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REPRESENTATION OF MEASURABLE FUNCTIONS BY MULTIPLE HAAR SERIES

BY

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Abstract. It is proved that for any finite a.e. measurable function f on the p-dimensional interval $[0,1]^p$ can be represented by an p-fold Haar series that is convergent to f a.e. summed by rectangles. This is an analogue for multiple Haar series corresponding to the representation of measurable functions by multiple trigonometric series.

1. Introduction. In answering a question, posed by Lusin, in connection with the representation of measurable functions, Men′šov and Bary [2] proved that for any finite almost everywhere (abbreviated a.e.) measurable function f on $[0,2\pi]$, there exists a trigonometric series convergent to f a.e.. The analogous result for Haar system proved by Bary [12] is as follows:

Theorem A. *If $f(x)$ is measurable and finite* a.e. *on $[0,1]$, then there is a series in terms of Haar functions which converges to $f(x)$* a.e. *on $[0,1]$.*

The representation of measurable functions of two variables by double trigonometric series was first studied by Dzhavarsheishvili [9]. Subsequently various representation problems for function of several variables by multiple series were discussed by Dzagnidze [7; 8] and Topuriya [15]. But their results are far from pointwise convergent representation. In connection with the Men′šov and Bary result for functions of several variables, Chen and Hwang [4; 5] proved the following theorem:

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Theorem B. *For any finite* a.e. *measurable function f on the p-torus $[0, 2\pi]^p$ there exists an p-fold trigonometric series which converges to f* a.e. *summed either by squares or by rectangles.*

In view of the above theorems, it is natural to ask whether every finite a.e. measurable function f on I^p can be represented by an p-fold Haar series convergent to f a.e., summed either by squares or by rectangles. In the present article we show that any such a function can be represented by an p-fold Haar series convergent to the given function a.e. summed by rectangles.

The result is much deeper in the case of functions of several variables because of the existence of integrable functions on $[0, 1]^p$ such that the rectangular partial sums of their p-tuple Haar-Fourier series divergent everywhere [6]. Although the basic procedures for proving our theorems are similar to those given by Bary [12], they need some substantial modifications. For convenience and for notational simplicity, we give the proof of our theorem explicity for $p = 2$.

2. Preliminaries and notations. Haar system can be defined as follows [1; 12]:

$$\chi_0^{(0)}(x) = +1 \quad \text{for} \quad x \in [0, 1],$$

$$\chi_0^{(1)}(x) = \begin{cases} +1 & \text{for} \quad x \in [0, \frac{1}{2}) \\ -1 & \text{for} \quad x \in (\frac{1}{2}, 1], \\ 0 & \text{for} \quad x = \frac{1}{2}, \end{cases}$$

and generally, for any natural number n, we defined 2^n functions as follows:

$$\chi_n^{(k)}(x) = \begin{cases} +\sqrt{2^n} & \text{for} \quad x \in \left[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right), \\ -\sqrt{2^n} & \text{for} \quad x \in \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right], \\ 0 & \text{otherwise in} \quad [0, 1]. \end{cases}$$

Here $n = 1, 2, 3, \ldots$ and $k = 1, 2, \ldots, 2^n$. The system $\{\chi_n^{(k)}(x)\}$ is a complete orthonormal system on $L^2([0, 1])$. We denote the Haar system arranged lexicographically by $\{\chi_m(x)\}$, i.e.,

$$\chi_1(x) = \chi_0^{(0)}(x), \;\; \chi_m(x) = \chi_n^{(k)}(x), \text{ if } m \geq 2 \text{ and } m = 2^n + k,\ 1 \leq k \leq 2^n.$$

Therefore, the double Haar system $\{\chi_m(x)\chi_\ell(y)\}$ is a complete orthonormal system on $L^2([0,1]^2)$.

