BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 19, Number 3, September 1991

GLOBAL REGULARITY OF THE $\bar{\partial}$-NEUMANN PROBLEM: A SUFFICIENT CONDITION

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Abstract. In this paper we give a sufficient condition for the global regularity of the $\bar{\partial}$-Neumann problem. This condition is satisfied, for instance, by any smoothly bounded convex domains in C^2.

1. Introduction. Let D be a smoothly bounded pseudoconvex domain in C^n with the standard Euclidean metric. The $\bar{\partial}$-Neumann problem on D is concerned with the regularity of the solution u to the following equation. Namely, given $f \in L^2_{p,q}(D)$, let $u \in L^2_{p,q}(D)$ be the solution that satisfies

$$Q(u,v) = (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^* u, \bar{\partial}^* v) = (f,v), \tag{1.1}$$

for all $v \in \widetilde{\mathcal{D}}_{p,q}(D)$. For definitions see the statements of Theorem 1. Then we ask

(i) (Local regularity) Is u smooth up to the boundary near $x_0 \in bD$ if f is smooth up to the boundary near x_0 ?

(ii) (Global regularity) Is $u \in C^\infty_{p,q}(\overline{D})$ if $f \in C^\infty_{p,q}(\overline{D})$?

Since the equation (1.1) is elliptic inside the domain, hence interior regularity causes no trouble by standard elliptic regularity theorem.

Local regularity of the $\bar{\partial}$-Neumann problem on finite type domain in the sense of D'Angelo [9] have been proved by Kohn [12] [13] [14] and Catlin [3]. On the other hand some techniques have been developed to establish the global regularity of the $\bar{\partial}$-Neumann problem on weakly pseudoconvex

Received by the editors May 31, 1991.

domains, for instance, see Boas [1], Boas and Straube [2], Catlin [4], Chen [6]. Recently the author also showed in [7] that the local geometry of the boundary presents no obstruction to the global regularity of the $\overline{\partial}$-Neumann problem.

The purpose of this article is to present a sufficient condition for the global regularity of the $\overline{\partial}$-Neumann problem. Here are our main results.

Let $D \subseteq C^n$, $n \geq 2$, be a smoothly bounded pseudoconvex domain with a normalized defining function r, namely, $\sum_{j=1}^{n}\left|\frac{\partial r}{\partial z_j}\right|^2 = 1$ on bD. If $\frac{\partial r}{\partial z_n}(x_0) \neq 0$ for some $x_0 \in bD$, then one can choose

$$L'_k = \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_n} - \frac{\partial r}{\partial z_n}\frac{\partial}{\partial z_k}, \quad k = 1, \ldots, n-1, \tag{1.2}$$

to be a basis of $T^{1,0}(bD)$ near x_0. Put $L'_n = \sum_{j=1}^{n}\frac{\partial r}{\partial \overline{z}_j}\frac{\partial}{\partial z_j}$.

Theorem 1. *Let $D \subseteq C^n$, $n \geq 2$, be a smoothly bounded pseudoconvex domain with a normalized defining function r. Suppose that the Levi form degenerates to infinite type on a compact subset M of the boundary and that D is of finite type outside M. Let V be an open neighborhood of M. Suppose also that D satisfies the following three conditions,*

(i) $\frac{\partial r}{\partial z_n} \neq 0$ *on V.*

(ii) *There exists a real (or purely imaginary) tangential vector field T defined on V and complimentary to $T^{1,0}(bD) \oplus T^{0,1}(bD)$. Set*

$$[T, L'_k] = a_k T + \sum_{j=1}^{n-1} a_{kj} L'_j + \sum_{j=1}^{n-1} b_{kj}\overline{L}'_j, \quad k = 1, \ldots, n-1, \tag{1.3}$$

$$[T, \overline{L}'_k] = \tilde{a}_k T + \sum_{j=1}^{n-1} \tilde{a}_{kj} L'_j + \sum_{j=1}^{n-1} \tilde{b}_{kj}\overline{L}'_j, \quad k = 1, \ldots, n-1, \tag{1.4}$$

$$[T, L'_n] = a_n L'_n + b_n \overline{L}'_n + \sum_{j=1}^{n-1} a_{nj} L'_j + \sum_{j=1}^{n-1} b_{nj}\overline{L}'_j, \tag{1.5}$$

$$(1.6) \qquad [T, \overline{L}'_n] = \widetilde{a}_n L'_n + \widetilde{b}_n \overline{L}'_n + \sum_{j=1}^{n-1} \widetilde{a}_{nj} L'_j + \sum_{j=1}^{n-1} \widetilde{b}_{nj} \overline{L}'_j,$$

where a_k, $\widetilde{a}_k$, b_k, $\widetilde{b}_k$, a_{kj}, $\widetilde{a}_{kj}$, b_{kj} and $\widetilde{b}_{kj}$, $k = 1, \dots, n$, are smooth functions defined on V.

