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SEMIPRIME RINGS WITH DIFFERENTIAL IDENTITIES

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Abstract. Let R be a semiprime ring, U its maximal right quotient ring and I_R be a dense R-submodule of U_R. It is shown that I and U satisfy the same differential identities. As a consequence of this result it is proved that if $f(x_i^{\Delta_j}) = 0$ is a differential identity on an ideal J of R, then $f(x_i^{\Delta_j})y = 0$ for all $x_i \in U$ and all $y \in J$.

In [4] Chuang proved a theorem to generalize the two main theorems on generalized polynomial identities in [11]. The result [4; Theorem 2] states: Let R be a prime ring, U its maximal right quotient ring and N_R a dense R-submodule of U_R. Then N and U satisfy the same generalized polynomial identities with coefficients in U. In fact, in a procede paper [2] Beidar proved that the result above remains true for semiprime rings. In this paper we shall generalize these results by considering differential identities instead of generalized polynomial identities. Explicitly, our main result states: Let R be a semiprime ring and U its maximal right quotient ring and I_R a dense R-submodule of U_R. Then I and U satisfy the same differential identities with coefficients in U. To prove this, our main tool is Kharchenko's result [9; Theorem 2]. For our need, Theorem 1 provides a convenient version of Kharchenko's result. In Theorem 4 we provide a unified approach to handle

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differential identities satisfied on ideals. Theorem 5, a generalization of [12], is then a standard application of our main theorem.

Throughout this paper, R always denotes an associative, left faithful ring (that is, for $a \in R$, $aR = 0$ implies $a = 0$) but not necessarily with 1. A right ideal ρ of R is said to be dense (or rational) if ρ_R is a dense (or rational) submodule of R_R (see p.58 [6]). In terms of dense right ideals the maximal right quotient ring of R (see p.57 [6] or [7]) can be characterized as a ring U satisfying the following axioms:

(1) R is a subring of U.

(2) For each $a \in U$ there exists a dense right ideal ρ of R such that $a\rho \subseteq R$.

(3) If $a \in U$ and $a\rho = 0$ for some dense right ideal ρ of R, then $a = 0$.

(4) For any dense right ideal ρ of R and for any right R-module map $\phi : \rho_R \to R_R$ there exists $a \in U$ such that $\phi(x) = ax$ for all $x \in \rho$.

The center of U, denoted by C, is called the extended centroid of R. Note that $a \in U$ is central (i.e., $a \in C$) if and only if there exist an ideal I of R, which is a dense right ideal of R, and an (R, R)-bimodule map $\phi : I \to R$ such that $\phi(x) = ax$ for all $x \in I$.

The following are well-known:

F1. If A is a subring of U which is a dense right R-submodule of U_R, then A itself is a left faithful ring and its maximal right quotient ring coincides with U.

F2. Let R be a prime ring; then U is also a prime ring with C a field.

Our first step is to give a convenient version of Kharchenko's theorem [9; Theorem 2] for the prime case. Our version is in three respects: (a) The coefficients of the generalized differential identities are allowed to lie in the maximal right quotient rings instead of in the Martindale two-sided quotient rings. (b) The derivation words are always formed from $Der U$, the set of all derivations on U. (c) The indeterminates occurring in differential identities assume their values in a dense R-submodule of U_R instead of in a nonzero ideal of R.

For the sake of completeness we recall some basis notations and definitions given in [8] or [5]. First, denote by $DerU$ the set of all derivations on U. By a derivation word we mean an additive map Δ in End(U,+,0) of the form $\Delta = d_1 d_2 \cdots d_n$ with each $d_i \in DerU$. A differential polynomial is then a generalized polynomial of the form $\phi(x_i^{\Delta_j})$ involving noncommutative indeterminates x_i which are acted by derivation words Δ_j as unary operations and with coefficients in U. $\phi(x_i^{\Delta_j}) = 0$ is said to be a differential identity on a subset T of U if it assumes the constant value 0 for any assignment of values from T to its indeterminates x_i.

