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ON THE BIFURCATION OF PERIODIC ORBITS OF SOME GENERALIZED HÉNON MAPPINGS

BY

BAU-SEN DU (杜寶生)

Abstract. In this note, we introduce some conservative diffeomorphisms H_A and K_A on the plane which contains the following types of conservative Hénon mappings as special cases: $F_A(x, y) = (A-y-x^2, x)$ and $G_A(x, y) = (A+y-x^2, x)$. We study the bifurcations of periodic orbits of H_A and K_A with some symmetry properties. For the special types of Hénon mappings F_A and G_A, we show that when we consider them as one-parameter families of diffeomorphisms on the plane with A as the parameter, the bifurcations of the first periodic orbits (fixed points in these cases) are more complicated than we expect.

1. Generalized conservative Hénon mappings. Let $v(x)$ be any fixed polynomial all of whose terms are of even degrees ≥ 2 and let $w(x)$ be any fixed polynomial whose leading term is of even degree ≥ 2. For any real number A, let $H_A(x, y) = (A-y-v(x), x)$ and $K_A(x, y) = (A-y-w(x), x)$. Then it is easy to see that both H_A and K_A are diffeomorphisms of the plane whose inverses are also given by polynomials. Furthermore, the Jacobian determinant of H_A is 1 and that of K_A is -1. So, for every A, H_A is a conservative orientation-preserving diffeomorphism and K_A is a conservative orientation-reversing diffeomorphism. We call attention to the fact that the following types [2-5] of Hénon mappings $F_A(x, y) = (A - y - x^2, x)$ and $G_A(x, y) = (A + y - x^2, x)$ are special cases of H_A and K_A respectively. These mappings will be treated in §2 and §3 separately.

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For the diffeomorphisms H_A and K_A, we have some results on the bifurcation of their periodic orbits with periods ≥ 3. These periodic orbits all have some symmetry property. Before we state these results, we need to define some real mappings first. For every positive integer n and any real number c, let $H_A^n(c,c) = (x_n(A,c), y_n(A,c))$ and $K_A^n(c,c) = (z_n(A,c), u_n(A,c))$. Also, let

$$g_{v,n}(c) = \frac{\partial}{\partial A}[x_n(A,c) - x_{n-1}(A,c)]_{|A=2c+v(c)},$$
$$h_{v,n}(c) = \frac{\partial}{\partial A}[x_n(A,c) - x_{n-2}(A,c)]_{|A=2c+v(c)},$$
$$p_{w,n}(c) = \frac{\partial}{\partial A}[z_n(A,c) - z_{n-1}(A,c)]_{|A=2c+w(c)}.$$

We can now state and prove our main results in this section.

Theorem 1.1. *For the diffeomorphism H_A, the following hold.*

(a) *Let $n \geq 2$ be a fixed integer and let $g_{v,n}(x)$ be defined as above. Then, for every (real) zero c of $g_{v,n}(x)$, there is a bifurcation of periodic orbit of period (may not be minimal) $2n$ for H_A at $A = 2c + v(c)$ from fixed points of H_A. If c is the largest zero of $g_{v,n}(x)$, then the bifurcated periodic orbit of H_A mentioned above has minimal period $2n$. Furthermore, all these bifurcated periodic orbits of H_A are symmetric with respect to the diagonal line $y = x$.*

(b) *Let $n \geq 2$ be a fixed integer and let $h_{v,n}(x)$ be defined as above. Then, for every (real) zero c of $h_{v,n}(x)$, there is a bifurcation of periodic orbit of period (may not be minimal) $2n-1$ for H_A at $A = 2c + v(c)$ from fixed points of H_A. If c is the largest zero of $h_{v,n}(x)$, then the bifurcated periodic orbit of H_A mentioned above has minimal period $2n-1$. Furthermore, all these bifurcated periodic orbits of H_A are symmetric with respect to the diagonal line $y = x$.*