(iii) *a_k and $\widetilde{a}_k$, $k = 1, \dots, n$, vanish on M.*

Then the $\overline{\partial}$-Neumann problem is globally regular on D. More precisely, if $f \in W^k_{p,q}(D)$, let $u \in L^2_{p,q}(D)$ be the solution to the equation (1.1) for all $v \in \widetilde{\mathcal{D}}_{p,q}(D)$, then $u \in W^k_{p,q}(D)$ and $||u||_k \leq C_k ||f||_k$, where $W^k_{p,q}(D)$ is the Sobolev space of order k for (p,q)-forms on D and $\widetilde{\mathcal{D}}_{p,q}(D)$ is the completion of all smooth (p,q)-forms with Neumann boundary conditions, denoted by $\mathcal{D}_{p,q}(D)$, under Q.

We would like to point out here that first, condition (i) is not necessary. It can be achieved by introducing a suitably chosen cut-off function. Secondly, the techniques employed in this article have been used before by M. Derridj, D.S. Tartakoff and the author to study the global analytic hypoellipticity of the $\overline{\partial}$-Neumann problem. (e.g. see Derridj and Tartakoff [10], Chen [5].) However in their works they need the vector field T to be defined globally and the functions a_k, $\widetilde{a}_k$, $k = 1, \dots, n$, defined in Theorem 1 vanishing on the whole boundary. Therefore we think our conditions are more reasonable.

2. Examples. The formulation of the $\overline{\partial}$-Neumann problem is well-known now, e.g. see [11]. So we omit it

Next we give some examples that satisfy the hypotheses of Theorem 1.

Example 1. Any smoothly bounded convex domain in C^2 satisfies the hypotheses of Theorem 1, hence the $\overline{\partial}$-Neumann problem is globally regular on such domains. The details of the proof of this fact will appear in Chen [8].

Example 2. Let B_n denote the unit ball in C^n, $n \geq 2$. Put

$$D_1 = B_n \cap \{(z_1, \dots, z_n) \in C^n | y_n < a \text{ with } 0 < a < 1 \text{ and } z_n = x_n + iy_n\},$$

and round the edge of D_1. Call this domain D. It is easy to see that D is a smoothly bounded pseudoconvex domain that is Levi-flat on a real $(2n-1)$-dimensional ball M sitting in $\{y_n = a\}$, and that D is of finite type outside M. In fact one can make D to be strictly pseudoconvex outside M. Note also that the domain D is not circular, not of finite type, and that D does not satisfy property (P) introduced in Catlin [4] either. Let $\rho(z)$ be a defining function of D. We see that $\rho(z) = y_n - a$ in some open neighborhood U of the interior of M. We normalize $\rho(z)$ as follows. Put

$$r(z) = \frac{\rho(z)}{\left(\sum_{j=1}^{n}\left|\frac{\partial\rho}{\partial z_j}(z)\right|^2\right)^{1/2}}.$$

On U we have $\frac{\partial\rho}{\partial z_j} = 0$ for $j = 1, \dots, n-1$ and $\frac{\partial\rho}{\partial z_n} = -\frac{i}{2}$. It shows that $r(z)$ is a normalized defining function for D and $r(z) = 2(y_n - a)$ on U. Condition (i) is clearly satisfied. For condition (ii), set

$$T = -i(L_n' - \overline{L}_n') = -i\left(\sum_{j=1}^{n}\frac{\partial r}{\partial \bar{z}_j}\frac{\partial}{\partial z_j} - \sum_{j=1}^{n}\frac{\partial r}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}\right),$$

and

$$L_k' = \frac{\partial r}{\partial z_n}\frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_n}, \text{ for } k = 1, \dots, n-1.$$