For $d \in DerU$ and $\alpha \in C$, define $d\alpha$ by $x^{d\alpha} = (x^d)\alpha$ for all $x \in U$. For $d, \delta \in DerU$, define their commutator $[d,\delta]$ by $x^{[d,\delta]} = (x^d)^\delta - (x^\delta)^d$ for all $x \in U$. Then $d\alpha$ and $[d,\delta]$ are also derivations on U. In this manner $DerU$ forms a right module over C and is a Lie algebra over the subring consisting of $\alpha \in C$ such that $\alpha^\sigma = 0$ for all $\sigma \in DerU$. $DerU$ is called the differential Lie C-algebra of derivations on U. The following basic differential identities hold in U:

B1. $(xy)^\sigma = x^\sigma y + xy^\sigma$, where $\sigma \in DerU$.

B2. $(x+y)^\sigma = x^\sigma + y^\sigma$, where $\sigma \in DerU$.

B3. $x^\sigma = ax - xa$, where σ is the inner derivation induced by $a \in U$.

B4. $x^{[\sigma,\mu]} = (x^\sigma)^\mu - (x^\mu)^\sigma$, where $\sigma, \mu \in DerU$.

B5. $(\cdots((x^\sigma)^\sigma)\cdots)^\sigma = x^{(\sigma^p)}$, where $\sigma \in DerU$ and where R has a characteristic p, which is a prime or 0. If $p = 0$, then this identity assumes the form $x = x$.

B6. $x^{\sigma\alpha+\mu\beta} = (x^\sigma)\alpha + (x^\mu)\beta$, where σ, $\mu \subset DerU$ and $\alpha, \beta \in C$.

Suppose now that R is a prime ring. In this case, since C is a field, $DerU$ forms a vector space over C. Let D_{int} be the subspace of $DerU$ consisting of all inner derivations of U. Choose a fixed basis M_0 for D_{int} and augment it to a basis M for $DerU$. Fix a total order $>$ in the set M such that $\mu_0 > \mu$ for $\mu_0 \in M_0$ and $\mu \in M - M_0$ and then extend this order to the set of all derivation words in M by assuming that a longer word is greater than a shorter one and that words of the same length are ordered lexicographically. By a regular word in M we mean a derivation word of the form $\Delta = \delta_1^{s_1}\delta_2^{s_2}\cdots\delta_m^{s_m}$ such that

(W1) $\delta_i \in M - M_0$ for $i = 1, 2, \cdots, m$,

(W2) $\delta_1 < \delta_2 < \cdots < \delta_m$ and

(W3) $s_i < p$ for $i = 1, 2, \cdots, m$, if $charR = p > 0$.

Note that by means of the basic identities (B1)–(B6) any differential identity can be transformed into a form $\phi(x_i^{\Delta_j})$ where

(R1) $\phi(z_{ij})$ is a generalized polynomial in distinct noncommutative indeterminates z_{ij} and with coefficients in U, and where

(R2) the Δ_j are distinct regular words in M.

A differential polynomial is called *reduced* if it assumes the form $\phi(x_i^{\Delta_j})$ satisfying (R1) and (R2). By a reduced differential identity we mean an identity $\phi(x_i^{\Delta_j}) = 0$ with $\phi(x_i^{\Delta_j})$ a reduced differential polynomial. Now we may state Kharchenko's theorem [9; Theorem 2] assuming the following form.

Theorem 1. *Let R be a prime ring, U its maximal right quotient ring and I_R a dense R-submodule of U_R. Assume that $\phi(x_i^{\Delta_j}) = 0$ is a reduced differential identity for I. Then $\phi(z_{ij}) = 0$ is a generalized polynomial identity for U, where the z_{ij} are distinct indeterminates.*

Proof. This theorem is essentially the result of Kharchenko, but assuming this form we can conveniently apply it to our later problems. For its proof we only give its sketch as follows. Assume first that $I = R$. In this case a careful verification shows that its proof is just the same as that of [9; Theorem 2] for the prime case (see p. 155–164 [8] and p. 67–74 [9]). For the general case, if I is a dense R-submodule of U_R, $I \cap R$ is then a dense right ideal of R. Since R is a prime ring, $I \cap R$ itself is also a prime ring and by F1 its maximal right quotient ring is just U. Thus, applying the first case, $\phi(z_{ij}) = 0$ on $I \cap R$. Finally, applying [4; Theorem 2,], $I \cap R$ and U satisfy the same generalized polynomial identies and hence $\phi(z_{ij}) = 0$ on U. This completes the proof.

