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ASYMPTOTIC AND OSCILLATORY BEHAVIOUR OF THE SOLUTIONS OF SECOND ORDER LINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAY

BY

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Abstract. In the paper some asymptotic and oscillatory properties of the solutions of the equation

$$\frac{d^2}{dt^2}\left[x(t)+\int_0^{\sigma(t)} x(t-s)d\tau_1(t,s)\right]+\int_0^{\sigma(t)} x(t-s)d\tau_2(t,s)=0$$

are investigated.

1. Introduction. Necessary and sufficient conditions for oscillation of all solutions of second order neutral differential equations were obtained in the paper [1] but the equation considered in it is with constant coefficients and constant delays. For such equations the oscillatory and asymptotic behaviour of their solutions is investigated in [2]. In [3] sufficient conditions for oscillation of the solutions of the linear neutral differential equations with variable coefficients are obtained. In the present paper the asymptotic and oscillatory behaviour of the solutions of the equation

$$\frac{d^2}{dt^2}\left[x(t)+\int_0^{\sigma(t)} x(t-s)d\tau_1(t,s)\right]+\int_0^{\sigma(t)} x(t-s)d\tau_2(t,s)=0 \tag{1}$$

is investigated. Some of the results obtained generalize the corresponding

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results of [3]. Similar results but about nonlinear equations with constant delay were obtained in [4].

2. Auxiliary results. We shall say that conditions (A) are met if the following conditions hold:

A1. $\sigma(t) \in C([t_0, +\infty), \mathcal{R})$,

A2. $\inf\limits_{[t_0,+\infty)} \{\sigma(t)\} > 0$,

A3. $\lim\limits_{t\to\infty} (t - \sigma(t)) = +\infty$.

We shall say that conditions (B) are met if the following conditions hold:

B1. $\tau_i(t, 0) = 0$ for $t \in [t_0, +\infty)$, $i = 1, 2$,

B2. $\tau_i(t, \sigma(t)) \in C([t_0, \infty), \mathcal{R})$, $i = 1, 2$,

B3. $\tau_2(t, s)$ is nondecreasing in s for $s \in [0, \sigma(t)]$.

Definition 1. The function f is said to eventually enjoy the property K if there exists t_0 such that for $t > t_0$ the function enjoys the property K.

Let

$$z(t) = x(t) + \int_0^{\sigma(t)} x(t-s)d\tau_1(t,s). \tag{2}$$

Then

$$z''(t) = -\int_0^{\sigma(t)} x(t-s)d\tau_2(t,s). \tag{3}$$

Definition 2. The function x defined for all sufficiently large values of t is said to be an *eventual solution* of (1) if for all sufficiently large t the functions x and z are continuous and x eventually satisfies equation (1).

Remark 1. In the paper no solutions are considered for which $x \equiv 0$ eventually.

Definition 3. The eventual solution $x(t)$ of (1) is said to *oscillate* if its set of zeros is unbounded from above. Otherwise it is said to be *nonoscillating.*

According to Definition 1 the nonoscillating solutions of (1) are characterized by the fact that they are eventually positive or eventually negative. We shall denote the set of all eventually positive solutions by Ω^+.

Lemma 1. *Let conditions* (A) *and* (B) *hold. Let* $\tau_1(t,s)$ *be nonincreasing in* s *for* $s \in [0,\sigma(t)]$ *and*

$$\int_{t_0}^{\infty} \tau_2(t,\sigma(t))dt = \infty, \tag{4}$$

$$\tau_1(t,\sigma(t)) \geq -1. \tag{5}$$

Then, if $x(t)$ *is an eventually positive solution of* (1), *then* z' *is a nonincreasing function and* $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0$, $z'(t) > 0$ *eventually and* $z(t) < 0$ *eventually.*