We see that T is a real tangential vector field and complimentary to

$$T^{1,0}(bD) \oplus T^{0,1}(bD).$$

On U we have

$$T = \frac{\partial}{\partial x_n},$$

and

$$L_k' = -i\frac{\partial}{\partial z_k}, \quad k = 1, \dots, n-1,$$
$$L_n' = i\frac{\partial}{\partial z_n}.$$

Therefore

$$[T, L'_k] = [T, \overline{L}'_k] \equiv 0 \text{ on } U \text{ for } k = 1, \dots, n.$$

It follows that $a_k = \widetilde{a}_k = 0$ on U for $k = 1, \dots, n$. Hence condition (iii) is satisfied. Then by Theorem 1 the $\overline{\partial}$-Neumann problem is globally regular on D.

3. Proof of the main results. The idea for proving Theorem 1 is very standard. We have to first obtain an a priori estimate for the solution u. Since the equation (1.1) is not elliptic up to the boundary, we modify the equation as in Kohn-Nirenberg [15] as follows. Define the form Q_δ, for $0 < \delta < 1$, by

$$Q_\delta(u, v) = Q(u, v) + \delta \sum_{j=1}^{n} \left(\left(\frac{\partial}{\partial z_j} u, \frac{\partial}{\partial z_j} v \right) + \left(\frac{\partial}{\partial \overline{z}_j} u, \frac{\partial}{\partial \overline{z}_j} v \right) \right), \tag{3.1}$$

for all $u, v \in \mathcal{D}_{p,q}(D)$. Then we extend Q_δ by continuity to $\widetilde{\mathcal{D}}^\delta_{p,q}(D)$, the completion of $\mathcal{D}_{p,q}(D)$ under Q_δ.

Lemma 3.2. *$\widetilde{\mathcal{D}}^\delta_{p,q}(D)$ is independent of $\delta > 0$, and is contained in $\widetilde{\mathcal{D}}_{p,q}(D) \cap W^1_{p,q}(D)$.*

Since $Q_\delta(\phi, \phi) \geq Q(\phi, \phi)$ for all $\phi \in \widetilde{\mathcal{D}}^\delta_{p,q}(D)$, given $f \in L^2_{p,q}(D)$, there exists an unique solution $u_\delta \in \widetilde{\mathcal{D}}^\delta_{p,q}(D)$, denoted by $u_\delta = N_\delta f$, such that

$$Q_\delta(u_\delta, v) = (f, v), \tag{3.3}$$

for all $v \in \widetilde{\mathcal{D}}^\delta_{p,q}(D)$. It is also obvious that elliptic-type estimate holds for Q_δ. Therefore if $f \in C^\infty_{p,q}(\overline{D})$, we have $u_\delta \in C^\infty_{p,q}(\overline{D})$. From now on we will assume that $f \in C^\infty_{p,q}(\overline{D})$, so is $u_\delta = N_\delta f$.

Next we choose a local orthonormal basis $w_1, \dots, w_{n-1}, w_n$ for (1,0)-forms on V. We may assume that

$$w_n = \frac{\partial r}{\sqrt{2} \cdot \left(\sum_{j=1}^{n} \left| \frac{\partial r}{\partial z_j} \right|^2 \right)^{1/2}}.$$

Let $L_1, \ldots, L_{n-1}, L_n$ be the dual basis for $T^{1,0}(C^n)$ on V. We see that $L_1, \ldots, L_{n-1}$ are in $T^{1,0}(bD)$ and

$$L_n = \sqrt{2}\left(\sum_{j=1}^{n}\left|\frac{\partial r}{\partial z_j}\right|^2\right)^{-1/2} \cdot \left(\sum_{j=1}^{n} \frac{\partial r}{\partial \bar{z}_j}\frac{\partial}{\partial z_j}\right).$$

Then locally on V one can express $\bar\partial$ and $\bar\partial^*$ as follows. If $u \in \mathcal{D}_{p,q}(D)$, write $u = \sum_{I,J} u_{IJ} w_I \wedge \overline{w}_J$ with $|I| = p$ and $|J| = q$, where I and J are strictly increasing multiindices. Then

$$\bar\partial u = (-1)^p \sum_{kIJK} \varepsilon^K_{kJ} \overline{L}_k(u_{IJ}) w_I \wedge \overline{w}_K + \text{terms of order zero,} \tag{3.4}$$

$$\bar\partial^* u = (-1)^{p+1} \sum_{kIHJ} \varepsilon^J_{kH} L_k(u_{IJ}) w_I \wedge \overline{w}_H + \text{terms of order zero,} \tag{3.5}$$

where $|I| = p$, $|J| = q$, $|K| = q+1$, $|H| = q-1$ and ε^K_{kJ} (or ε^J_{kH}) is the sign of the permutation taking kJ (or kH) to K (or J respectively).