Since any differential identity can be transformed into some reduced differential identity via the basic identities (B1)–(B6), we have the following

Theorem 2. *Let R be a prime ring, U its maximal right quotient ring and I_R a dense R-submodule of U_R. Then I and U satisfy the same differential identities.*

Our main objective is to generalize Theorem 2 to the case of semiprime rings. To arrive at this aim we need some results about orthogonal completions for semiprime rings given in [3]. For the sake of convenience we summarize these here.

Let R be a semiprime ring, U its maximal right quotient ring and C the extended centroid of R. A subset $T \subseteq U$ is called orthogonally complete if $0 \in T$ and given any set of orthogonal idempotents $\{e_\omega\} \subseteq C$, $\omega \in \Omega$, any subset $\{x_\omega\} \subseteq T$, $\omega \in \Omega$, there exists $x \in T$ such that $e_\omega x = e_\omega x_\omega$ for all $\omega \in \Omega$. For any subset $K \subseteq U$, denote by $\hat{K}$ the orthogonal completion of K which is defined as the intersection of all orthogonally complete subsets of U containing K. Note that $\hat{K}$ itself is an orthogonally complete subset of U.

F3. Let $T \subseteq U$, $r(C;T) = \{\alpha \in C | \alpha t = 0 \text{ for all } t \in T\}$. Then there exists a unique central idempotent $E[T]$ such that $r(C;T) = (1-E[T])C$ (see [3; point 2]).

F4. C is a commutative regular self-injective ring (see [1; Theorem 1]).

F5. U is an orthogonally complete nonsingular C-module (see [3; Lemma 1]).

Suppose that $\{e_\omega\}$, $\omega \in \Omega$, is a set of orthogonal idempotents of C, $\{x_\omega\} \in U$, $\omega \in \Omega$ and $P = \sum e_\omega C$; then, by F3, $r(C,P) = eC$ for some $e^2 = e \in C$, and there is a unique element $x \in U$ such that $e_\omega x = e_\omega x_\omega$ for all $\omega \in \Omega$ and $ex = 0$. We denote the element x by $\sum_{\omega\in\Omega}^{\perp} e_\omega x_\omega$.

F6. If $x = \sum_\omega^\perp e_\omega x_\omega \in U$, $y = \sum_\delta^\perp u_\delta y_\delta$ and $z \in U$, then $x+y = \sum_{\omega,\delta}^\perp (e_\omega u_\delta)(x_\omega + y_\delta)$, $xy = \sum_{\omega,\delta}^\perp (e_\omega u_\delta)(x_\omega y_\delta)$ and $zx = \sum_\omega^\perp e_\omega(zx_\omega)$ (see [3; Lemma 1]).

F7. For any $T \subseteq U$, $\hat{T}$ consists of all elements $q \in U$ of the form $q = \sum_\omega^\perp e_\omega x_\omega$, where $\{e_\omega\}$, $\omega \in \Omega$, is a set of orthogonal diempotents in C and $\{x_\omega\}$, $\omega \in \Omega$, is contained in $T \cup \{0\}$ (see [3; Lemma 1]).

Denote by B the Boolean algebra of all idempotents in C. For $e, f \in B$ we define $e \oplus f = e + f - 2ef$ and $e \cdot f = ef$. Also, $M(C)$ and $M(B)$ stand for the set of all maximal ideals of C and for the set of all maximal ideals of B, respectively.

F8. If $P \in M(C)$ and set $m = P \cap B$, we have $m \in M(B)$ and $mC = P$; if U_p is the localization of the C-algebra U w.r.t. $C - P$, then U_p is canonically isomorphic to U/mU. For the obvious map $\phi_p : U \to U_p \cong U/mU$ it has the following properties:

(i) for any orthogonally complete, dense right ideal ρ of $\hat{R}$, $\phi_p(\hat{\rho})$ is a dense right ideal of $\phi_p(\hat{R})$;

(ii) $\phi_p(\hat{R})$ is a prime ring and U_p is a right quotient ring of $\phi_p(\hat{R})$. Also, the extended centroid of U_p is just C_p;

(iii) PU is a prime ideal of U invariant under any derivation $d \in DerU$. Furthermore, $\cap\{PU | P \in M(C)\} = 0$ (see [3; Lemma 1 and Theorem 1]).