Proof. From (3) and B3 it follows that $z''(t) \leq 0$ eventually. Hence $z'(t)$ is an eventually nonincreasing function. Let for $t \geq t_1$, $z'(t)$ be a nonincreasing function. Then $z(t)$ is a monotone function. Suppose that there exists t_2 such that $t_2 \geq t_1$ and $z'(t_2) \leq 0$. From (4) and B2 it follows that $\tau_2(s,\sigma(s)) \not\equiv 0$ in any interval $[t,\infty)$ and since $x(t)$ is an eventually positive solution, then there exists $t_3 > t_2$ such that $z'(t) \leq z'(t_3) < 0$ for $t \geq t_3$. Then $\lim_{t\to\infty} z(t) = -\infty$ and from (2) and (5) it follows that the solution $x(t)$ is unbounded. From $\lim_{t\to\infty} z(t) = -\infty$ it follows that $z(t) < 0$ eventually. Then from (2), (5) and $\tau_1(t,s)$ nonincreasing in s there follows the estimate

$$\begin{aligned} 0 > x(t) + \int_0^{\sigma(t)} x(t-s)d\tau_1(t,s) &\geq x(t) + \max_{[t-\sigma(t),t]} x(s) \cdot \tau_1(t,\sigma(t)) \\ &\geq x(t) - \max_{[t-\sigma(t),t]} x(s). \end{aligned}$$

From the above inequalities it follows that there exists $t_4 > t_3$ such that for $t > t_4$ the following estimate is valid

$$x(t) < \max_{[t-\sigma(t),t]} x(s). \tag{6}$$

From the fact that $x(t)$ is an unbounded function it follows that there exists an increasing sequence $\{\tau_n\}_0^\infty$ such that

$$\lim_{n\to\infty} \tau_n = +\infty, \ \lim_{n\to\infty} x(\tau_n) = +\infty \text{ and } \max_{[\tau_0,\tau_n]} x(s) = x(\tau_n).$$

From A3 it follows that there exists $t_5 > t_4$ such that for $t > t_5$ we have $t-\sigma(t) > \tau_0$. Then $\max\limits_{[t-\sigma(t),t]} x(s) \leq \max\limits_{[\tau_0,t]} x(s)$ and for sufficiently large $n \in \mathcal{N}$ we have $\max\limits_{[\tau_n-\sigma(\tau_n),\tau_n]} x(s) = x(\tau_n)$, which contradicts (6). The contradiction obtained shows that $z'(t) > 0$ for $t \geq t_1$. Since z' is a nonincreasing function, then there exists the finite limit $\lim\limits_{t\to\infty} z'(t) = c \geq 0$. Integrate (3) from t_1 to t and obtain

$$z'(t) = z'(t_1) - \int_{t_1}^{t} \int_0^{\sigma(v)} x(v-s)d\tau_2(v,s)dv.$$

Suppose that $\liminf\limits_{t\to\infty} x(t) = d > 0$. Then

$$z'(t) \leq z'(t_1) - d\int_{t_1}^{t} \tau_2(v,\sigma(v))dv.$$

From (4) it follows that the right-hand side of this inequality tends to $-\infty$ as $t \to \infty$, which contradicts the existence of the finite limit $\lim\limits_{t\to\infty} z'(t) = c$. Hence $\liminf\limits_{t\to\infty} x(t) = 0$. We shall prove that $c = 0$. Suppose that this is not true, i.e. $c > 0$. Hence $z'(t) \geq c$ and then $\lim\limits_{t\to\infty} z(t) = +\infty$. From (2) and $\tau_1(t,s)$ nonincreasing in s it follows that $x(t) \geq z(t)$ and, consequently, $\lim\limits_{t\to\infty} x(t) = +\infty$, which contradicts the fact that $\liminf\limits_{t\to\infty} x(t) = 0$. Thus we proved that $\lim\limits_{t\to\infty} z'(t) = 0$. In order to prove the assertion of Lemma 1 it remains to show that $z(t) < 0$ eventually and $\lim\limits_{t\to\infty} z(t) = 0$. Suppose that there exists $\bar{t} \geq t_1$ such that $z(\bar{t}) \geq 0$. From $z'(t) > 0$ eventually it follows that there exists $\bar{\bar{t}}$ such hat $z(t) \geq z(\bar{\bar{t}}) > 0$ for $t \geq \bar{\bar{t}} > \bar{t}$ which again contradicts the fact that $\liminf\limits_{t\to\infty} x(t) = 0$ since $x(t) \geq z(t)$. Hence $z(t) < 0$ for $t \geq t_1$. We shall prove that $\lim\limits_{t\to\infty} z(t) = 0$. From $z(t) < 0$ and from the fact that $z(t)$ is an eventually increasing function there follows the existence of the finite limit $\lim\limits_{t\to\infty} z(t) = \ell \leq 0$. Suppose that $\ell < 0$. From $z(t) < 0$, as at the beginning of the proof of the lemma it is shown that inequality (6) is valid, from which again as above it follows that the function $x(t)$ is bounded.