For $f(x,y) \in L([0,1]^2)$, denote the double Haar-Fourier series for f by

$$S[(x,y);f] = \sum_{m,\ell=1}^{\infty} \hat{f}(m,\ell)\chi_m(x)\chi_\ell(y).$$

Here $$\hat{f}(m,\ell) = \iint_{[0,1]^2} f(u,v)\chi_m(u)\chi_\ell(v)dudv.$$

By a rectangular partial sum for $S[(x,y);f]$ we mean a partial sum $S_{m\ell}[(x,y);f]$ of $S[(x,y);f]$ according to:

$$S_{m\ell}[(x,y);f] = \sum_{i=1}^{m}\sum_{j=1}^{\ell} \hat{f}(i,j)\chi_i(x)\chi_j(y).$$

Similar to the case of Haar-Fourier series for functions in $L([0,1])$, we have the following identity [10]:

$$S_{m\ell}[(x,y);f] = \frac{\iint_{\Delta_{m\ell}} f(u,v)dudv}{|\Delta_{m\ell}|} \tag{2.1}$$

where $m = 2^n + k$, $\ell = 2^q + r$, $1 \leq k \leq 2^n$, $1 \leq r \leq 2^q$; $(x,y) \in \Delta_{m\ell}$ and $\Delta_{m\ell}$ is one of the following open intervals:

$$\begin{aligned}
&\alpha_{+}{}^{(k,r)}_{(n,q)} = \delta_{+}{}^{(k)}_{n} \times \delta_{+}{}^{(r)}_{q}, && \alpha_{-}{}^{(k,r)}_{(n,q)} = \delta_{+}{}^{(k)}_{n} \times \delta_{-}{}^{(r)}_{q},\\
&\beta_{+}{}^{(k,r)}_{(n,q)} = \delta_{-}{}^{(k)}_{n} \times \delta_{-}{}^{(r)}_{q}, && \beta_{-}{}^{(k,r)}_{(n,q)} = \delta_{-}{}^{(k)}_{n} \times \delta_{+}{}^{(r)}_{q},\\
&\gamma_1{}^{(k,r)}_{(n,q)} = \delta_n^{(k)} \times \delta_{+}{}^{(r)}_{q}, && \gamma_2{}^{(k,r)}_{(n,q)} = \delta_n^{(k)} \times \delta_{-}{}^{(r)}_{q},\\
&\delta_1{}^{(k,r)}_{(n,q)} = \delta_{+}{}^{(k)}_{n} \times \delta_q^{(r)}, && \delta_2{}^{(k,r)}_{(n,q)} = \delta_{-}{}^{(k)}_{n} \times \delta_q^{(r)},\\
&J^{(k,r)}_{(n,q)} = \delta_n^{(k)} \times \delta_q^{(r)};
\end{aligned} \tag{2.2}$$

where

$$\delta_{+}{}^{(t)}_{s} = \left(\frac{2t-2}{2^{s+1}}, \frac{2t-1}{2^{s+1}}\right), \;\; \delta_{-}{}^{(t)}_{s} = \left(\frac{2t-1}{2^{s+1}}, \frac{2t}{2^{s+1}}\right), \;\; \delta_s^{(t)} = \left(\frac{2t-2}{2^{s+1}}, \frac{2t}{2^{s+1}}\right).$$

We also need the following generalization of a theorem of Lusin:

Theorem C (S. Saks [13; 14 (p. 218)]). *If f is a measurable* a.e. *finite function in* $[0,1]^p$, *there exists an additive continuous function of intervals* $F(I)$ *such that the strong derivatives* $F'_s(x) = f(x)$ a.e. $x \in [0,1]^p$.

3. Main theorem. Now we can prove our representation on $[0,1]^2$.

Theorem 1. *For any finite* a.e. *measurable function f on* $[0,1]^2$, *there is a double Haar series of functions* $\sum_{m\ell} a_{m,\ell}\chi_m(x)\chi_\ell(y)$ *such that*

$$\lim_{m,\ell\to\infty}\Big(\sum_{i=1}^{m}\sum_{j=1}^{\ell} a_{ij}\chi_i(x)\chi_j(y)\Big) = f(x,y) \quad \textit{for a.e.} \quad (x,y)\in[0,1]^2.$$

Proof. By Sak's theorem (Theorem C), there exists an additive continuous function of intervals $F(I)$ on $[0,1]^2$ such that the strong derivatives $F'_s(x,y) = f(x,y)$ a.e. $(x,y)\in[0,1]^2$.