Lemma 1. *Let R be a semiprime ring, ρ a dense right ideal of R and $P \in M(C)$. Then $\hat{\rho}$ is a dense right ideal of $\hat{R}$ such that $\phi_p(\hat{\rho})$ is a dense right $\phi_p(\hat{R})$-submodule of U_p.*

Proof. Applying F8 (i), (ii) the rest is to prove $\hat{\rho}$ to be a dense right ideal of $\hat{R}$. For $x \in \hat{\rho}$, $y \in \hat{\rho}$ and $z \in \hat{R}$ it follows from F7 that there exist $\{e_\omega | \omega \in \Omega\}$, $\{f_\delta | \delta \in \Delta\}$ and $\{g_\gamma | \gamma \in \Gamma\}$ three sets of orthogonal idempotents in C and $\{x_\omega | \omega \in \Omega\} \subseteq \rho$, $\{y_\delta | \delta \in \Delta\} \subseteq \rho$ and $\{z_\gamma | \gamma \in \Gamma\} \subseteq R$ such that $x = \sum_\omega^\perp e_\omega x_\omega$, $y = \sum_\delta^\perp f_\delta y_\delta$ and $z = \sum_\gamma^\perp g_\gamma z_\gamma$. By F6 we have that $x + y = \sum_{\omega,\delta}^\perp (e_\omega f_\delta)(x_\omega + y_\delta)$ and $xz = \sum_{\omega,\gamma}^\perp (e_\omega g_\gamma)(x_\omega z_\gamma)$. Since $x_\omega + y_\delta \in \rho$ and $x_\omega z_\gamma \in \rho$ for all $\omega \in \Omega$, $\delta \in \Delta$ and $\gamma \in \Gamma$, it follows that $x + y \in \hat{\rho}$ and $xz \in \hat{\rho}$. Therefore $\hat{\rho}$ is a right ideal of $\hat{R}$.

Let $x = \sum_\omega^\perp e_\omega x_\omega \neq 0$ in $\hat{R}$, $y = \sum_\delta^\perp f_\delta y_\delta \in \hat{R}$, where $\{e_\omega | \omega \in \Omega\}$ and $\{f_\delta | \delta \in \Delta\}$ are two set of orthogonal idempotents of C and $x_\omega \in R$, $y_\delta \in R$ for all $\omega \in \Omega$, $\delta \in \Delta$. We may assume further that $\sum_\delta f_\delta C$ is an essential ideal of C. Since $x \neq 0$, there exists $\omega_0 \in \Omega$ such that $e_{\omega_0} x_{\omega_0} \neq 0$ and hence $f_{\delta_0} e_{\omega_0} x_{\omega_0} \neq 0$ for some $\delta_0 \in \Delta$ by F5. By assumption ρ is a dense right ideal of R. Thus ρ_R is a dense R-submodule of U_R. So for each $\delta \in \Delta$ there exists $t_\delta \in R$ such that $(f_{\delta_0} e_{\omega_0} x_{\omega_0}) t_\delta \neq 0$ in U and $y_\delta t_\delta \in \rho$. Consider the element $t = \sum_\delta^\perp f_\delta t_\delta \in \hat{R}$. Then, by F6, $yt = \sum_\delta^\perp f_\delta (y_\delta t_\delta) \in \hat{\rho}$ and $xt = \sum_{\omega,\delta}^\perp (e_\omega f_\delta)(x_\omega t_\delta) \in \hat{R}$, since $y_\delta t_\delta \in \rho$, $x_\omega t_\delta \in R$ for all $\delta \in \Delta$, $\omega \in \Omega$. Finally we must prove that $xt \neq 0$. Indeed, $(f_{\delta_0} e_{\omega_0}) xt = (f_{\delta_0} e_{\omega_0}) x_{\omega_0} t_{\delta_0} \neq 0$ and hence, in particular, $xt \neq 0$ follows. This completes the proof.