Proof. For every positive integer n, let $H_A^n(c,c) = (x_n(A,c), y_n(A,c))$. By direct computation, we easily obtain that, if $x_n = x_{n-1}$, then $x_{n+1} = x_{n-2}$, $x_{n+2} = x_{n-3}, \ldots, x_{2n-2} = x_1 = A - c - v(c)$, $x_{2n-1} = x_0 = c$. Consequently, $H_A^{2n}(c,c) = (c,c)$ and the orbit of (c,c) under H_A is symmetric with respect to the diagonal line $y = x$. Since, for some polynomial $Q_n(A, c)$ in A and c, we have $x_n - x_{n-1} = (A - 2c - v(c))[Q_n(A, c)(A - 2c - v(c)) + g_{v,n}(c)]$, we obtain that $(x_n - x_{n-1})/(A - 2c - v(c)) = Q_n(A, c)(A - 2c - v(c)) + g_{v,n}(c)$ which, for every fixed real number c, is a polynomial in A of odd degree. Hence, the equation $(x_n - x_{n-1})/(A - 2c - v(c)) = 0$ has, for every fixed c, some real solutions in A and these solutions depend continuously on c. If we choose c to be any fixed real zero of $g_{v,n}(c)$ and let $A = 2c + v(c)$, then this point (A, c) satisfies the equation $(x_n - x_{n-1})/(A - 2c - v(c)) = 0$. So, these period $2n$ (may not be minimal, but > 1) points are bifurcated from the fixed points of H_A. If c^* is the largest real zero of $g_{v,n}(c)$ and $A = 2c^* + v(c^*)$, then it is easy to see that the bifurcated periodic orbit of H_A at $A = 2c^* + v(c^*)$ has minimal period $2n$. This completes the proof of part (a). Part (b) can be proved similarly.

Theorem 1.2. *For the diffeomorphism K_A, the following hold.*

(a) *Let n be any fixed positive integer and let $p_{w,n}(x)$ be defined as above. Then, for every (real) zero c of $p_{w,n}(x)$, there is a bifurcation of periodic orbit of period (may not be minimal) $2n$ for K_A at $A = w(c)$ from symmetric period 2 orbit of K_A. If c is the largest zero of $p_{w,n}(x)$, then the bifurcated periodic orbit of K_A has minimal period $2n$. Furthermore, all these bifurcated periodic orbits of K_A are symmetric with respect to the diagonal line $y = -x$.*

(b) *For every odd integer $m \geq 3$, there is a real number A_m such that the point $(0, -A_m/2)$ is a periodic point of K_{A_m} with minimal period m whose orbit is symmetric with respect to the diagonal line $y = -x$.*

Proof. We only give a proof of part (b). For every integer $n \geq 2$, let $K_A^n(0, -A/2) = (s_n(A), t_n(A))$. If $s_n = -s_{n-1}$, then, by direct computation, we obtain $s_{n+1} = -s_{n-2}$, $s_{n+2} = -s_{n-3}, \ldots, s_{2n-3} = -s_2 = -(A - A^2/4)$, $s_{2n-2} = -s_1 = -A/2$. Consequently, $K_A^{2n-1}(0, -A/2) = (0, -A/2)$. Since the equation $s_n - s_{n-1} = 0$ is a polynomial equation in A of event degree with zero constant term, we see that the equation $s_n - s_{n-1} = 0$ has nonzero real solutions in A. If A_n^* is the largest real zero of $s_n - s_{n-1} = 0$, then we easily see that the point $(0, -A_n^*/2)$ is a periodic point of $K_{A_n^*}$ with minimal period $2n-1$ whose orbit is symmetric with respect to the line $y=-x$.

2. The conservative orientation-preserving Hénon mapping. In this section we consider the conservative orientation-preserving Hénon mapping $F_A(x,y) = (A - x^2 - y, x)$. This mapping is exactly the mapping H_A discussed in §1 with $v(x) = x^2$. We can find the explicit formulas for all periodic points of F_A with minimal periods 1, 2, 3, 4 and some periodic points with minimal period 6. The following result can be verified by direct computation.

