. As shown in Fig. 1, branch (iii) indicates the uniqueness of type III solution for each given $R > 0$. This can be verified by the followings. For convenience, we rewrite $\Gamma_3 = \{\vec{x}(1,r) : -\infty < r < N\}$ where $\vec{x}$ denotes $\vec{x}^+$.

Lemma 2.5.1. $\partial R(1,r)/\partial r > 0$ *for* $r \in (-\infty, N)$.

Proof. By differentiating $R(1,r) = a(1,r)g(a(1,r);1,r)$ with respect to r, we get that

$$\partial R(1,r)/\partial r = \partial a(1,r)/\partial r \cdot g(a(1,r);1,r) + a(1,r) \cdot \varphi(a;1,r)$$

since $g'(a(1,r);1,r) = 0$. Similarly, we get that

$$g''(a(1,r);1,r)\partial a(1,r)/\partial r + \varphi'(a(1,r);1,r) = 0.$$

From Lemma 1.7 and Theorem 1.5, we obtain that $g'' < 0$ and $\varphi', g, \varphi > 0$ at $y = a(1,r)$. Hence $\partial R^{+}(1,r)/\partial r > 0$.

Theorem 2.5.2. *For each given $R > 0$, there exists a unique β such that the* problem (1), (2) *has a type* III *solution.*

Proof. Suppose that (1), (2) possesses two solutions f_1 and f_2, with corresponding β_1 and β_2. In fact, $f_i''(0) > 0$ and $\beta_i < 0$ for $i = 1,2$. Let $u_i = f_i''(0)$, $\delta_i = (Ru_i)^{1/3}$ and $g_i(\zeta) = (R/\delta_i)f(\zeta/\delta_i)$. Then g_i solves (3), (4) with $B_i = 1$, $C_i = (\beta_i/u_i) \cdot (Ru_i)^{-1/3}$ and the corresponding $y_*(1,C_i) = \delta_i$. Hence $C_1 = C_2$ since $\partial R(1,r)/\partial r > 0$. Consequently, $\delta_1 = \delta_2$ and $u_1 = u_2$. Thus, $\beta_1 = \beta_2$ and $f_1 = f_2$.

Note that the uniqueness of β implies that Γ_3 is the graph of a continuous function $\beta(R)$ for $R > 0$ such that (1), (2) possesses a type III solution at $(R, \beta(R))$.

3.6. Concluding Remark. Let $\Gamma = \bigcup_{i=1}^{4} \Gamma_i \cup \{(0,-12)\}$. Then Γ connects two limit points $(+\infty,-\infty)$, $(-\infty,+\infty)$. Hence we have the second part of main result.

Theorem B. *For every real R, there exists at least one β such that the* problem (1), (2) *has a solution of either type* I, II, III or IV. *Moreover, existence of β is unique when $R \geq 0$.*

Moreover, it is also valid that there is a R such that (1), (2) has a solution for every real β. Unfortunately, the mathematical evidence for the occurrence of multiple solutions is yet clear and it requires further delicate study. However, Hwang and Wang [7] verified the existence of multiple solutions for a similar problem, Berman's problem, arising from the study of fluid in a channel with two-sided porous wall.

References

1. C. L. Huang, *Application of quasilinearization technique to the vertical channel flow and heat convection*, Int. J. Non-Linear Mechanics, **13**, 55-60.

2. C. Y. Wang and R. Skalak, *Fluid injection through one side long vertical channel*, AIChE J. **20** (1974), 603.

3. P. deuflhard and G. Bader, *Multiple shooting technique revised*, Univ. Heidelberg, SFB 123, **163** (1982).

4. G. Bader and P. Deuflhard, *A semi-implicit midpoint rule for stiff systems of ordinary differential equations*, Univ. Heidelberg, SFB 123, Tech. Rep. 114, (1981).

5. P. Deuflhard, *Order and stepsize control in extrapolation methods*, Univ. Heidelberg, SFB 123, Tech. Rep. 93, (1980).