Let $x(t) \leq A$. Choose $n \in \mathcal{N}$ such that $-n\ell > A$. Since $\lim_{t\to\infty} z(t) = \ell$ and $z(t)$ is an eventually increasing function, then $z(t) < \ell$ eventually. Then the following estimate is valid

$$\ell > x(t) + \int_0^{\sigma(t)} x(t-s)d\tau_1(t,s) \geq x(t) - \max_{[t-\sigma(t),t]} x(s) \cdot \tau_1(t,\sigma(t))$$
$$\geq x(t) - \max_{[t-\sigma(t),t]} x(s).$$

Hence

$$\max_{[t-\sigma(t),t]} x(s) - x(t) > -\ell. \tag{7}$$

From A3 it follows that for $t = t_1$ there exists t_2 such that for $t \geq t_2$ we have $t - \sigma(t) > t_1$. Moreover, let $t_2 > t_1$. For the so chosen t_2 there exists $t_3 > t_2$ such that $t - \sigma(t) > t_2$ for $t \geq t_3$. We can perform this procedure infinitely, thus defining the sequence $\{t_n\}_1^\infty$. Consider the interval $[t_n - \sigma(t_n), t_n]$ (n is such that $-n\ell > A$). Let $x(t_n) = a$. Then in this interval by (7) there exists a point τ_1 such that $x(\tau_1) > a - \ell$. Consider the interval $[\tau_1 - \sigma(\tau_1), \tau_1]$. Again by (7) in it there exists a point τ_2 such that $x(\tau_2) > x(\tau_1) - \ell > a - 2\ell$. Repeating this process n times, we shall get to a point τ_n such that $x(\tau_n) > a - n\ell$. The way in which the sequence $\{t_n\}$ is defined guarantees that $\tau_n > t_1$. But from the choice of n it follows that $x(\tau_n) > A$. The contradiction obtained shows that $c = 0$ i.e. $\lim_{t\to\infty} z(t) = 0$. This completes the proof of Lemma 1.

Lemma 2. *Let conditions* (A) *and* (B) *hold and let the function* $\tau_1(t,s)$ *be nondecreasing in* s. *Then, if* $x(t) \in \Omega^+$, *then* $z(t) > 0$, $z'(t) \geq 0$, $z''(t) \leq 0$ *eventually and* $\lim_{t\to\infty} z(t) > 0$.

Proof. From $x(t) \in \Omega^+$ and (2) it follows that $z(t) > 0$ eventually. From $x(t) \in \Omega^+$, (3) and B3 it follows that $z''(t) \leq 0$ eventually. Suppose that $z'(t) < 0$ eventually, but then from $z''(t) \leq 0$ eventually it follows that $\lim_{t\to\infty} z(t) = -\infty$, which contradicts $z(t) > 0$ eventually. Hence $z'(t) \geq 0$ eventually which implies that $z(t)$ is a nondecreasing function and since $z(t) > 0$ eventually, then $\lim_{t\to\infty} z(t) > 0$. This completes the proof of the lemma.