Theorem 2.1. *For the diffeomorphism F_A, the following hold.*

(1) *For any $A < -1, F_A$ has no periodic point.*

(2) *For all $A \geq -1$, $(-1+\sqrt{A+1}, -1+\sqrt{A+1})$ and $(-1-\sqrt{A+1}, -1-\sqrt{A+1})$ are the only fixed points of F_A.*

(3) *For all $A > 3$, $\{(1+\sqrt{A-3}, 1-\sqrt{A-3}), (1-\sqrt{A-3}, 1+\sqrt{A-3})\}$ is the only period 2 orbit of F_A. It is bifurcated from the branch $(-1+\sqrt{A+1}, -1+\sqrt{A+1})$ of fixed point of F_A at $A=3$.*

(4) *For all $A > 1$, $\{(\sqrt{A-1}, \sqrt{A-1}), (1-\sqrt{A-1}, \sqrt{A-1}), (\sqrt{A-1}, 1-\sqrt{A-1})\}$ and $\{(-\sqrt{A-1}, -\sqrt{A-1}), (1+\sqrt{A-1}, -\sqrt{A-1}), (-\sqrt{A-1}, 1+\sqrt{A-1})\}$ are the only period 3 orbits of F_A. They are bifurcated spontaneously at $A=1$ and the first branch intersects the branch $(-1+\sqrt{A+1}, -1+\sqrt{A+1})$ of fixed point of F_A at $A=5/4$.*

(5) (a) *For all $A > 0$, let $B = A + 2\sqrt{A}$. Then $\{(\sqrt{A}, \sqrt{A}), (-\sqrt{A}, \sqrt{A}), (\sqrt{A}, -\sqrt{A}), (-\sqrt{A}, -\sqrt{A})\}$ and $\{(-\sqrt{A}, \sqrt{B}), (-\sqrt{B}, -\sqrt{A}), (-\sqrt{A}, -\sqrt{B}), (\sqrt{B}, -\sqrt{A})\}$ are period 4 orbits of F_A. They are bifurcated from the branch $(-1+\sqrt{A+1}, -1+\sqrt{A+1})$ of fixed point of F_A at $A=0$.*

(b) *For all $A > 4$, let $C = A - 2\sqrt{A}$. Then $\{(\sqrt{A}, -\sqrt{C}), (-\sqrt{C}, \sqrt{A}), (\sqrt{A}, \sqrt{C}), (-\sqrt{C}, \sqrt{A})\}$ is a period 4 orbit of F_A. It is bifurcated from the branch $\{(1+\sqrt{A-3}, 1-\sqrt{A-3}), (1-\sqrt{A-3}, 1+\sqrt{A-3})\}$ of period 2 orbit of F_A at $A=4$.*

(6) (a) *For all* $A > -3/4$, *let* $D = 1 + 4(A + \sqrt{A+1})$. *Then* $\{(-\sqrt{A+1}, (-1-\sqrt{D})/2), ((-1+\sqrt{D})/2, -\sqrt{A+1}), ((-1+\sqrt{D})/2, (-1+\sqrt{D})/2), (-\sqrt{A+1}, (-1+\sqrt{D})/2), ((-1-\sqrt{D})/2, -\sqrt{A+1}), ((-1-\sqrt{D})/2, (-1-\sqrt{D})/2)\}$ *is a period* 6 *orbit of* F_A. *It is bifurcated from the branch* $(-1+\sqrt{A+1}, -1+\sqrt{A+1})$ *of fixed point of* F_A *at* $A = -3/4$.

(b) *For all* $A > 5/4$, *let* $E = 1 + 4(\sqrt{A} - \sqrt{A-1})$. *Then* $\{(\sqrt{A+1}, (-1-\sqrt{E})/2), ((-1+\sqrt{E})/2, \sqrt{A+1}), ((-1+\sqrt{E})/2, (-1+\sqrt{E})/2), (\sqrt{A+1}, (-1+\sqrt{E})/2), ((-1-\sqrt{E})/2, \sqrt{A+1}), ((-1-\sqrt{E})/2, (-1-\sqrt{E})/2)\}$ *is a period* 6 *orbit of* F_A. *It is bifurcated from the branch* $\{(-\sqrt{A-1}, -\sqrt{A-1}), (1+\sqrt{A-1}, -\sqrt{A-1}), (-\sqrt{A-1}, 1+\sqrt{A-1})\}$ *of period* 3 *orbit of* F_A *at* $A = 5/4$.

