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FORWARD SELF-SIMILAR SOLUTIONS OF A SEMILINEAR HEAT EQUATION

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Abstract. We are interested in finding the forward self-similar solutions of the equation $u_t = \Delta u + u^{-\beta}$ in $\mathbf{R}^n \times (0,\infty)$, where $\beta > 0$. We show that there are solutions of this equation with initial conditions of the form $c|x|^{2\gamma}$ for some positive constant c.

1. Introduction. In this paper, we consider the Cauchy problem for the equation:

$$u_t = \Delta u + u^{-\beta} \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{1.1}$$

with the initial condition:

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}^n, \tag{1.2}$$

where $\beta > 0, n \geq 1$, and $u_0 \geq 0$.

In [1] Brezis and Friedman considered the initial–boundary value problem for the equation:

$$u_t = \Delta u - |u|^{p-1}u \quad \text{in} \quad \Omega \times (0,T), \quad \Omega \subset \mathbf{R}^n,$$

with the Dirac mass at 0 as initial condition. They proved, among other things, that a solution of this initial–boundary value problem exists if and only if $0 < p < (n+2)/n$. They also considered a similar problem for the porous medium equation. For related results, we refer the readers to the references cited therein (see also [2, 5, 13]).

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A natural question arises for (1.1), namely whether there is a solution of (1.1) with singular initial condition. Note that if u_0 assumes the zero value at some point(s) then it is singular. We shall use the similarity solutions approach to obtain such solutions.

We are interested in finding the forward self-similar solutions of the form

$$u(x,t) = t^{\gamma} w(|x|/\sqrt{t}), \quad \gamma = 1/(\beta+1). \tag{1.3}$$

Let $r = |x|/\sqrt{t}$. Then $w \in C^2([0,\infty))$ and satisfies

$$w'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)w' - f(w) = 0, \quad r > 0, \tag{1.4a}$$

$$w'(0) = 0, \quad w > 0, \tag{1.4b}$$

where $f(w) = \gamma w - w^{-\beta}$. We shall study the solutions of (1.4) in Section 2. Let $\kappa = \gamma^{-\gamma}$. We show that solutions of (1.4) exist if and only if $w(0) \geq \kappa$. In particular, we found that there are solutions of (1.1) with initial conditions $u_0(x) = c|x|^{2\gamma}$ for some positive constants c.

We remark that a related problem to (1.1) for the equation

$$u_t = \Delta u - u^{-\beta}, \tag{1.5}$$

which is the so called quenching problem, has been studied extensively by many authors (see, for example, the survey paper of Levine [12]). Using the backward self-similar variable $u(x,t) = (-t)^{\gamma} w(|x|/\sqrt{-t})$, we see that $w = w(y)$ satisfies the equation

$$\Delta w - \frac{1}{2} y \cdot \nabla w + f(w) = 0 \quad \text{in} \quad \mathbf{R}^n. \tag{1.6}$$

It has been shown that every nonconstant positive radial solution of (1.6) must be increasing ultimately to infinity. Also, the asymptotic behavior of these solutions at $|y| = \infty$ has been obtained (cf. [8, 9]). Using this information we are able to derive the quenching rate estimate for solutions of (1.5) by using the method of Giga and Kohn [6, 7]. For details we refer the interested readers to references [3, 4, 8, 9, 10].

Finally, we deal with a similar equation to (1.4a) as follows:

$$w'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)w' + f(w) = 0, \quad r > 0. \tag{1.7}$$

We show in Section 3 that every solution w of (1.7) is monotone ultimately. Moreover, $w \to (\beta+1)^{1/(\beta+1)}$ and $w' \to 0$ as $r \to \infty$.

2. Forward self-similar solutions. In this section we shall study the forward self-similar solution of (1.1) in the form (1.3). We note that (1.4a) can be rewritten as the following system:

$$w' = v, \tag{2.1a}$$

$$v' = -\left(\frac{n-1}{r} + \frac{r}{2}\right)v + f(w), \tag{2.1b}$$

where $f(w) = \gamma w - w^{-\beta}$ with $\gamma = 1/(\beta+1)$. To solve (1.4) is equivalent to solve the integral system:

$$w(r) = \eta + \int_0^r v(s)ds, \tag{2.2a}$$

$$v(r) = \int_0^r \left(\frac{s}{r}\right)^{n-1} \exp\left(\frac{s^2 - r^2}{4}\right) f(w(s))ds, \tag{2.2b}$$

if $w(0) = \eta > 0$.