To generalize Theorem 2 to the semiprime case we need a well-known fact that any derivation on a left faithful ring can be uniquely extended to a derivation on its maximal right quotient ring [10; p101, Exercise 10]. For the sake of our later application we will give a slight generalization of this result.

Let R be a left faithful ring and U its maximal right quotient ring. By a derivation from R into U we mean an additive mapping $d : R \to U$ satisfying $d(xy) = d(x)y + xd(y)$ for all x, y in R. Denote by $Der(R, U)$ the set of all derivations from R into U.

Lemma 2. *Every $d \in Der(R, U)$ can be uniquely extended to a derivation in $DerU$.*

Proof. For $q \in U$, choose a dense right ideal ρ of R such that $q\rho \subseteq R$. Note that ρU is a dense right ideal of U. Define the map $\phi : \rho U \to U$ as follows: for $z = \sum x_i y_i \in \rho U$ with $x_i \in \rho$, $y_i \in U$ then $\phi(z) = \sum (d(qx_i) - qd(x_i)) y_i$. Claim first that ϕ is well-defined. Suppose that $\sum x_i y_i = 0$. Choose a dense right ideal ρ_1 of R such that $y_i \rho_1 \subseteq R$ for each i. For $t \in \rho_1$, applying d to $\sum x_i (y_i t) = 0$ and to $\sum (qx_i)(y_i t) = 0$, we yield

(1) $\sum (d(x_i) y_i t + x_i d(y_i t)) = 0$, and

(2) $\sum (d(qx_i) y_i t + qx_i d(y_i t)) = 0$.

It follows from (1) and (2) that $(\sum (d(qx_i) - qd(x_i)) y_i) t = 0$ for all $t \in \rho_1$. So $\sum (d(qx_i) - qd(x_i)) y_i = 0$, which implies ϕ to be well-defined.

Obviously, ϕ is a right U-module map. Since the maximal right quotient ring of U is just itself, the map ϕ defines an element of U. Also if $q \in R$ we can easily check that this element is just $d(q)$. Hence in this manner d can be extended to a map from U into itself. For brevity we denote this map by d also. The fact that d is a derivation on U and that the uniqueness of d are easily checked. So we omit these.

We are now ready to prove our main result.

Theorem 3. *Let R be a semiprime ring, U its maximal right quotient ring and I_R a dense R-submodule of U_R. Then I and U satisfy the same differential identities.*

Proof. Since I_R is a dense R-submodule of U_R, so is $(I \cap R)_R$. Thus by considering $I \cap R$ instead of I from the start we may assume that I is a dense right ideal of R. Suppose that $f(x_i^{\Delta_j}) = 0$ is a differential identity on I. For any $P \in M(C)$, consider the map ϕ_p as given in F8. Denote by $W(P)$ the maximal right quotient ring of $\phi_p(\hat{R})$. Since, by F8 (ii), U_p is a right quotient ring of $\phi_p(\hat{R})$, there is a cononical inclusion $U_p \subseteq W(P)$ and each $d \in DerU$ naturally induces a derivation $\hat{d}$ on $W(P)$ such that $\hat{d} = \overline{d}$ on U_p, where $\overline{d}$ is the canonical derivation on U_p defined by F8 (iii). In this way every derivation word $\Delta = d_1 d_2 \cdots d_n$, where $d_i \in DerU$, has a corresponding derivation word $\hat{\Delta} = \hat{d}_1 \hat{d}_2 \cdots \hat{d}_n$ with $\hat{d}_i \in DerW(P)$. Let $\phi_p(f)(x_i^{\Delta_j})$ denote the differential polynomial obtained by applying ϕ_p to the coefficients of $f(x_i^{\Delta_j})$ and by replacing Δ_j by $\hat{\Delta}_j$. Thus $\phi_p(f)(x_i^{\Delta_j})$ is a differential polynomial in noncommutative indeterminates x_i which are acted by derivation words $\hat{\Delta}_j$ and has coefficients in $W(P)$ (in fact, in U_p).