Remark. We call attention to the fact as shown in Theorem 2.1(4) and (6b) that, for every $A > 1$ and $A \neq 5/4$, F_A has exactly two distinct period 3 orbits. They are bifurcated spontaneously at $A = 1$, and at $A = 5/4$, one of them coalences into and then separates from the branch $\{(-1+\sqrt{A+1}, -1+\sqrt{A+1})\}$ of fixed point of F_A while the other undergoes a period-doubling bifurcation. This provides an example of period 3 to period 3 bifurcation (see also [6]).

Let $g_n(x) = g_{v,n}(x)$ and $h_n(x) = h_{v,n}(x)$ with $v(x) = x^2$. As indicated in Theorem 1.1, the zeros of the mapping g_n (h_n resp.) are the bifurcation values of some periodic orbits of H_A (K_A resp.) with periods ≥ 3. These mappings g_n and h_n can also be obtained by some simple recurrence relations which are shown in the following result. This result also includes some relations and properties of g_n's and h_n's which can be easily proved by induction.

Theorem 2.2. *Let* $g_1(x) \equiv 1$, $g_2(x) = -2x$, $h_1(x) \equiv -1$, $h_2(x) = 2x - 1$, *and for all integers* $n \geq 3$, *let* $g_n(x) = -2xg_{n-1}(x) - g_{n-2}(x)$ *and* $h_n(x) = -2xh_{n-1}(x) - h_{n-2}(x)$. *Then the following hold.*

(1) *For every integer* $n \geq 0$, *we have*

$$g_{2n+2}(x) = -2x \cdot \sum_{k=0}^{n} (-1)^k \binom{2n-k+1}{k} (4x^2)^{n-k} \text{ and}$$

$$g_{2n+3}(x) = \sum_{k=0}^{n+1} (-1)^k \binom{2n-k+2}{k} (4x^2)^{n-k+1}$$

(2) $g_m(x) = g_{k+1}(x)g_{m-k}(x) - g_k(x)g_{m-k-1}(x)$ *for* $1 \le k < m$.

(3) $h_m(x) = g_{k+1}(x)h_{m-k}(x) - g_k(x)h_{m-k-1}(x)$ *for* $1 \le k < m$.

(4) *For all positive integers* m *and* n *with* m *dividing* n, $g_m(x)$ *divides* $g_n(x)$.

(5) $g_{2k+1}(x) = [g_{k+1}(x)]^2 - [g_k(x)]^2 = [-g_{k+1}(x) - g_k(x)][-g_{k+1}(x) + g_k(x)]$.

(6) *For every positive integer* k, $h_{k+1}(x) = -g_{k+1}(x) - g_k(x)$ *and* $h_{k+1}(x)$ *divides* $g_{2k+1}(x)$.

(7) *For all positive integers* k *and* $n, h_n(x)$ *divides* $h_{(2n-1)k+n}(x)$.

(8) *For every integer* $n \ge 2, g_n(x)$ *has exactly* $n-1$ *distinct real zeros and these zeros all lie in the interval* $(-1, 1)$. *Furthermore, between any two consecutive zeros of* $g_n(x)$, *there is a zero of* $g_{n+1}(x)$, *and vice versa.*

Proof. We only give a sketchy proof of part (8). When $n = 2$, the result is clear. So, assume that $n \ge 3$. Now, by definition, we have $g_n(x) = -2xg_{n-1}(x) - g_{n-2}(x)$. Consequently, it follows, by induction, that at the zeros of $g_{n-1}(x)$, the signs of $g_n(x)$ (which are determined by $g_{n-2}(x)$) change alternatively. This, together with the sign of $g_n(1)$ and that of $g_n(0)$ or that of $[g_n(x)/x]_{|x=0}$ depending on whether n is odd or even, implies the desired result.

Remark. It seems that the zeros of all the h_n's are dense in the interval $(-1, 1)$ (see [1, 3]). However, we are unable to show this. Instead, we show in the following that $A = -1$ is a limit point of the set of the zeros of all the h_n's. In the meantime, we also show that $A = 0$ is a limit point of the set of the zeros of all the g_n's.

Theorem 2.3. *For every positive integer* k, *let* $g_k(x)$ *and* $h_k(x)$ *be defined as in Theorem 2.2. Then the following hold.*

(a) *For every integer* $n \ge 7$, $h_n(-1+6/[(n-1)n])h_n(-1) < 0$. *Consequently,* $A = -1$ *is a left accumulation point of bifurcations of minimal period* $2m-1$ *points of* F_A *for infinitely many positive integers* m.