3. Main results.

Theorem 1. *Let conditions* (A), (B), (4) *hold and let the function* $\tau_1(t,s)$ *be nonincreasing in* s. *Let*

$$\tau_1(t,\sigma(t)) \geq \rho > -1. \tag{8}$$

Then for each nonoscillating eventual solution $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t) \in \Omega^+$. From Lemma 1 it follows that $z(t) < 0$ eventually. Suppose that $\limsup_{t\to\infty} x(t) = \mathcal{C} > 0$. Then there exists an increasing sequence $\{t_n\}_1^\infty$ such that $\lim_{n\to\infty} t_n = +\infty$, $\lim_{n\to\infty} x(t_n) = \mathcal{C}$. From $z(t) < 0$ and (2) it follows that

$$0 > x(t_n) + \int_0^{\sigma(t_n)} x(t_n - s)d\tau_1(t_n, s) \geq x(t_n) + \max_{[t_n-\sigma(t_n),t_n]} x(s)\cdot\tau_1(t_n,\sigma(t_n))$$
$$\geq x(t_n) + \rho\cdot \max_{[t_n-\sigma(t_n),t_n]} x(s).$$

Hence the inequality $x(t_n) < (-\rho)\cdot \max_{[t_n-\sigma(t_n),t_n]} x(s)$ holds. Let us pass to the limit in this inequality as $n\to\infty$. From A3 it follows that

$$\lim_{n\to\infty} \max_{[t_n-\sigma(t_n),t_n]} x(s) = \mathcal{C}$$

and we obtain that $\mathcal{C} \leq (-p)\cdot\mathcal{C}$. Hence $p \leq -1$, which contradicts (8). Consequently, $\mathcal{C} = 0$ and then $\lim_{t\to\infty} x(t) = 0$. If $x(t)$ is an eventually negative solution, then $-x(t) \in \Omega^+$ due to the linearity of equation (1), that is why in this case as well $\lim_{t\to\infty} x(t) = 0$. This completes the proof of Theorem 1.

Theorem 2. *Let conditions* (A), (B), (4) *and* (5) *hold and let the function* $\tau_1(t,s)$ *be nonincreasing in* s *for* $s \in [0,\sigma(t)]$. *Then each unbounded solution of* (1) *oscillates.*

Proof. It suffices to show that equation (1) has no unbounded nonoscillating solutions. If $x(t) \in \Omega^+$, this follows for Lemma 1. If $x(t)$ is an eventually negative solution of (1), then $-x(t) \in \Omega^+$ and, consequently, in this case as well $x(t)$ is a bounded function. This completes the proof of Theorem 2.

The case when $p \leq \tau_1(t,\sigma(t)) \leq q < -1$ will be considered only if the function $\tau_1(t,s)$ has the special form $\tau_1(t,s) = p(t)\cdot e(s-\tau(t))$, where $e(t) = \{ {}^{0\ -\infty<t\leq 0}_{1\ 0<t<+\infty}$. Then equation (1) takes the form

$$(1') \qquad \frac{d^2}{dt^2}[x(t)+p(t)\cdot x(t-\tau(t))] + \int_0^{\sigma(t)} x(t-s)d\tau_2(t,s) = 0$$

and respectively

$$(2') \qquad z(t) = x(t) + p(t),\ x(t-\tau(t)).$$

Theorem 3. *For the function* $\tau_2(t,s)$ *let conditions* (B) *and* (4) *hold, and for* $\sigma(t)$ *so do conditions* (A). *Let* $p(t) \in C([t_0,+\infty),\mathcal{R})$ *and*

$$(9) \qquad p \leq p(t) \leq q < -1.$$

Let $\tau(t) \in C([t_0,+\infty),\mathcal{R}^+)$ *and* $\lim_{t\to\infty}(t-\tau(t)) = +\infty$.