Let $\alpha_{+(n,q)}^{(k,r)}$, $\alpha_{-(n,q)}^{(k,r)}$, $\beta_{+(n,q)}^{(k,r)}$, $\beta_{-(n,q)}^{(k,r)}$, be the open intervals defined in (2.2), and $\tilde{\alpha}_{+(n,q)}^{(k,r)}$, $\tilde{\alpha}_{-(n,q)}^{(k,r)}$, $\tilde{\beta}_{+(n,q)}^{(k,r)}$, $\tilde{\beta}_{-(n,q)}^{(k,r)}$, be the closure of these intervals. For $m,\ell \ge 2$, $m = 2^n + k$, $\ell = 2^q + r$, $1 \le k \le 2^n$, $1 \le r \le 2^q$, set

$$\begin{aligned}
a_{m\ell} &= \sqrt{2^{n+q}}\Big[F\Big(\tilde{\alpha}_{+(n,q)}^{(k,r)}\Big) + F\Big(\tilde{\beta}_{+(n,q)}^{(k,r)}\Big) - F\Big(\tilde{\alpha}_{-(n,q)}^{(k,r)}\Big) - F\Big(\tilde{\beta}_{-(n,q)}^{(k,r)}\Big)\Big],\\
a_{11} &= F([0,1]^2),\\
a_{m1} &= \sqrt{2^n}\Big[F\Big(\Big[\frac{2k-2}{2^{n+1}},\frac{2k-1}{2^{n+1}}\Big]\times[0,1]\Big) - F\Big(\Big[\frac{2k-1}{2^{n+1}},\frac{2k}{2^{n+1}}\Big]\times[0,1]\Big)\Big],\\
a_{1\ell} &= \sqrt{2^q}\Big[F\Big([0,1]\times\Big[\frac{2r-2}{2^{q+1}},\frac{2r-1}{2^{q+1}}\Big]\Big) - F\Big([0,1]\times\Big[\frac{2r-1}{2^{q+1}},\frac{2r}{2^{q+1}}\Big]\Big)\Big].
\end{aligned}$$

In this way, we obtain the double Haar series:

$$\sum_{m,\ell} a_{m\ell}\chi_m(x)\chi_\ell(y). \tag{3.1}$$

We shall prove that the series (3.1) converges to $f(x,y)$ a.e. $(x,y)\in[0,1]^2$, summed by rectangles, i.e., if

$$T_{m\ell}(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{\ell} a_{ij}\chi_i(x)\chi_j(y),$$

then

$$\lim_{m,\ell\to\infty} T_{m\ell}(x,y) = f(x,y) \text{ for a.e. } (x,y)\in[0,1]^2.$$

Now, write $m = 2^k + k'$, $\ell = 2^q + r'$, where $m,\ell \geq 2$ and $1 \leq k' \leq 2^n$, $1 \leq r' \leq 2^q$. We define a simple function $f^*(x,y)$ as follows:

$$f^*(x,y) = 2^{n+q+2} F\left(\tilde{\Delta}^{(k,r)}_{(n,q)}\right) \text{ for } (x,y)\in\Delta^{(k,r)}_{(n,q)}$$

and $$\Delta^{(k,r)}_{(n,q)} = \alpha_{+(n,q)}^{(k,r)},\quad \alpha_{-(n,q)}^{(k,r)},\quad \beta_{+(n,q)}^{(k,r)},\quad \beta_{-(n,q)}^{(k,r)}.$$

Since $\left|\tilde{\Delta}^{(k,r)}_{(n,q)}\right| = 2^{n+q+2}$, so

$$f^*(x,y) = \frac{F\left(\tilde{\Delta}^{(k,r)}_{(n,q)}\right)}{\left|\tilde{\Delta}^{(k,r)}_{(n,q)}\right|} \text{ if } (x,y)\in\Delta^{(k,r)}_{(n,q)}.$$

By computing the double Haar-Fourier coefficients $\hat{f}^*(i,j)$ as given in [12; p. 110–111], we have $\hat{f}^*(i,j) = a_{ij}$ for $1\leq i\leq m$, $1\leq j\leq \ell$. Therefore we obtain

$$T_{m\ell}(x,y) = S_{m\ell}[(x,y);f^*],$$

where $S_{m\ell}[(x,y);f^*] = \sum_{i=1}^{m}\sum_{j=1}^{n} \hat{f}^*(i,j)\chi_i(x)\chi_j(y)$, the rectangular partial sum for the double Haar-Fourier series of f^*.