(b) *For every positive integer n, $g_{2n+2}(1/\sqrt{(n+1)(n+2)})g_{2n+2}(0) < 0$ and $g_{2n+3}(1/\sqrt{(n+1)(n+2)})g_{2n+3}(0) < 0$. Consequently, $A = 0$ is a left accumulation point of bifurcations of period $4m$ points of F_A for infinitely many positive integers m.*

Proof. (a) For every positive integer n, we have

$$\begin{aligned} h_n(-1+\epsilon/2) &= (2n-1)\left\{-1+\sum_{m=1}^{n-1}\left[(-1)^{m+1}\frac{(n+m-1)!}{(2m+1)!}\right]\epsilon^m\right\} \\ &= (2n-1)\left\{-1+\frac{n(n-1)}{3!}\epsilon-\frac{(n+1)n(n-1)(n-2)}{5!}\epsilon^2\right. \\ &\quad \left.+\frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{7!}\epsilon^3-\dots\right\} \end{aligned}$$

So, by taking $\epsilon = 6/[(n-1)n]$ with $n \geq 7$, we obtain that $h_n(-1+6/[(n-1)n]) > .025(2n-1)$ for all $n \geq 7$. Consequently, $h_n(-1+6/[(n-1)n])h_n(-1) < 0$ for all $n \geq 7$. Therefore, if $n \geq 7$ is a fixed integer, then there is a bifurcation in $(-1, -1+6/[(n-1)n])$ of periodic orbit of F_A of period $2k-1$ (may not be minimal) for every positive integer $k \geq n$. In particular, the point $A = -1$ is a left accumulation point of bifurcations of periodic orbits of F_A of minimal period $2m-1$ for infinitely many positive integers m.

(b) For every positive integer n, we have

$$g_{2n+3}(x) = \sum_{m=1}^{n+1}(-1)^m\frac{(2n-m+2)!}{(2n-2m+2)!m!}(2x^2)^{n-m+1}.$$

So, if $x = 1/\sqrt{(n+1)(n+2)}$, then it can be easily shown that, for all integers $n \geq 1$, $|g_{2n+3}(1/\sqrt{(n+1)(n+2)})| > .41$ and $g_{2n+3}(1/\sqrt{(n+1)(n+2)})g_{2n+3}(0) < 0$. On the other hand, for every positive integer n, we have

$$g_{2n+2}(x) = \sum_{m=1}^{n}(-1)^{m+1}\frac{(2n-m+1)!}{(2n-2m+1)!m!}(2x^2)^{n-m}.$$

Consequently, it can also be easily shown that, for all integers $n \geq 1$,

$$|g_{2n+2}(1/\sqrt{(n+1)(n+2)})| > .41 \qquad \text{and}$$

$$g_{2n+2}(1/\sqrt{(n+1)(n+2)})g_{2n+2}(0) < 0.$$

Therefore, for every positive integer n, there is a bifurcation in $(0, 1/\sqrt{(n+1)(n+2)})$ of periodic orbits of F_A of period $4k$ (may not be minimal) for every integer $k \geq 4$. In particular, the point $A = 0$ is a left accumulation point of bifurcations of periodic orbits of F_A of period $4m$ for infinitely many positive integers m.

3. The conservative orientation-reversing Hénon mapping. In this section we consider the conservative orientation-reversing Hénon mapping $G_A(x, y) = (A - x^2 + y, x)$. This mapping is exactly the mapping K_A discussed in §1 with $w(x) = x^2$. For this mapping G_A, we have the following easy result which can be proved by direct computation.