6. D. Kahaner, C. Moler and S. Nash, *Numerical Methods and Software*, Prentice Hall Inc., New York (1989).

7. T. W. Hwang and C. A. Wang, *On multiple solutions of Berman's problem*, Royal Soc. Edinburgh: Sec. A, to appear.

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Q	β	
291.2306	−765.747	
59.3593	−170.092	
32.6156	−99.3122	
11.0935	−41.8079	
5.6978	−26.4432	
3.0815	−19.5652	
2.0273	−16.8917	
0.0	−12.0	
−1.1	−9.5235	
−3.1	−5.4174	
−6.3038	0.0	
−10.1	4.9738	
−13.1	9.3481	
−14.5	15.9326	
−14.6169*	20.8329	Turning point
−14.3279	50.0	
−14.1*	73.0091	Turning point
−14.1015	73.8808	
−14.4	122.827	
−20.8479	441.013	
−30.3588	788.318	
−40.2645	1136.41	
−50.6316	152.08	
−60.9619	2310.56	

Table 1. Selected data of (Q, β) for the problem (1), (2)

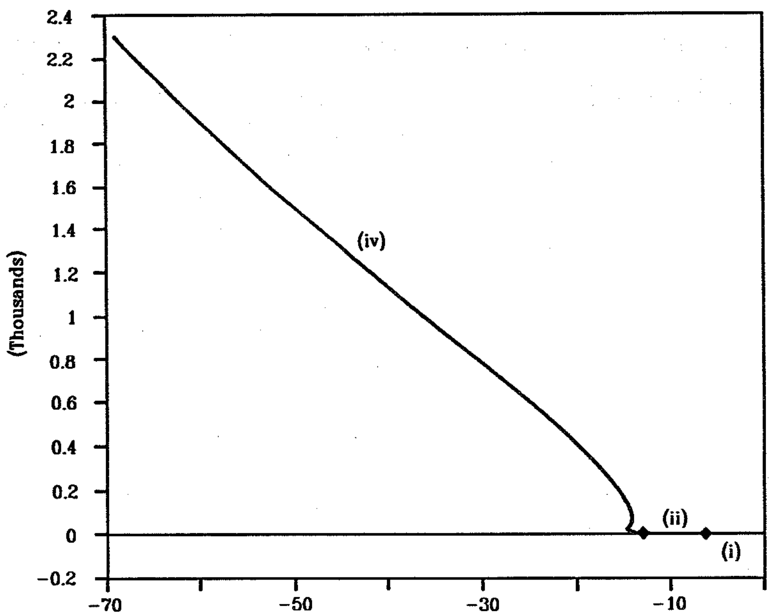


Figure 1(a). Bifurcation diagram of the problem (1), (2): $R \leq 0$.

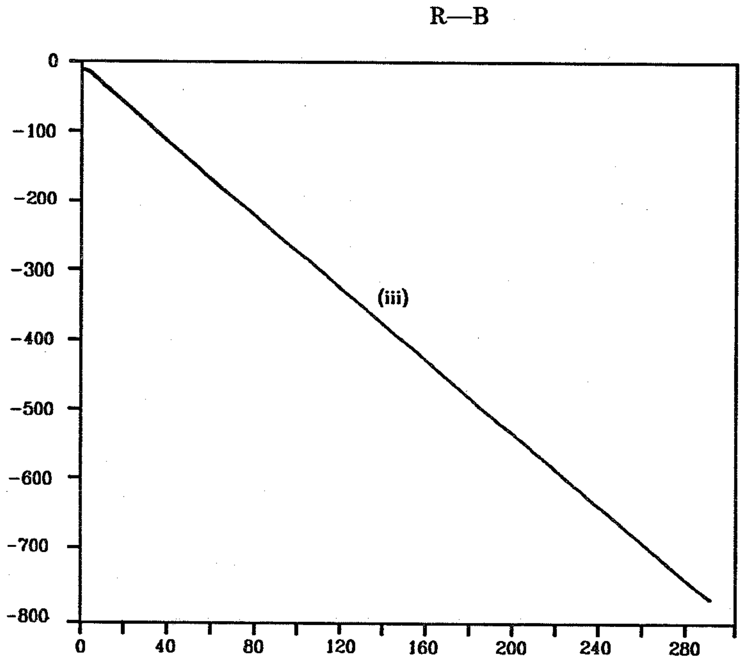


Figure 1(b). Bifurcation diagram of the problem (1), (2): $R \geq 0$.