Now choose an open neighborhood V_1 of M such that $V_1 \subseteq \overline{V}_1 \subset\subset V$. Also choose a cut-off function φ, $0 \le \varphi \le 1$, such that $\varphi \equiv 1$ in some open neighborhood of M and such that the support of φ is contained in V_1. Denote by $Op(s,k)$ any tangential differential operator of order k formed out of the $L_i, \overline{L}_i, i = 1, \ldots, n-1$ and T in all order with precisely s L_i's or $\overline{L}_i$'s. Denote also by $\widetilde{Op}(k)$ any differential operator of order k, i.e., it may involve the normal differentiation. Define

$$\varphi Op(s,k) u_\delta = \sum_{I,J} (\varphi Op(s,k)(u_\delta)_{IJ}) w_I \wedge \overline{w}_J, \tag{3.6}$$

and

$$\varphi \widetilde{Op}(k) u_\delta = \sum_{I,J} (\varphi \widetilde{Op}(k)(u_\delta)_{IJ}) w_I \wedge \overline{w}_J. \tag{3.7}$$

Then inductively we will show that

$$I_k = \sum_{\text{finite sum}} \|\varphi \widetilde{Op}(k) u_\delta\|^2 + II_k \le C(k)\|f\|^2_k, \tag{3.8}$$

where

$$II_k = \sum_{j=1}^{n} ||\overline{L}_j(\varphi T^k u_\delta)||^2 + \sum_{j=1}^{n-1} ||L_j(\varphi T^{k-1} u_\delta)||^2 + ||\varphi T^k u_\delta||^2 + Q_\delta(\varphi T^k u_\delta, \varphi T^k u_\delta), \tag{3.9}$$

and in the first term of I_k we sum over a finite basis of k-th order differential operators $\widetilde{Op}(k)$, and the constant $C(k)$ depends on k, but will be independent of δ. The second and third terms of II_k in fact are included in the first term of I_k, we single them out just for clarity and technical reasons.

The initial step $k = 0$ is easy to check simply by observing that the derivative of the cut-off function φ is supported at finite type points of D where we have a stronger local estimate, namely, subelliptic estimate of order ε, $0 < \varepsilon \le \frac{1}{2}$. So we assume that the estimate (3.8) holds up to $k-1$. Then we will show that it also holds for k. First we estimate II_k.

Lemma 3.10.

$$\sum_{j=1}^{n-1} ||L_j(\varphi T^{k-1} u_\delta)||^2 \le C(k-1, S)||f||_{k-1}^2 + \frac{1}{S}(\sup|\lambda_j|^2)||\varphi T^k u_\delta||^2,$$

where $\sup|\lambda_j|^2 = \sup(|\lambda_1|^2, \dots, |\lambda_{n-1}|^2)$ *on* $\overline{V}_1$, *and* λ_j'*s are defined by*

$$[L_j, \overline{L}_j] = \lambda_j T + \sum_{i=1}^{n-1} c_{ji} L_i + \sum_{i=1}^{n-1} d_{ji} \overline{L}_i, \tag{3.11}$$

and S could be any positive number. Here c_{ji} and d_{ji} are smooth functions defined on V.

Proof. The proof is straight forward simply by the integration by parts and by using the following well-known trick

$$|AB| \le \frac{1}{S}|A|^2 + S|B|^2,$$

where S could be any positive number. So we are done.