Theorem 3.1. *For the mapping G_A, the following hold.*

(1) *For any $A < 0$, the mapping G_A has no periodic point.*

(2) *For $A = 0$, the point $(0,0)$ is a fixed point and there are no other periodic points for G_A.*

(3) *For all $A > 0$, $(\sqrt{A}, \sqrt{A})$ and $(-\sqrt{A}, -\sqrt{A})$ are the only fixed points of G_A and $\{(\sqrt{A}, -\sqrt{A}), (-\sqrt{A}, \sqrt{A})\}$ is the only periodic orbit of G_A of minimal period 2.*

(4) *For all $A > 1$, $\{(-1+\sqrt{A-1}, 1-\sqrt{A-1}), (1+\sqrt{A-1}, -1+\sqrt{A-1}), (-1-\sqrt{A-1}, 1+\sqrt{A-1}), (1-\sqrt{A-1}, -1-\sqrt{A-1})\}$ is a periodic orbit of G_A with minimal period 4 which is bifurcated from the period 2 orbit $\{(\sqrt{A}, -\sqrt{A}), (-\sqrt{A}, \sqrt{A})\}$ of G_A at $A = 1$.*

For every positive integer n, let $p_n(x) = p_{w,n}(x)$ with $w(x) = x^2$. As indicated in Theorem 1.2, the zeros of p_n are the bifurcation values of some periodic orbits of G_A of some even periods ≥ 4. These periodic orbits are all symmetric with respect to the diagonal line $y = -x$. These mappings can also be obtained by some simple recurrence relation as is shown in the following result. This result also lists some relations and properties of p_n's which can be easily proved (by induction).

Theorem 3.2. *Let $p_1(x) \equiv 1$, $p_2(x) = 2x + 2$, and, for all integers $n \geq 3$, let $p_n(x) = p_{n-1}(x)[(-1)^n(2x) + 2] - p_{n-2}(x)$. Then the following* *hold.*

(1) (a) *For all odd integers $n > 1$, we have*

$$p_n(x) = n\left\{1 + \sum_{j=1}^{(n-1)/2} (-1)^j \left(\left(\prod_{m=1}^{j} [n^2 - (2m-1)^2] \right) / [2^{2j}(2j+1)!] \right) x^{2j} \right\}$$

(b) *For all even integers $n > 0$, we have*

$$p_n(x) = [(x+2)n/2]\left\{1 + \sum_{j=1}^{n/2} (-1)^j \left(\left[\prod_{m=1}^{j} (n^2 - 4m^2) \right] / \left[2^{2j}(2j+1)! \right] \right) x^{2j} \right\}.$$

(2) *For all integers $k \geq 2$, we have*

$$p_{2k}(x) = p_2(x) \sum_{j=1}^{k} (-1)^{j+1} p_{2k-2j+1}(x) \qquad \text{and}$$

$$p_{2k+1}(x) = p_3(x) \sum_{j=1}^{k} (-1)^{j+1} p_{2k-2j+1}(x) - \sum_{j=2}^{k} (-1)^{j+1} p_{2k-2j+1}(x).$$

(3) *For all integers k and m with $1 \leq k < m$, we have*

$$p_{2m}(x) = p_{2k+1}(x) p_{2m-2k}(x) - p_{2k}(x) p_{2m-2k-1}(x) \qquad \text{and}$$

$$p_{2m+1}(x) = p_{2k+1}(x) p_{2m-2k-1}(x) - (2 - 2x) p_{2m-2k}(x) \sum_{j=1}^{k} (-1)^{j+1} p_{2k-2j+1}(x).$$

Consequently, if m and n are positive integers with m dividing n, then $p_m(x)$ divides $p_n(x)$.

(4) *For every integer $n \geq 2$, $p_n(x)$ has exactly $n - 1$ distinct real zeros. All these zeros lie in the interval $[-1, 1)$ and those lying in the interval $(-1, 1)$ are symmetric with respect to the origin. Furthermore, when $n \geq 5$, between any two consecutive positive zeros of $p_n(x)$, there is a positive zero of $p_{n+1}(x)$, and vice versa.*

Remark. It seems that the zeros of all the p_n's are dense in the interval $(-1, 1)$. However, we are unable to show this. Instead, we show in the following that $A = 0$ and $A = 1$ are limit points of the set of the zeros of all the p_n's.