Let $\kappa = \gamma^{-\gamma}$. Since $f(w) > 0$ for $w > \kappa$ and $f(w) < 0$ for $w < \kappa$, it follows from (2.2b) that $w'(r) > 0$, $\forall r > 0$ if $\eta > \kappa$; and $w'(r) < 0$, $\forall r > 0$ if $\eta < \kappa$. Using (2.2), the local existence of the solution of (1.4) with $w(0) = \eta > 0$ follows from the standard fixed point argument. The uniqueness is clearly. We denote the solution of (1.4) with $w(0) = \eta > 0$ by $w(r;\eta)$. Let (w, v) exist in the interval $[0, \epsilon]$ for some $\epsilon > 0$. Clearly, $w \in C^1([0,\epsilon]) \cap C^2((0,\epsilon])$. Since $\lim_{r\to 0} v(r)/r = \lim_{r\to 0} v'(r) = f(\eta)/n$, we conclude that $w \in C^2([0,\epsilon])$.

We shall denote $\sigma(r) = r^{n-1}\exp(r^2/4)$.

For $\eta \in (0,\kappa)$ we have:

Proposition 2.1. *The solution $w(r;\eta)$ of (1.4) with $\eta \in (0,\kappa)$ cannot extended to all $r > 0$.*

Proof. Recall that $w' < 0$. Suppose that for some $\eta \in (0, \kappa)$ the solution $w(r;\eta)$ of (1.4) exists for all $r > 0$. Then we have

$$(\sigma w')' = \sigma f(w) \leq \sigma f(\eta). \tag{2.3}$$

Since

$$\lim_{r\to\infty}\left[\frac{r}{\sigma(r)}\int_0^r \sigma(s)ds\right] = 2, \tag{2.4}$$

it follows from (2.3) that $w'(r) \leq f(\eta)/r$ for all $r \geq r_0$ for some $r_0 > 0$. Hence $w(r) \leq w(r_0) + f(\eta)\ln\left(\frac{r}{r_0}\right) \to -\infty$ as $r \to \infty$, a contradiction.

From now on we shall assume that $\eta > \kappa$. First, we have the following global existence result.

Proposition 2.2. *The local solution $w(r;\eta)$ of (1.4) can be continued to all $r > 0$.*

Proof. Recall that $w'(r) > 0$ for all $r > 0$. Fix $r_0 \in (0, \epsilon)$. Then we have

$$w(r) = w(r_0) + \int_{r_0}^r v(s)ds, \tag{2.5a}$$

$$v(r) = v(r_0) + \int_{r_0}^r \left(\frac{s}{r}\right)^{n-1} \exp\left(\frac{s^2 - r^2}{4}\right) f(w(s))ds. \tag{2.5b}$$

Suppose first that $n \geq 2$. For $r \geq r_0$, we compute that

$$\begin{aligned} v(r) &= v(r_0) + \int_{r_0}^r \left(\frac{s}{r}\right)^{n-2} (r^{-1}e^{-r^2/4})(se^{s^2/4})f(w(s))ds \\ &\leq v(r_0) + (r^{-1}e^{-r^2/4})f(w(r))2\left[e^{r^2/4} - e^{r_0^2/4}\right] \\ &\leq v(r_0) + 2\gamma w(r)/r, \end{aligned} \tag{2.6}$$

where that fact f is increasing is used. Hence from (2.5a) it follows that

$$w(r) \leq w(r_0) + v(r_0)(r - r_0) + \int_{r_0}^r \frac{2\gamma}{s} w(s)ds, \quad r \geq r_0. \tag{2.7}$$

From (2.7) and the Gronwall inequality it follows that $w(r)$ and $v(r)$ are bounded for r finite. Notice that $w(r) \geq \eta$ for all $r > 0$. Therefore, the result follows by the standard continuation theorem.

For the case $n = 1$, by writing $\exp(s^2/4) = s^{-1}[s\exp(s^2/4)]$ in (2.5b) and noting that $1/s \le 1/r_0$ for $r \ge r_0$, we have:

$$v(r) \le v(r_0) + 2\gamma w(r)/r_0. \tag{2.8}$$

Hence we obtain

$$w(r) \le w(r_0) + v(r_0)(r - r_0) + \int_{r_0}^{r} \frac{2\gamma}{r_0} w(s)ds. \tag{2.9}$$

Then the proposition follows by the same argument as above.

By a similar argument to the proof of Proposition 2.1, we have $w'(r) \ge f(\eta)/r$ for all $r \ge r_0$ for some $r_0 > 0$. Therefore, we obtain:

$$w(r;\eta) \to \infty \quad \text{as} \quad r \to \infty. \tag{2.10}$$

From (2.10) it follows that

$$\lim_{r\to\infty} [rw'(r)] = \infty. \tag{2.11}$$

Indeed, given $K > 0$ there is a $R > 0$ such that $f(w(R)) \ge K$ and hence $w'(r) \ge K/r, \forall r \ge R$, by a similar argument to the proof of Proposition 2.1. Therefore, (2.11) follows.