Then for each nonoscillating bounded eventual solution of (1′)

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. Without loss of generality we assume that $x(t) \in \Omega^+$. First we shall show that in this case as well the assertions of Lemma 1 are valid. In fact all conditions under which Lemma 1 was proved are met, only condition (5) is replaced by condition (9). Just as in the proof of Lemma 1, from the assumption that z' is not an eventually positive function it follows that $x(t)$ is an unbounded function which contradicts the condition of the theorem. In the proof of the assertions $\lim_{t\to\infty} z'(t) = 0$ and $z(t) < 0$ eventually we did not use condition (5). Hence they are valid in this case as well. It remains to show that $\lim_{t\to\infty} z(t) = 0$. Suppose that this is not true, i.e. $\lim_{t\to\infty} z(t) = -\ell$ $(\ell < 0)$. From $z' > 0$ eventually it follows that $z(t)$ is an eventually increasing function. Hence $z(t) < -\ell$. Then from (2′) and (9) there follows the estimate

$$x(t) + p \cdot x(t - \tau(t)) \leq x(t) + p(t) \cdot x(t_1 - \tau(t)) < -\ell$$

In the proof of Lemma 1 we showed (without having used condition (5)) that $\lim_{t \to \infty} \inf x(t) = 0$. Then, since $\tau(t)$ is a continuous function and $\lim_{t \to \infty} (t - \tau(t)) = +\infty$, we can choose a sequence $\{t_n\}_1^\infty$ such that

$$\lim_{n \to \infty} t_n = +\infty, \ \lim_{n \to \infty} (t_n - \tau(t_n)) = +\infty \text{ and } \lim_{n \to \infty} x(t_n - \tau(t_n)) = 0.$$

Then we can choose a number N such that for $n \geq N$ we should have $x(t_n - \tau(t_n)) < \frac{-\ell}{2p}$. Then from the inequality $x(t) + px(t - \tau(t)) < -\ell$ we obtain that $x(t_n) < \frac{-\ell}{2}$ for $n \geq N$ and since $\lim_{n \to \infty} t_n = +\infty$, this inequality contradicts the fact that $x(t) \in \Omega^+$. Hence $\lim_{t \to \infty} z(t) = 0$. Suppose that $\lim_{t \to \infty} \sup x(t) = C > 0$. Then we can choose an increasing sequence $\{\bar{t}_n\}^\infty$, such that $\lim_{n \to \infty} \bar{t}_n = +\infty$, $\lim_{n \to \infty} (\bar{t}_n - \tau(\bar{t}_n)) = +\infty$ and $\lim_{n \to \infty} x(\bar{t}_n - \tau(\bar{t}_n)) = C$. Then from (2′) and (9) there follows the estimate

$$z(\bar{t}_n) = x(\bar{t}_n) + p(\bar{t}_n) \cdot x(\bar{t}_n - \tau(\bar{t}_n)) \leq x(\bar{t}_n) + q \cdot x(\bar{t}_n - \tau(\bar{t}_n)).$$

Let $\{\bar{t}_{n_k}\}$ be a subsequence of the sequence $\{t_n\}$ tending to ℓ, the upper accumulation point of $\{x(\bar{t}_n)\}_1^\infty$. Then $z(\bar{t}_{n_k}) \leq x(\bar{t}_{n_k}) + q \cdot x(\bar{t}_{n_k} - \tau(\bar{t}_{n_k}))$. Let us pass to the limit in this inequality as $k \to \infty$. We obtain that $0 \leq \ell + q \cdot C \leq C + q \cdot C$. Hence $C \geq (-q) \cdot C > C$. The contradiction obtained shows that $\lim_{t \to \infty} \sup x(t) = 0$ and hence $\lim_{t \to \infty} x(t) = 0$. This completes the proof of Theorem 3.

The above results concern the case when the function $\tau_1(t, s)$ is nonincreasing in s. Further on in the work we shall consider the function $\tau_1(t, s)$ nondecreasing in s.