Since $\sup|\lambda_j|^2$ is finite on $\overline{V}_1$, by choosing S large enough we may assume that $\frac{1}{S}(\sup|\lambda_j|^2) \le 1$, so we have

$$II_k \le C(k-1,S)||f||^2_{k-1} + CQ_\delta(\varphi T^k u_\delta, \varphi T^k u_\delta), \tag{3.12}$$

where the constants C and $C(k-1,S)$ are independent of δ. Put

$$\begin{aligned} III_k &= ||\overline{\partial}\varphi T^k u_\delta||^2 + ||\overline{\partial}^*\varphi T^k u_\delta||^2 \\ &\quad + \delta \sum_{j=1}^n \left(\left\| \frac{\partial}{\partial z_j}\varphi T^k u_\delta \right\|^2 + \left\| \frac{\partial}{\partial \overline{z}_j}\varphi T^k u_\delta \right\|^2 \right) \\ &= (\varphi T^k f, \varphi T^k u_\delta) + E + E^* + \delta \sum_{j=1}^n (E_{z_j} + \widetilde{E}_{z_j}), \end{aligned} \tag{3.13}$$

where $E = E_1 + E_2 + E_3 + E_4$, and

$$\begin{aligned} E_1 &= ([\overline{\partial}, \varphi]T^k u_\delta, \overline{\partial}\varphi T^k u_\delta), \\ E_2 &= (\overline{\partial}T^k u_\delta, [\varphi, \overline{\partial}]\varphi T^k u_\delta), \\ E_3 &= ([\overline{\partial}, T^k]u_\delta, \overline{\partial}\varphi^2 T^k u_\delta), \\ E_4 &= (\overline{\partial}u_\delta, [(T^*)^k, \overline{\partial}]\varphi^2 T^k u_\delta). \end{aligned}$$

Similarly there are four terms for each E^*, E_{z_j} or $\widetilde{E}_{z_j}$, and in E^*, E_{z_j} or $\widetilde{E}_{z_j}$ we simply replace $\overline{\partial}$ by $\overline{\partial}^*$, $\frac{\partial}{\partial z_j}$ or $\frac{\partial}{\partial \overline{z}_j}$ respectively.

Estimates for E_1 and E_2.

$$|E_j| \le C(k, S_0)||f||^2_k + \frac{1}{S_0}||\overline{\partial}\varphi T^k u_\delta||^2, \quad j = 1, 2,$$

where the constant $S_0 > 0$ will be determined later.

Estimate for E_3.

Lemma 3.14.

(i)
$$\begin{aligned} [\overline{\partial}, T^k] &= \sum_{j=1}^k \binom{k}{j} \underbrace{[\ldots[[\overline{\partial}, T], T]\ldots]}_{j-\text{brackets}} T^{k-j} \\ &= \sum_{j=1}^k \binom{k}{j} T^{k-j} \underbrace{[\ldots[[\overline{\partial}, T], T]\ldots]}_{j-\text{brackets}} \\ &= k[\overline{\partial}, T]T^{k-1} + \sum_{j=2}^k \widetilde{Op}(k-j+1). \end{aligned}$$

(ii)
$$[\overline{\partial},(T^*)^k]=\sum_{j=1}^{k}\underline{\binom{k}{j}}(T^*)^{k-j}\underbrace{[\ldots[[\overline{\partial},T^*],T^*]\ldots]}_{j-\text{brackets}}$$
$$=k(T^*)^{k-1}[\overline{\partial},T^*]+\sum_{j=2}^{k}(T^*)^{k-j}\cdot\widetilde{Op}(\leq 1),$$

where the underline means that there are at most $\binom{k}{j}$ *such terms with suitable plus or minus signs. Similar equations also hold if we replaces* $\overline{\partial}$ *by* $\overline{\partial}^*$.

Proof. Obvious.

So if we apply equation (i) in Lemma 3.14, then by induction hypotheses we obtain

$$|E_3|\leq C(k,S_0)||f||_k^2+\frac{1}{S_0}||\overline{\partial}\varphi T^k u_\delta||^2+k\sum_{j=1}^{n}|([\overline{L}_j,T]T^{k-1}u_\delta,\overline{\partial}\varphi^2T^ku_\delta)|$$
$$\leq C(k,S_0)||f||_k^2+(\frac{1+nk}{S_0})||\overline{\partial}\varphi T^k u_\delta||^2+(kS_0)\sum_{j=1}^{n}||\varphi[\overline{L}_j,T]T^{k-1}u_\delta||^2.$$