Observe first that $f(x_i^{\Delta_j})$ may be assumed to be blended in each indeterminate x_i, that is, each x_i occurs in every monomial occurring in f. Since I satisfies $f(x_i^{\Delta_j}) = 0$ and since central idempotents in C are constants of each derivation word Δ_j, applying F7 and F6 and using the fact that $f(x_i^{\Delta_j})$ is blended in each x_i we get that $f(x_i^{\Delta_j}) = 0$ on $\hat{I}$. Applying ϕ_p to this identity we have that $\phi_p(f)(x_i^{\Delta_j}) = 0$ on $\phi_p(\hat{I})$. By Lemma 1, $\phi_p(\hat{I})$ is a dense $\phi_p(\hat{R})$-submodule of U_p and hence is also a dense $\phi_p(\hat{R})$-submodule of $W(P)$. Since, by F8 (ii), $\phi_p(\hat{R})$ is a prime ring, Theorem 2 implies that $\phi_p(f)(x_i^{\Delta_j}) = 0$ on $W(P)$ and, in particular, on U_p. So $f(x_i^{\Delta_j}) \in PU$ for all $x_i \in U$. Since $\cap\{PU | P \in M(C)\} = 0$, we have that $f(x_i^{\Delta_j}) = 0$ on U, which completes the proof.

As a consequence of Theorem 3, we have

Theorem 4. *Let R be a semiprime ring, U its maximal right quotient ring and I a nonzero ideal of R. Suppose that $f(x_i^{\Delta_j}) = 0$ is a differential identity on I. Then $f(x_i^{\Delta_j})y = 0$ for all $x_i \in U$, $y \in I$.*

Before giving the proof of Theorem 4, we recall some basic properties about annihilators in semiprime rings. For a semiprime ring A, $L \subseteq A$ is called a left annihilator if $L = l_A(S) = \{x \in A | xs = 0 \text{ for all } x \in S\}$ for some $S \subseteq A$. Similarly, define $r_A(S) = \{x \in A | sx = 0 \text{ for all } s \in S\}$ and call such a set a right annihilator of A. By an annihilator ideal of A we mean an ideal L of A which is a left annihilator of A. In fact, for any ideal K of A we have that $l_A(K) = r_A(K)$. In this case, denote by $Ann_A(K)$ the left or right annihilator of K in A. Thus every annihilator ideal of A assumes the form $Ann_A(K)$ for some ideal K of A. Also, $A/Ann_A(K)$ always remains a semiprime ring and $d(Ann_A(K)) \subseteq Ann_A(K)$ for all $d \in Der A$.

Lemma 3. *Given R, U and I as in Theorem 4, then the following hold:*

(1) *for $x \in U$, $xI = 0$ if and only if $Ix = 0$, and*

(2) *set $J = Ann_U(I)$, an ideal of U, $\overline{U} = U/J$, $\overline{R} = (R + J)/J$ and $\overline{I} = (I + J)/J$; then $\overline{I}_{\overline{R}}$ is a dense $\overline{R}$-submodule of $\overline{U}_{\overline{R}}$ and $\overline{U}$ is a semiprime ring.*

Proof. (1) Suppose that $xI = 0$, where $x \in U$. Choose a dense right ideal ρ of R such that $x\rho \subseteq R$. Thus $x\rho I = 0$ and so $Ix\rho = 0$. Hence $Ix = 0$ as desired. Similarly, $Ix = 0$ implies $xI = 0$. (2) By (1), $l_U(I) = r_U(I)$ and hence J is an ideal of U such that $\overline{U}$ is a semiprime ring. The rest is to prove that $\overline{I}_{\overline{R}}$ is a dense $\overline{R}$-submodule of $\overline{U}_{\overline{R}}$. Let $\overline{x} = x + J \neq 0$ in $\overline{U}$ and $\overline{y} = y + J$ in $\overline{U}$, where $x, y \in U$. Then $xI \neq 0$. Choose a dense right ideal ρ of R such that $x\rho \subseteq R$ and $y\rho \subseteq R$. Since $xI \neq 0$, it follows that $x\rho I \neq 0$. Thus there exists $t \in \rho I \subseteq I$ such that $xt \neq 0$. Then $\overline{xt} \neq 0$ in $\overline{U}$, otherwise $xt \in J$, which implies $xt \in J \cap I = 0$, a contradiction. Also, $yt \in y\rho I \subseteq I$ and hence $\overline{yt} \in \overline{I}$. So $\overline{I}_{\overline{R}}$ is a dense $\overline{R}$-submodule of $\overline{U}_{\overline{R}}$.