Theorem 4. *Let conditions* (A), (B) *hold and let* $\tau_2(t, \sigma(t_1)) \not\equiv 0$ *in each half-interval* $[\bar{t}, +\infty)$. *Let* $\tau_1(t, s)$ *is nondecreasing in* s. *Then, if* $x(t)$ *is a nonoscillating eventual solution of* (1), *then* $|x(t)| \leq C \cdot t$ *eventually in* t *for some constant* $C > 0$.

Proof. Without loss of generality let $x(t) \in \Omega^+$. Then from Lemma 2 it follows that $z(t) > 0$, $z'(t) \geq 0$ and $z''(t) \leq 0$ eventually. From the fact that $\tau_2(t,\sigma(t)) \not\equiv 0$ in each half-interval $[\tilde{t},+\infty)$ it follows that $z''(t) \not\equiv 0$ eventually. Then $z'(t) > 0$ eventually. Integrate the inequality $z''(t) \leq 0$ twice from t_1 to t, where t_1 is large enough. We obtain that $z(t) \leq z(t_1) + z'(t_1)(t-t_1)$. If t_1 is large enough, then $z'(t_1) > 0$ and there exists a constant $C > 0$ such that $z(t) \leq C \cdot t$ eventually. From (2) and the fact that the function $\tau_1(t,s)$ is nondecreasing in s it follows that $x(t) \leq C \cdot t$ eventually. This completes the proof of Theorem 4.

Theorem 5. *Let conditions* (A), (B) *and* (4) *hold. Let* $\tau_1(t,s)$ *be nondecreasing in* s. *Then each nonoscillating eventual solution of* (1) *enjoys the property* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Without loss of generality we shall assume that $x(t) \in \Omega^+$. Suppose that $\liminf_{t\to\infty} x(t) = a > 0$. Integrate (3) from t_1 to t, where t_1 is large enough, $t_1 > t_0$ and obtain

$$z'(t) = z'(t_1) - \int_{t_1}^{t}\int_{0}^{\sigma(v)} x(v-s)d\tau_2(v,s)dv \leq z'(t_1) - a\int_{t_1}^{t} \tau_2(v,\sigma(v))dv.$$

Hence $\int_{t_1}^{t} \tau_2(v,\sigma(v))dv \leq \frac{1}{a}[z'(t_1) - z'(t)]$. From Lemma 2 it follows that $z'(t) \geq 0$ eventually and $z'(t)$ is an eventually nonincreasing function. Then there exists the finite limit $\lim_{t\to\infty} z'(t)$ which implies that the right-hand side of the above inequality has a finite limit as $t \to \infty$, which contradicts (4). This completes the proof of Theorem 5.

Corollary 1. *Under the conditions of* Theorem 5 *let, moreover, the following inequality hold*

$$\sup_{t\geq t_0} \tau_1(t,\sigma(t)) < \infty. \tag{10}$$

Then each nonoscillating eventual solution $x(t)$ *of* (1) *enjoys the property* $\limsup_{t\to\infty} |x(t)| > 0$.

Proof. Without loss of generality we shall assume that $x(t) \in \Omega^+$. Suppose that the assertion is not true, i.e. $\limsup_{t\to\infty} x(t) = 0$. Then from Theorem 5 it follows that $\lim_{t\to\infty} x(t) = 0$. From the estimate

$$z(t) = x(t) + \int_0^{\sigma(t)} x(t-s)d\tau_1(t,s) \le x(t) + \max_{[t-\sigma(t),t]} x(s) \cdot \tau_1(t,\sigma(t))$$

and from (11) it follows that $\lim_{t\to\infty} z(t) = 0$, which contradicts Lemma 2. This completes the proof of Corollary 1.

Remark 2. It is immediately seen that if conditions (4) and (10) are met simultaneously, then no eventual solution of (1) can be eventually monotone.