Theorem 3.3. *For the one-parameter family $G_A(x,y) = (A-x^2+y,x)$ with A as the parameter, the following hold.*

(a) *$A = 0$ is a left accumulation point of bifurcations of periodic orbits of G_A with period $4m+2$ for infinitely many positive integers m.*

(b) *$A = 1$ is a right accumulation point of bifurcations of periodic orbits of G_A with minimal period $4m$ for infinitely many positive integers m.*

Proof. (a) For every odd integer n, we have

$$p_n(x) = n\Big\{1 + \sum_{j=1}^{(n-1)/2} (-1)^j \Big(\Big(\prod_{k=1}^{j}[n^2 - (2k-1)^2]\Big)/[2^{2j}(2j+1)!]\Big)x^{2j}\Big\}.$$

So, if $x = 2\epsilon/\sqrt{(n-1)(n+1)}$, then

$$\begin{aligned} p_n(x) &= n\Big\{1 - \frac{\epsilon^2}{3!} + \frac{(m^2-9)}{(m^2-1)}\frac{\epsilon^4}{5!} - \frac{(m^2-9)(m^2-25)}{(m^2-1)(m^2-1)}\frac{\epsilon^6}{7!} + \cdots\Big\} \\ &< n\Big\{1 - \frac{\epsilon^2}{3!} + \frac{\epsilon^4}{5!} - \frac{(m^2-9)(m^2-25)}{(m^2-1)(m^2-1)}\frac{\epsilon^6}{7!} + \cdots\Big\} < -0.00041 \end{aligned}$$

for $\epsilon = 3.2$ and all odd integers $n > 20$. Therefore $p_n(6.4/\sqrt{(n-1)(n+1)})$ $p_n(0) < 0$ for all odd integers $n > 20$. Consequently, if $n > 20$ is a fixed odd integer, then there is a bifurcation in $(0, 6.4/\sqrt{(n-1)(n+1)})$ of periodic orbit of G_A of period $4k+2$ (may not be minimal) for every odd integer $k \geq n$. In particular, the point $A = 0$ is a left accumulation point of bifurcations of periodic orbits of G_A of period $4m+2$ for infinitely many positive integers m.

(b) For every positive integer k, let $q_k(x) = p_{2k}(x)/(x+2)$. Then it is easy to see that $q_k(x)$ satisfies the following recursive formula: $q_1(x) \equiv 1$, $q_2(x) = -x^2+2$ and $q_{k+1}(x) = (-x^2+2)q_k(x) - q_{k-1}(x)$. Therefore, if $0 < \epsilon < 4$ and $x = \sqrt{4-\epsilon}$, then $q_1(\sqrt{4-\epsilon}) = 1$, $q_2(\sqrt{4-\epsilon}) = -2+\epsilon$ and

$q_{k+1}(\sqrt{4-\epsilon}) = (-2+\epsilon)q_k(\sqrt{4-\epsilon}) - q_{k-1}(\sqrt{4-\epsilon})$. Consequently, we have

$$\begin{aligned} q_n(\sqrt{4-\epsilon}) &= (-1)^{n-1}q_n(\sqrt{\epsilon}) \\ &= (-1)^{n-1}n\Big(1+\sum_{j=1}^{n-1}(-1)^j\Big\{\prod_{i=1}^{j}(n^2-i^2)/[(2j+1)!]\Big\}\epsilon^j\Big). \end{aligned}$$

Let $\epsilon = x^2/(n^2-1)$. Then

$$\begin{aligned} &1-\frac{x^2}{3!}+\frac{(n^2-4)}{(n^2-1)}\frac{x^4}{5!}-\frac{(n^2-4)(n^2-9)}{(n^2-1)(n^2-1)}\frac{x^6}{7!}+\cdots \\ \le&1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{(n^2-4)(n^2-9)}{(n^2-1)(n^2-1)}\frac{x^6}{7!}+\cdots \\ <&-0.00041 \text{ for } x=3.2 \text{ and all integers } n\ge 10. \end{aligned}$$

Therefore, $q_n(\sqrt{4-(3.2)^2/(n^2-1)})q_n(2) < 0$ for all integers $n \ge 10$. Consequently, if $n \ge 10$ is a fixed integer, then there is a bifurcation in $(1-2.56/(n^2-1), 1)$ of periodic orbit of G_A of period (may not be minimal) $4m$ for every integer $m \ge n$. In particular, the point $A = 1$ is a right accumulation point of bifurcations of periodic orbits of G_A of period $4m$ for infinitely many positive integers m.

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Institute of Mathematics, Academia Sinica, Taipei, Taiwan, 11529, R.O.C.