In the rest of this section we shall study the asymptotic behavior of w at $r = \infty$. First, we have:

Lemma 2.3. *There holds*

$$\lim_{r\to\infty} \frac{v(r)}{w(r)} = 0. \tag{2.12}$$

Proof. For $n \ge 2$, (2.12) follows from (2.6) and (2.10). For $n = 1$, using (2.8) and noting that r_0 in (2.8) can be chosen to be arbitrary large, the lemma follows.

Lemma 2.4. *There holds*

$$\lim_{r\to\infty} \frac{rv(r)}{w(r)} = 2\gamma. \tag{2.13}$$

Proof. Let $z(r) = v(r)/w(r)$. Then z satisfies the equation

$$z' + \frac{r}{2}z = \gamma + a(r), \tag{2.14}$$

where $a(r) = -(n-1)z(r)/r - z(r)^2 - w(r)^{-\beta-1} \to 0$ as $r \to \infty$. On the other hand, by (2.14) $z(r)$ can be expressed by

$$z(r) = \exp\left(-\frac{r^2}{4}\right)\int_0^r [\gamma + a(s)] \exp\left(\frac{s^2}{4}\right) ds,$$

the lemma follows by applying L'Hôpital's rule.

Lemma 2.5. *For any $\epsilon > 0$ there are $K = K(\epsilon) > 0$ and $R = R(\epsilon) > 0$ such that*

$$w(r) \geq K r^{2\gamma-\epsilon}, \quad \forall r > R. \tag{2.15}$$

Proof. Given $\epsilon > 0$ it follows from (2.13) that there is an $R > 0$ such that

$$\frac{w'(r)}{w(r)} \geq \frac{1}{r}(2\gamma - \epsilon)$$

for all $r \geq R$. An integration gives (2.15).

Finally, we state the main result of this section as follows.

Theorem 2.6. *The limit, $\lim_{r\to\infty}[r^{-2\gamma}w(r)]$, exits and is positive.*

Proof. Take any positive constant $\lambda < 2$. Applying L'Hôpitoal's rule to

$$r^\lambda[rz(r) - 2\gamma] = \frac{\{\int_0^r [\gamma + a(s)]\exp(s^2/4)ds\} - 2\gamma r^{-1}\exp(r^2/4)}{r^{-1-\lambda}\exp(r^2/4)},$$

we obtain that

$$\lim_{r\to\infty}\{r^\lambda[rz(r) - 2\gamma]\} = 2\lim_{r\to\infty}[r^\lambda a(r)], \tag{2.16}$$

if that last limit exists.

We claim that the limit on the right–hand side of (2.16) is zero. Since $\lambda < 2$, there is an $\epsilon > 0$ such that $\lambda\gamma < 2\gamma - \epsilon$. It follows from Lemma 2.5 taht

$$\lim_{r\to\infty}[r^{\lambda\gamma}w(r)^{-1}] = 0. \tag{2.17}$$

On the other hand, using $r^\lambda a(r) = -(n-1)r^{\lambda-2}[rz(r)] - r^{\lambda-2}[rz(r)]^2 - [r^{\lambda\gamma}w(r)^{-1}]^{\beta+1}$, by (2.13) and (2.17), we see that the limit on the right–hand side of (2.16) is zero.

From (2.16) we conclude that there is a positive constant A such that $w(r) = Ar^{2\gamma}[1 + o(r^{-\lambda})]$ as $r \to \infty$. Hence the theorem follows.

Recall that the function $u(x,t) = t^{\gamma} w(|x|/\sqrt{t})$ is a solution of (1.1) for any function $w(r)$ satisfies (1.4) with $w(0) > \kappa$. For any $x \neq 0$, we write

$$u(x,t) = |x|^{2\gamma} r^{2\gamma} w(r), \quad r = |x|/\sqrt{t}. \tag{2.18}$$

Lettint $t \to 0$ in (2.18), it follows from Theorem 2.6 that there exists a positive constant c such that

$$u(x,0) = c|x|^{2\gamma}, \quad x \neq 0. \tag{2.19}$$

On the other hand, since $w(0) = \eta$, $u(0,0) = 0$. This shows that there is a solution of (1.1) with initial condition $u_0 = c|x|^{2\gamma}$. Notice that for $w \equiv \kappa$ the function $u(x,t) = \kappa t^{\gamma}$ is a solution of (1.1) with initial condition $u_0 \equiv 0$.