In view of the result of Corollary 1, it is natural to ask what can be said about the behaviour of the nonoscillating solutions when $\sup_{t>t_0} \tau_1(t,\sigma(t)) = +\infty$. A partial answer to this question is given by the following theorem. We shall formulate the theorem about the case when $\tau_1(t,s) = p(t)\cdot\tau(s)$. Then the equation takes the form

$$(1'') \qquad \frac{d^2}{dt^2}\left[x(t) + p(t)\int_0^a x(t-s)d\tau(s)\right] + \int_0^{\sigma(t)} x(t-s)d\tau_2(t,s) = 0.$$

Theorem 6. *For the function $\tau_2(t,s)$ let conditions* (B) *and* (4) *hold and for the function $\sigma(t)$-conditions* (A). *Let $\tau(0) = 0$ and let $\tau(s)$ be a nondecreasing function with at least one point of discontinuity and $p(t) \in C((t_0,+\infty),\mathcal{R})$. Then the following assertions are valid:*

(i) *if $\frac{t}{p(t)}$ is bounded, then each nonoscillating eventual solution of* $(1'')$ *is bounded.*

(ii) *if $\lim_{t\to\infty} \frac{t}{p(t)} = 0$, then for any nonoscillating eventual solution $x(t)$ of* $(1'')$ *we have $\lim_{t\to\infty} x(t) = 0$.*

Proof. Without loss of generality assume that $x(t) \in \Omega^+$. We shall prove (i). From the proof of Theorem 4 it follows that

$$z(t) = x(t) + p(t)\cdot\int_0^a x(t-s)d\tau(s) \le C\cdot t$$

for some constant $C > 0$. Hence

$$p(t)\cdot\int_0^a x(t-s)d\tau(s) \le C\cdot t.$$

Then

$$\int_0^a x(t-s)d\tau(s) \le C \cdot \frac{t}{p(t)}$$

and we obtain that the function $\int_0^a x(t-s)d\tau(s)$ is bounded. Suppose that $x(t)$ is an unbounded function. Then there exists an increasing sequence $\{t_n\}_1^\infty$ such that $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{n\to\infty} x(t_n) = +\infty$. Let $\tau(s)$ have a discontinuity at the point $c \in (0,a)$ and let the magnitude of the jump of the function be δ. Consider the sequence $\{\tau_n\}_1^\infty$, $\tau_n = \int_0^a x(t_n + c - s)d\tau(s)$. From the very definition of the Riemann-Stieltjes integral there follows the estimate $\int_0^a x(t_n + c - s)d\tau(s) \ge x(t_n) \cdot \delta$. Hence the sequence $\{\tau_n\}_1^\infty$ is unbounded, which contradicts the boundedness of the function $\int_0^a x(t-s)d\tau(s)$. Thus assertion (i) of Theorem 6 is proved. The proof of (ii) is carried out by the same scheme. As above,

$$\int_0^a x(t-s)d\tau(s) \le C \cdot \frac{t}{p(t)}$$

which implies that $\lim_{t\to\infty} \int_0^a x(t-s)d\tau(s) = 0$. Suppose that $\lim_{t\to\infty} \sup x(t) = c > 0$. Then there exists an increasing sequence $\{\tilde{t}_n\}_1^\infty$ such that $\lim_{n\to\infty} \tilde{t}_n = +\infty$ and $\lim_{n\to\infty} x(\tilde{t}_n) = c$. Again as above consider the respective sequence $\{\tau_n\}_1^\infty$, and obtain the estimate

$$\int_0^a x(\tilde{t}_n + c - s)d\tau(s) \ge x(\tilde{t}_n) \cdot \delta > \frac{c}{2} \cdot \delta$$

for n large enough, which contradicts $\lim_{t\to\infty} \int_0^a x(t-s)d\tau(s) = 0$. This completes the proof of Theorem 6.

Remark 3. Theorem 6 is also valid for equation (1) if $\tau_1(t,s)$ satisfies the following requirements: $\tau_1(t,s)$ is continuous in t, there exists t_1 such that for $t \ge t_1$ in each interval $[t-\sigma(t), t]$, $\tau_1(t,s)$ has at least one point of discontinuity s_t and the set of the magnitudes of the jumps at all points of discontinuity of $\tau_1(t,s)$ is bounded from below by the constant $\alpha > 0$. The proof is carried out in the same way.