Lemma 3.15. *By induction hypotheses we have*

(i) $$||\varphi[L_j,T]T^{k-1}u_\delta||^2\leq C(k-1,S)||f||_{k-1}^2+\frac{\gamma}{S}(\sup|\lambda_j|^2)||\varphi T^k u_\delta||^2$$
$$+\gamma(\sup|a|^2)||\varphi T^k u_\delta||^2,\ \text{for } 1\leq j\leq n-1,$$

(ii) $$||\varphi[L_n,T]T^{k-1}u_\delta||^2\leq C(k-1,S)||f||_{k-1}^2+\frac{\gamma}{S}(\sup|\lambda_j|^2)||\varphi T^k u_\delta||^2$$
$$+\gamma A||\varphi T^k u_\delta||^2,$$

where γ = *the supremum of the square of the absolute value of the coefficient functions on* $\overline{V}_1$ *that results from taking commutators or changing basis, and* $\sup|a|^2=\sup(|a_k|^2,|\widetilde{a}_k|^2),k=1,\ldots,n$ *on* $\overline{V}_1$, *and* $A=\sup(|R\widetilde{a}_n|^2,|[R,T]-Ra_n|^2)$ *on* $\overline{V}_1$ *with* $R=\sqrt{2}\left(\sum_{j=1}^{n}\left|\frac{\partial r}{\partial z_j}\right|^2\right)^{-\frac{1}{2}}$ *Same estimates also hold if we replace* L_j *by* $\overline{L}_j$ *for* $1\leq j\leq n$.

Proof. If $1\leq j\leq n-1$, then

$$L_j=\sum_{i=1}^{n-1}g_{ji}L_i',\ \text{and}\ L_i'=\sum_{\ell=1}^{n-1}h_{i\ell}L_\ell, \tag{3.16}$$

where g_{ji} and $h_{i\ell}$ are smooth functions defined on V. Hence

$$\begin{aligned}[L_j,T] &= \sum_{i=1}^{n-1}[g_{ji},T]L'_i + \sum_{i=1}^{n-1} g_{ji}[L'_i,T] \\ &= \sum_{i=1}^{n-1}[g_{ji},T]L'_i + \sum_{i=1}^{n-1}(-g_{ji}a_i)T + \sum_{i=1}^{n-1}\sum_{\ell=1}^{n-1}(-g_{ji}a_{i\ell})L'_\ell \\ &\quad + \sum_{i=1}^{n-1}\sum_{\ell=1}^{n-1}(-g_{ji}b_{i\ell})\overline{L}'_\ell.\end{aligned}$$

Therefore by Lemma 3.10 and induction hypotheses, we have (i). For (ii) we observe that T is complimentary to $T^{1,0}(bD) \oplus T^{0,1}(bD)$ on V, so one can write

$$T = g(L'_n - \overline{L}'_n) - \sum_{i=1}^{n-1}(g_iL'_i + \overline{g}_i\overline{L}'_i) \text{ with } g \neq 0 \text{ on } V,$$

and

$$L'_n = g^{-1}T + \overline{L}'_n + \sum_{i=1}^{n-1}(g^{-1}g_iL'_i + g^{-1}\overline{g}\overline{L}'_i).$$

Hence we have $L_n = RL'_n$, $R = \sqrt{2}$ on the boundary, and

$$\begin{aligned}(3.17)\,[L_n,T] &= [R,T]L'_n + R[L'_n,T] \\ &= [R,T]L'_n - R\Big(a_nL'_n + b_n\overline{L}'_n + \sum_{j=1}^{n-1} a_{nj}L'_j + \sum_{j=1}^{n-1} b_{nj}\overline{L}'_j\Big) \\ &= ([R,T] - Ra_n)\Big(g^{-1}T + \overline{L}'_n + \sum_{i=1}^{n-1}(g^{-1}g_iL'_i + g^{-1}\overline{g}_i\overline{L}'_i)\Big) \\ &\quad - R\Big(b_n\overline{L}'_n + \sum_{j=1}^{n-1} a_{nj}L'_j + \sum_{j=1}^{n-1} b_{nj}\overline{L}'_j\Big).\end{aligned}$$

Again by Lemma 3.10 we obtain (ii). This completes the proof of the lemma.

Therefore by Lemma 3.15 one can estimate $|E_3|$ as follows:

$$|E_3| \leq C(k,S_0,S)||f||_k^2 + \Big(\frac{1+nk}{S_0}\Big)||\overline{\partial}\varphi T^k u_\delta||^2 + G_1||\varphi T^k u_\delta||^2,$$

where $G_1 = \gamma(kS_0)(\frac{n}{S}(\sup|\lambda_j|^2) + (n-1)(\sup|a|^2) + A)$.