Proof of Theorem 4. Remain these notations given in Lemma 3. Since J is an annihilator ideal of U, $d(J) \subseteq J$ for all $d \in DerU$. Therefore, every $d \in DerU$ canonically induces a derivation $\overline{d} \in Der\overline{U}$ and so does every derivation word Δj. Denote by $\overline{\Delta}_j$, the derivation word induced by Δ_j. Consider the canonical map $\phi : U \to \overline{U}$ and let $\phi(f)(x_i^{\overline{\Delta}_j})$ denote the differential polynomial obtained by applying ϕ to the coefficients of f and by replacing Δ_j by $\overline{\Delta}_j$. Since $f(x_i^{\Delta_j}) = 0$ is a differential identity on I, we have that $\phi(f)(x_i^{\overline{\Delta}_j}) = 0$ is a differential identity on $\overline{I}$. By Lemma 3, $\overline{I}_{\overline{R}}$ is a dense $\overline{R}$-submodule of $\overline{U}_{\overline{R}}$. Thus $\overline{U}$ is contained in the maximal right quotient ring of $\overline{R}$. Now considering the derivation words of $\overline{U}$ as those of the maximal right quotient ring of $\overline{R}$ by Lemma 2 and applying Theorem 3 we yield that $\phi(f)(x_i^{\overline{\Delta}_j}) = 0$ on $\overline{U}$, that is, $f(x_i^{\Delta_j}) \in J$ for all $x_i \in U$. Thus $f(x_i^{\Delta_j})y = 0$ for all $x_i \in U$ and all $y \in I$. This completes the proof.

We conclude this paper with a generalization of [14]. In [14] Lee and Lee proved that if d is a derivation on a prime ring R such that $d^n(R) \subseteq Z$, the center of R, where n is a fixed integer, then either $d^n(R) = 0$ or R is a commutative integral domain. In any ring S we denote by $[a, b] = ab - ba$ the commutator of a and b in S and by $[A, B]$ the additive subgroup of S generated by elements of the form $[a, b]$ with $a \in A$, $b \in B$. First we deal with the prime case.

Theorem 5. *Let R be a prime ring, U its maximal right quotient ring, ρ a right ideal of R and $d \in DerD$. Suppose that $d^n(\rho) \subseteq C$, the extended centroid of R, where n is a fixed positive integer. Then either $d^n(\rho) = 0$ or R is a commutative domain.*

Proof. We claim first that $d^n(\rho U) \subseteq C$. For any $a_1, \cdots, a_t \in \rho$, since $\rho R \subseteq \rho$, $d^n(\sum_{i=1}^t a_i x_i) = \sum_{i=1}^t \sum_{k=0}^n \binom{k}{n} d^k(a_i) d^{n-k}(x_i) \in C$ for all $x_i \in R$. Note that R_R is a dense R-submodule of U_R. Applying Theorem 3, $\sum_{i=1}^t \sum_{k=0}^n \binom{k}{n} d^k(a_i) d^{n-k}(x_i) \in C$ for all $x_i \in U$. Thus $d^n(\sum_{i=1}^t a_i x_i) \in C$ for all $x_i \in U$. That is, $d^n(\rho U) \subseteq C$. By considering ρU instead of ρ we may assume from the start that ρ is a right ideal of U. Set $I = \rho + d(\rho) + d^2(\rho) + \cdots$. Then I is a right ideal of U which is invariant under d. Let

$\overline{I} = I/(I \cap l(I))$, where $l(I) = \{x \in U | xI = 0\}$. Then $\overline{I}$ is also a prime ring. Also, d induces a derivation, denoted by $\overline{d}$, on $\overline{I}$ such that $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $\overline{x} \in \overline{I}$. Then $\overline{d}^n(\overline{I}) \subseteq Z(\overline{I})$, the center of $\overline{I}$. By [12; Theorem 1] either $[\overline{I}, \overline{I}] = 0$ or $\overline{d}^n(I) = 0$. Suppose that $d^n(\rho) \neq 0$. If $\overline{d}^n(I) = 0$, then $d^n(I)I = 0$. In particular, $d^n(\rho)\rho = 0$. Since $d^n(\rho) \subseteq C$, we have $\rho d^n(\rho) = 0$. So $\rho = 0$, a contradiction. If $[\overline{I}, \overline{I}] = 0$, then $[I, I]I = 0$. In particular, $[\rho, \rho]d^n(\rho) = 0$. Since $0 \neq d^n(\rho) \subseteq C$, this implies $[\rho, \rho] = 0$ and hence $[U, U] = 0$ by the primeness of U. This completes the proof.