Theorem 7. *Let conditions* (A), (B) *hold, let* $\tau_1(t,s)$ *be nondecreasing in s and* $\tau_1(t,\sigma(t)) \le 1$ *eventually in t. Let the following relation be valid*

$$\int_{t_1}^{\infty}\int_0^{\sigma(t)}[1-\tau_1(t-s,\sigma(t-s))]d\tau_2(t,s)dt = +\infty. \tag{11}$$

Then each eventual solution $x(t)$ *of* (1) *oscillates.*

Proof. Suppose that this is not true and let $x(t)$ be a nonoscillating solution of (1). Without loss of generality let $x(t) \in \Omega^+$. Then from (2) and Lemma 2 it follows that eventually the following inequality holds

$$x(t) \ge z(t) - \int_0^{\sigma(t)} z(t-s)d\tau_1(t,s) \ge [1-\tau_1(t,\sigma(t))]\cdot z(t).$$

Then from (3), making use of $x(t) \ge [1-\tau_1(t,\sigma(t))]\cdot z(t)$ it follows that eventually the following inequality holds

$$z''(t) + \int_0^{\sigma(t)}[1-\tau_1(t-s,\sigma(t-s))]\cdot z(t-s)d\tau_2(t,s) \le 0.$$

Integrate this inequality from t_1 to t, where t_1 is large enough and obtain

$$z'(t) + \int_{t_1}^{t}\int_0^{\sigma(v)}[1-\tau_1(v-s,\sigma(v-s))]\cdot z(v-s)d\tau_2(v,s)dt \le z'(t_1).$$

From Lemma 2 it follows that there exists a constant $a > 0$ such that $z(t) \ge a$ eventually. Then

$$z'(t) + a\int_{t_1}^{t}\int_0^{\sigma(v)}[1-\tau_1(v-s,\sigma(v-s))]d\tau_2(v,s)dt \le z'(t_1).$$

Passing to the limit in this inequality as $t \to +\infty$ and taking into account Lemma 2, we obtain that

$$\int_{t_1}^{\infty}\int_0^{\sigma(v)}[1-\tau_1(t-s,\sigma(t-s))]d\tau_2(t,s)dt < +\infty$$

which contradicts (11). Theorem 7 is thus proved.

Theorem 8. *Let conditions* (A), (B) *and* (4) *hold. Let* $\tau_1(t,s)$ *be nondecreasing in s and* $\tau_1(t,\sigma(t)) \le p < 1$. *Then each eventual solution* $x(t)$ *of equation* (1) *oscillates.*

Proof. Suppose that this is not true, i.e. there exists an eventual solution $x(t)$ of (1) which is nonoscillating. Without loss of generality let $x(t) \in \Omega^+$. Then as in the proof of Theorem 7 we conclude that eventually the inequality $x(t) \geq [1 - \tau_1(t, \sigma(t))] \cdot zt$ holds. From this and from (3) we obtain that

$$z''(t) + \int_0^{\sigma(t)} [1 - \tau_1(t-s, \sigma(t-s))] \cdot z(t-s) d\tau_2(t,s) \leq 0.$$

From Lemma 2 it follows that there exists a constant $a > 0$ such that $z(t) \geq a$ eventually. Then from the above inequality it follows that

$$z''(t) + (1-p) \cdot a \cdot \tau_2(t, \sigma(t)) \leq 0.$$

Integrate this inequality from t_1 to t, where t_1 is large enough and obtain

$$z'(t) - z'(t_1) + (1-p) \cdot a \cdot \int_{t_1}^{t} \tau_2(v, \sigma(v)) dv \leq 0.$$

Passing to the limit in this inequality as $t \to \infty$ and taking into account Lemma 2, we obtain that $\int_{t_1}^{\infty} \tau_2(v, \sigma(v)) dv < \infty$, which contradicts (4). Theorem 8 is thus proved.

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