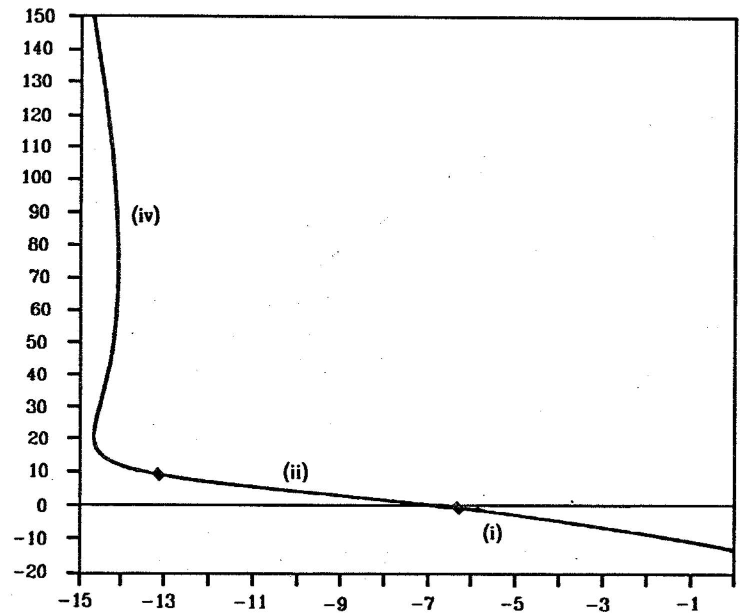


Figure 1(c). Detail diagram which exhibits multiple solutions of the problem (1), (2).