As noted in (2.1), we have

$$S_{m\ell}[(x,y);f^*] = \frac{\iint_{\Delta_{m\ell}} f^*(u,v)dudv}{|\Delta_{m\ell}|} \text{ for } (x,y)\in\Delta_{m\ell},$$

where $\Delta_{m\ell}$ is an interval defined in (2.2). Since f^* is constant in each $\Delta^{(k,r)}_{(n,q)}$ and $F(I)$ is an additive continuous function, we have

$$S_{m\ell}[(x,y);f^*] = \frac{F(\tilde{\Delta}_{m\ell})}{|\tilde{\Delta}_{m\ell}|} \text{ for } (x,y)\in\Delta_{m\ell}.$$

Let (x,y) be a point of $[0,1]^2$ which is not in the boundary of any of the rectangles $\Delta^{(k,r)}_{(n,q)}$, n, $q = 0,1,2,\ldots$, $1\leq k\leq 2^n$, $1\leq r\leq 2^q$, and for which $F_s'(x,y) = f(x,y)$. Then (x,y) belongs to a double sequence $\{\Delta_{m\ell}\}$ of intervals for which

$$T_{m\ell}(x,y) = \frac{F(\tilde{\Delta}_{m\ell})}{|\tilde{\Delta}_{m\ell}|},$$

where $m, \ell \geq 2$, $m = 2^n + k'$, $\ell = 2^q + r'$; $1 \leq k' \leq 2^n$; $1 \leq r' \leq 2^q$; $\Delta_{m\ell}$ is one of the intervals defined in (2.2). Therefore

$$\lim_{m,\ell\to\infty} \frac{F(\tilde{\Delta}_{m\ell})}{|\tilde{\Delta}_{m\ell}|} = F'_s(x,y) = f(x,y).$$

So we obtain

$$\lim_{m,\ell\to\infty} T_{m\ell}(x,y) = \lim_{m,\ell\to\infty} \frac{F(\tilde{\Delta}_{m\ell})}{|\tilde{\Delta}_{m\ell}|} = f(x,y).$$

Hence the double Haar series (3.1) converges to $f(x,y)$ a.e., and the proof of Theorem 1 is finished.

4. Some remarks. If the function f is integrable, then the additive continuous function $F(I)$ of intervals in Theorem C can be chosen such that $F(I)$ is of bounded variation in $[0,1]^p$ ([13]). In this case, we can write $a_{m\ell}$ as the double Haar Lebesgue-Stieltjes Fourier coefficients ([14], p. 64):

$$a_{m\ell} = \iint_{[0,1]^2} \chi_m(u)\chi_\ell(v)dF.$$

Therefore

$$T_{m\ell}(x,y) = \iint_{[0,1]^2} K_m(x,u)K_\ell(y,v)dF,$$

where $K_m(x,u) = \sum_{i=1}^m \chi_i(x)\chi_i(u)$, $K_\ell(y,v) = \sum_{i=1}^\ell \chi_i(y)\chi_i(v)$. Write $m = 2^n + k'$, $\ell = 2^q + r'$, $1 \leq k' \leq 2^n$, $1 \leq r' \leq 2^q$, then as in [1] or [10] for the values of $K_m(x,u)$ and $K_\ell(y,v)$, we have

$$T_{m\ell}(x,y) = \frac{F(\tilde{\Delta}_{m\ell})}{|\tilde{\Delta}_{m\ell}|}.$$

Therefore it is easy to obtain $\lim_{m,\ell\to\infty} T_{m\ell}(x,y) = f(x,y)$ for a.e. $(x,y) \in [0,1]^2$.

The basic procedures and the fundamental tools still work for higher dimensions, the only difficulty being that the notations becomes slightly complicated, so we have the following theorem.

Theorem 2. *For any finite* a.e. *measurable function* f *on* $[0,1]^p$, *there is an p-fold Haar series such that its rectangular partial sum converges to* f a.e. *in* $[0,1]^p$.

Finally we note that from our previous paper [5], for the case $f(x) = \pm\infty$ on a set of positive measure in $[0,1]$, Theorem A can not hold. In view of the relationship (2.1) and the theory of strong differentiation [14, p. 147], there exists $g \in L^+([0,1]^2)$ such that $\overline{\lim}_{m.\ell\to\infty} S_{m\ell}[(x,y);g] = \infty$ a.e. $(x,y) \in [0,1]^2$. Hence it is interesting to see whether our representation theorem holds for functions assigned infinity on a set of positive measure in $[0,1]^p$.

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