Estimate for E_4.

By Lemma 3.14 (ii) we have

$$\begin{aligned} E_4 &= -k(\overline{\partial} u_\delta, (T^*)^{k-1}[\overline{\partial}, T^*]\varphi^2 T^k u_\delta) + \sum_{j=1}^{k} (\overline{\partial} u_\delta, (T^*)^{k-j} \widetilde{Op}(\le 1)\varphi^2 T^k u_\delta) \\ &= -k(T^{k-1}\overline{\partial} u_\delta, [\overline{\partial}, T^*]\varphi^2 T^k u_\delta) + \sum_{j=2}^{k} (T^{k-j}\overline{\partial} u_\delta, \widetilde{Op}(\le 1)\varphi^2 T^k u_\delta) \\ &= T_1 + T_2. \end{aligned}$$

In order to estimate T_2 we will throw one T (or two Ts) from the right to the left. Since $T^* = -T + h$, where h is a smooth function defined on V, we have

$$\begin{aligned} (3.18) \qquad |T_2| &\le C(k)||f||_k^2 + |(\varphi T^k \overline{\partial} u_\delta, \widetilde{Op}(\le 1)\varphi T^{k-2} u_\delta)| \\ &\le C(k, S_0)||f||_k^2 + \frac{1}{S_0}||\varphi\overline{\partial} T^k u_\delta||^2 + \frac{1}{S_0}||\varphi[T^k, \overline{\partial}]u_\delta||^2 \\ &\le C(k, S_0, S)||f||_k^2 + \frac{1}{S_0}||\overline{\partial}\varphi T^k u_\delta||^2 + G_2||\varphi T^k u_\delta||^2, \end{aligned}$$

where $G_2 = \frac{\gamma k^2}{S_0}(\frac{n}{S}(\sup|\lambda_j|^2) + (n-1)(\sup|a|^2) + A)$.

For term T_1 we throw one T from the right to the left, and get

$$\begin{aligned} (3.19) \qquad |T_1| &\le C(k, S_0)||f||_k^2 + \frac{1}{S_0}||\varphi T^k u_\delta||^2 \\ &\quad + k\sum_{j=1}^{n} |(\varphi T^{k-1}\overline{\partial} u_\delta, \varphi[\overline{L}_j, T]T^k u_\delta)| \\ &\le C(k, S_0)||f||_k^2 + \frac{1}{S_0}||\varphi T^k u_\delta||^2 \\ &\quad + k\sum_{j=1}^{n} \{|(\varphi T^{k-1}\overline{\partial} u_\delta, \varphi[[\overline{L}_j, T], T]T^{k-1} u_\delta)| \\ &\qquad + |(\varphi T^{k-1}\overline{\partial} u_\delta, [\varphi, T][\overline{L}_j, T]T^{k-1} u_\delta)| \\ &\qquad + |(h\varphi T^{k-1}\overline{\partial} u_\delta, \varphi[\overline{L}_j, T]T^{k-1} u_\delta)| \\ &\qquad + |(([\varphi, T]T^{k-1}\overline{\partial} u_\delta, \varphi[\overline{L}_j, T]T^{k-1} u_\delta)| \\ &\qquad + |(\varphi\overline{\partial} T^k u_\delta, \varphi[\overline{L}_j, T]T^{k-1} u_\delta)| \end{aligned}$$

$$+ |(\varphi[T^k, \overline{\partial}]u_\delta, \varphi[\overline{L}_j, T]T^{k-1}u_\delta)|\}$$
$$\leq C(k, S_0, S)||f||_k^2 + \frac{nk}{S_0}||\overline{\partial}\varphi T^k u_\delta||^2 + G_3||\varphi T^k u_\delta||^2,$$

where

$$G_3 = \frac{1+2\gamma}{S_0} + \frac{2\gamma}{S_0 S}(\sup|\lambda_j|^2) + ((k + nk^3 + kS_0)\gamma)\Big(\frac{n}{S}(\sup|\lambda_j|^2) + (n-1)(\sup|a|^2) + A\Big).$$