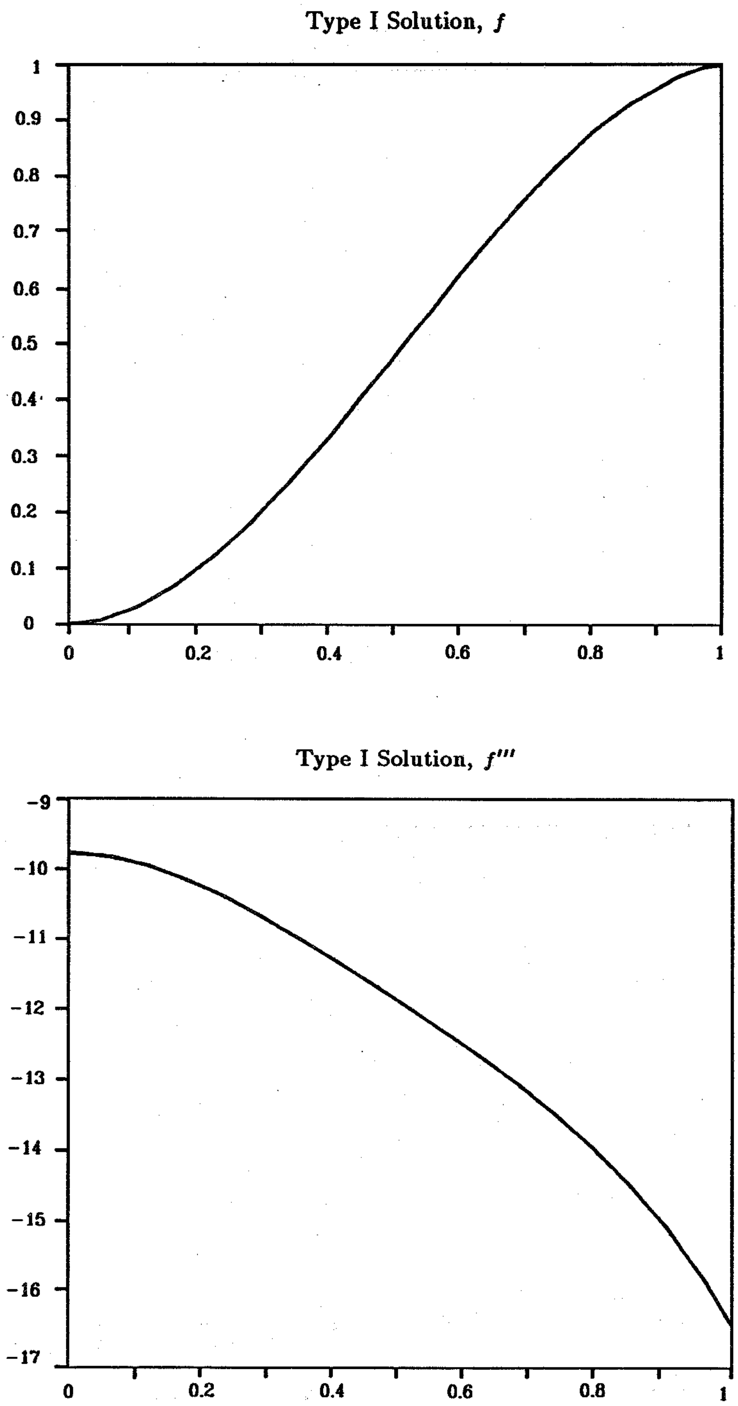
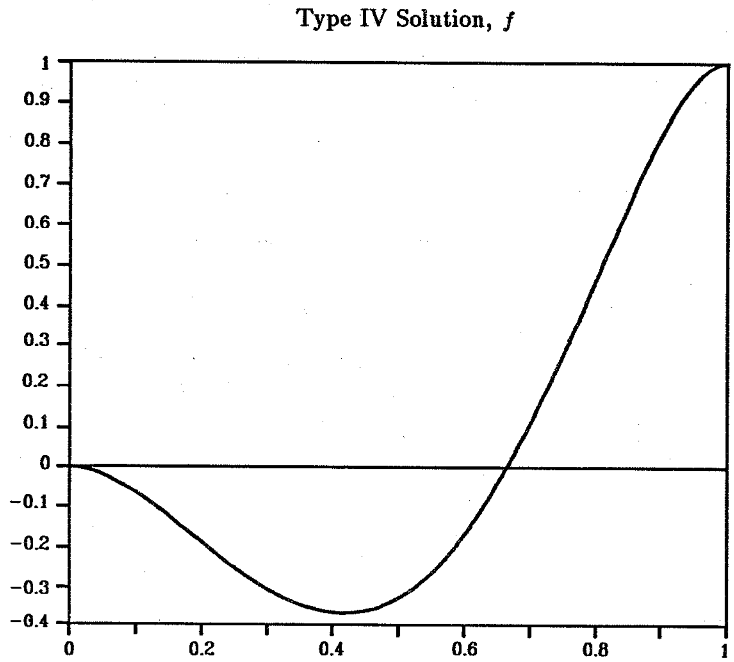


Figure 2(a). The graph of solution f with $f > 0$ and $f''' < 0$ on $(0, 1]$.