Next, we handle the semiprime case.

Theorem 6. *Let R be a semiprime ring, U its maximal right quotient ring, ρ a right ideal of R and $d \in DerU$. Suppose that $d^n(\rho) \subseteq C$, the extended centroid of R, where n is a fixed positive integer. Then there is a ring decomposition $U = U_1 \oplus U_2$ such that $d^n(\rho) \subseteq U_2 \subseteq C$.*

Proof. As the argument given in Theorem 5 we may assume that ρ is a right ideal of U. Let $P \in M(C)$ and ϕ_p be the map given in F8. By F8 (iii), d induces a derivation $\overline{d} \in DerU_p$ satisfying $\overline{d}^n(\phi_p(\rho)) \subseteq C_p$. Since ρ is a right ideal of U and since ϕ_p is a surjective map, $\phi_p(\rho)$ is a right ideal of U_p. From the primeness of U_p, applying Theorem 5 either $\overline{d}^n(\phi_p(\rho)) = 0$ or U_p is a commutative integral domain. That is, either $d^n(\rho) \subseteq PU$ or $[U, U] \subseteq PU$. It follows from F8 (iii) that $d^n(\rho)[U, U] = 0$. Set $e = E[[U, U]]$ and $f = 1 - e$ (see F3). Then we have $[fU, U] = f[U, U] = 0$ and hence $fU \subseteq C$. Note that $d^n(\rho) \subseteq fU$. Set $U_1 = eU$ and $U_2 = fU$; then $d^n(\rho) \subseteq U_2 \subseteq C$. This completes the proof.

Remark. Theorem 5 and Theorem 6 remain true if $d \in DerU$ is replaced by the condition $d \in Der(R, U)$, since by Lemma 2 every derivation from R into U can be uniquely extended to a derivation on U.

References

1. K. I. Beidar, *Rings with generalized identities I*, Vestnik Moskov. Univ. Ser I Mat. Meh. (Engl. Transl. Moscow Univ. Math. Bull.) **32**(1977), 19–26.

2. K. I. Beidar, *Rings with generalized identities III*, Vestnik Moskov. Univ. Ser I Mat. Meh. (Engl. Transl. Moscow Univ. Math. Bull.) **33**(1978), 66–73.

3. K. I. Beidar, *Rings of quotients of semiprime rings*, Vestnik Moskov. Univ. Ser I Mat. Meh. (Engl. Transl. Moscow Univ. Math. Bull.) **33**(1978), 36–42.

4. C. L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103**(1988), 723–728.

5. C. L. Chuang, *$*$-Differential identities on prime ring with involvution*, Trans. Amer. Math. Soc. **316**(1989), 251–279.

6. C. Faith, *Lectures on injective modules and quotient rings*, Lectures Notes in Mathematics, no.**49**, Springer–Verlag, Berlin/New York (1967).

7. K. R. Goodearl, *Ring Theory*, Marcel Dekker, INC. New York and Basel (1976).

8. V. K. Kharchenko, *Differential identities of prime rings*, Algebra and Logic **17**(1978), 155–168.

9. V. K. Kharchenko, *Differential identities of semiprime rings*, Algebra and Logic **18**(1979), 86–119.

10. J. Lambek, *Lectures on Rings and Modules*, Waltham, Mass., Blaisdell(1966).

11. C. Lanski, *A note on GPIs and their coefficients*, Proc. Amer. Math. Soc. **98**(1986), 17–19.

12. P. H. Lee and T. K. Lee, *Note on nilpotent derivations*, Proc. Amer. Math. Soc **98**(1986), 31–32.

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