3. Solutions of (1.7). In this section we consider the positive solution of the problem:

$$w'' + \left(\frac{n-1}{r} + \frac{r}{2}\right) w' + g(w) = 0, \quad r > 0, \tag{3.1a}$$

$$w(0) = \eta > 0, \quad w'(0) = 0, \tag{3.1b}$$

where $g(w) = \lambda w - w^{-\beta}$ with $\lambda > 0$ and $\beta \geq 1$. Note that (1.7) is a special case of (3.1a). Define

$$G(w) = \int_{\kappa}^{w} g(s)ds, \quad \kappa = \lambda^{-\gamma}, \quad \gamma = 1/(\beta + 1). \tag{3.2}$$

Proposition 3.1. *For any $\eta > 0$, there exists a unique bounded positive global solution $w(r;\eta)$ of* (3.1).

Proof. Rewrite (3.1a) as

$$(\sigma(r)w'(r))' + \sigma(r)g(w(r)) = 0. \tag{3.3}$$

Integrating (3.3) twice, we see that a solution $w(r;\eta)$ exists if and only if w satisfies the integral equation

$$w(r) = \eta - \int_0^r \int_0^t \left(\frac{s}{t}\right)^{n-1} \exp\left(\frac{s^2 - t^2}{4}\right) g(w(s))dsdt. \tag{3.4}$$

Since $(s/t)^{n-1}\exp[(s^2-t^2)/4] \le 1$ for $s \le t$, the local existence and uniqueness follow from the contraction mapping principle.

Multiplying (3.1a) by w' and integrating it from 0 to r, we obtain:

$$\frac{w'(r)^2}{2} + \int_0^r \left(\frac{n-1}{s} + \frac{s}{2}\right) w'(s)^2 ds + G(w(r)) = G(\eta). \tag{3.5}$$

From this equality we see that w' is bounded and w is bounded form above and away from zero. Hence the local solution can be continued to all $r > 0$ and the result follows.

Next, we have the following nonoscillatory result.

Lemma 3.2. *Every nonconstant positive solution of* (3.1) *can only assume the value* κ *at finitely many points.*

Proof. Notice that any critical point of w with $w \neq \kappa$ is a local maximum point if $w > \kappa$; is a local minimum point if $w < \kappa$. It is easily to see that the function $v = w'$ satisfies the equation

$$v'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)v' + \left[\frac{1}{2} - \frac{n-1}{r^2} + \lambda + \beta w^{-\beta-1}\right]v = 0. \tag{3.6}$$

Rewriting (3.6) as

$$(\sigma(r)v'(r))' + \sigma(r)\left[\frac{1}{2} - \frac{n-1}{r^2} + \lambda + \beta w^{-\beta-1}\right]v = 0, \tag{3.7}$$

and using a nonoscillation criterion [6, p.362], we conclude that v is nonoscillatory. The lemma follows.

We now state and prove the main result of this section.

Theorem 3.3. *Every nonconstant positive solution of* (3.1) *is monotone ultimately. Moreover,* $w \to \kappa$ *and* $w' \to 0$ *as* $r \to \infty$.

Proof. Let w be any nonconstant positive solution of (3.1). Since

$$\left(\frac{w'^2}{2} + G(w)\right)' = -\left(\frac{n-1}{r} + \frac{r}{2}\right)w'^2 \le 0,$$

the limit

$$\lim_{r\to\infty}\left(\frac{w'^2}{2}+G(w)\right)=l \tag{3.8}$$

exists and is nonegative. Note that w is monotone ultimately by Lemma 3.2. Hence w tends a limit l_1 as $r\to\infty$. Since w is bounded, we see that $l_1<\infty$. Therefore, by (3.8) and the monotonicity of w, we conclude that w' tends to a limit l_2. Furthermore, since w is bounded, the limit l_2 must be zero.

Suppose that $w'>0$ for $r>r_0$ for some $r_0>0$. Dividing the equation (3.1a) by r and integrating it from r_0 to r, we obtain

$$\frac{w'(r)}{r}-\frac{w'(r_0)}{r_0}+\int_{r_0}^{r}\left(\frac{n}{s^2}+\frac{1}{2}\right)w'(s)ds+\int_{r_0}^{r}\frac{1}{s}g(w(s))ds=0. \tag{3.9}$$

Since w' is integrable to ∞ and $w'\to 0$ as $r\to\infty$, the integral

$$\int_{r_0}^{\infty}\frac{1}{s}g(w(s))ds \tag{3.10}$$

exists. Suppose for contradiction that $l_1\neq\kappa$. Then there exist $r_1>r_0$ and $\delta>0$ such that $|g(w(r))|>\delta$ for all $r>r_1$. This implies that the integral in (3.10) diverges, a contradiction. Therefore, we must have $l_1=\kappa$.

The case for which w is decreasing ultimately is similar and the theorem follows.

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