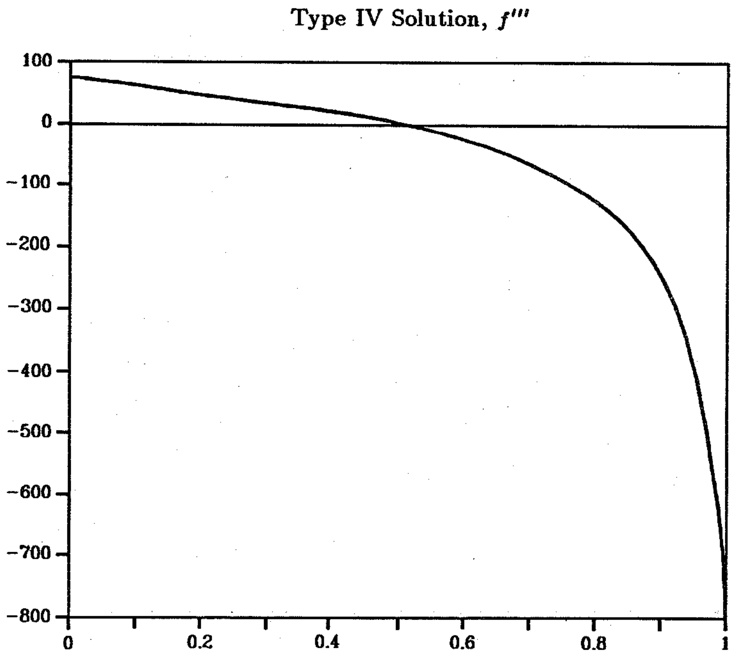
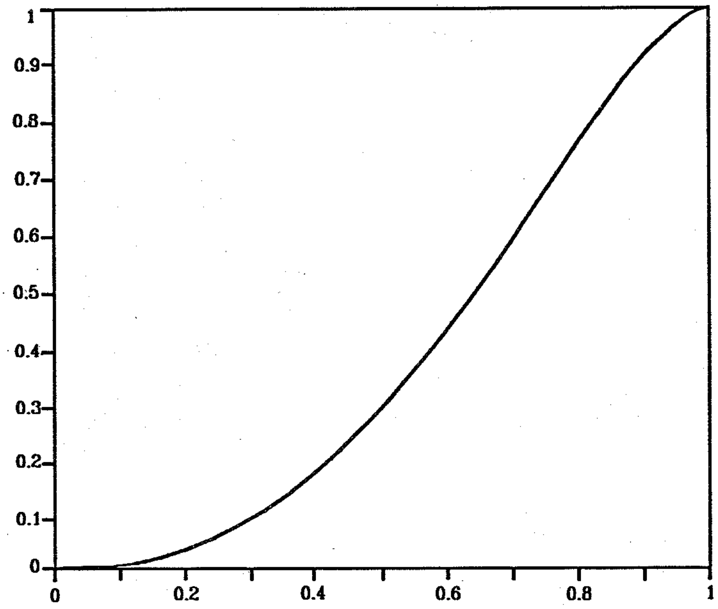


Figure 2(b). The graph of solution f with $f > 0$ and f''' changes signs on $(0,1]$.

Type II Solution, f



Type II Solution, f'''

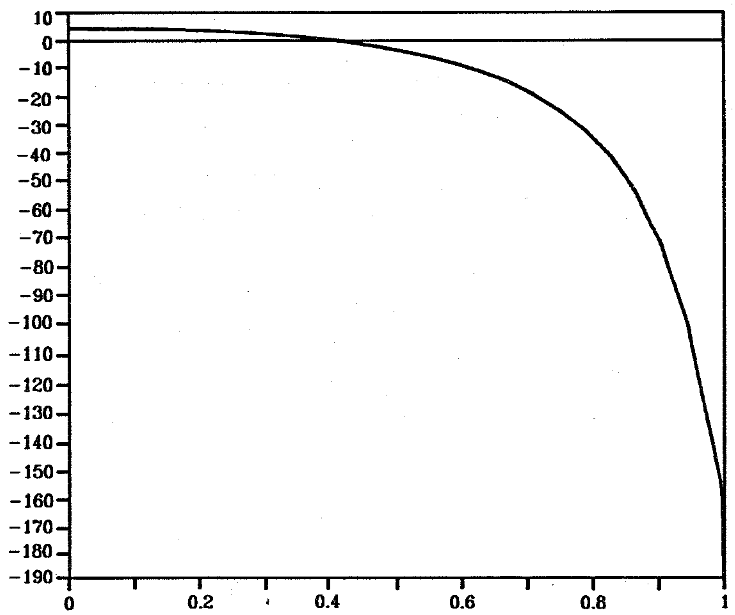


Figure 2(c). The graph of solution f with f changes signs once on $(0,1]$.