It follows that we have

$$\begin{aligned}(3.20)\qquad |E_4| &\leq |T_1| + |T_2| \\ &\leq C(k, S_0, S)||f||_k^2 + \Big(\frac{1+nk}{S_0}\Big)||\overline{\partial}\varphi T^k u_\delta||^2 \\ &\quad + (G_2 + G_3)||\varphi T^k u_\delta||^2.\end{aligned}$$

Put these estimates together, we obtain

$$\begin{aligned}(3.21)\qquad |E| &\leq |E_1| + |E_2| + |E_3| + |E_4| \\ &\leq C(k, S_0, S)||f||_k^2 + \Big(\frac{4+2nk}{S_0}\Big)||\overline{\partial}\varphi T^k u_\delta||^2 \\ &\quad + (G_1 + G_2 + G_3)||\varphi T^k u_\delta||^2.\end{aligned}$$

This completes the estimates for E.

Now we can estimate E^* by applying exactly the same arguments, so we get

$$\begin{aligned}(3.22)\qquad |E^*| &\leq C(k, S_0, S)||f||_k^2 + \Big(\frac{4+2nk}{S_0}\Big)||\overline{\partial}^*\varphi T^k u_\delta||^2 \\ &\quad + (G_1 + G_2 + G_3)||\varphi T^k u_\delta||^2.\end{aligned}$$

The estimates for E_{z_j} will be done along the same line developed above, so we get

$$\begin{aligned}(3.23)\qquad |E_{z_j 1}| &\leq C(k, S_0)||f||_k^2 + \frac{1}{S_0}\Big|\Big|\frac{\partial}{\partial z_j}\varphi T^k u_\delta\Big|\Big|^2, \\ |E_{z_j 2}| &\leq C(k, S_0)||f||_k^2 + \frac{1}{S_0}\Big|\Big|\frac{\partial}{\partial z_j}\varphi T^k u_\delta\Big|\Big|^2, \\ |E_{z_j 3}| &\leq C(k, S_0, S)||f||_k^2 + \frac{1}{S_0}\Big|\Big|\frac{\partial}{\partial z_j}\varphi T^k u_\delta\Big|\Big|^2\end{aligned}$$

$$+(\gamma k^2 S_0)\Big(1+\frac{1}{S}(\sup|\lambda_j|^2)\Big)\Big\|\varphi T^k u_\delta\Big\|^2,$$

$$|E_{z_j4}|\le k\Big|\Big(T^{k-1}\frac{\partial}{\partial z_j}u_\delta,\Big[T^*,\frac{\partial}{\partial z_j}\Big]\varphi^2T^k u_\delta\Big)\Big|$$
$$+\sum_{i=2}^{k}\Big|\Big(T^{k-i}\frac{\partial}{\partial z_j}u_\delta,\widetilde{Op}(\le 1)\varphi^2 T^k u_\delta\Big)\Big|$$
$$\le C(k,S_0,S)\|f\|_k^2+\frac{1+k}{S_0}\Big\|\frac{\partial}{\partial z_j}\varphi T^k u_\delta\Big\|^2$$
$$+\Big(\frac{1}{S_0}+\gamma\Big(1+\frac{1}{S}(\sup|\lambda_j|^2)\Big)$$
$$\Big(2k^2+kS_0+\frac{2+k^2}{S_0}\Big)\Big)\|\varphi T^k u_\delta\|^2.$$

It follows that we have for $1\le j\le n$,

$$|E_{z_j}|\le C(k,S_0,S)\|f\|_k^2+\Big(\frac{4+k}{S_0}\Big)\Big\|\frac{\partial}{\partial z_j}\varphi T^k u_\delta\Big\|^2+G_4\|\varphi T^k u_\delta\|^2, \tag{3.24}$$

where $G_4=\frac{1}{S_0}+\gamma(1+\frac{1}{S}(\sup|\lambda_j|^2))(2k^2+kS_0+k^2S_0+\frac{2+k^2}{S_0})$. It is clear that the estimate (3.24) works also for all $\widetilde{E}_{z_j}$, $1\le j\le n$.

Finally we obtain

$$\begin{aligned} III_k &= Q_\delta(\varphi T^k u_\delta,\varphi T^k u_\delta)\\ &\le C(k,S_0,S)\|f\|_k^2+\frac{1}{S_0}\|\varphi T^k u_\delta\|^2+S_0\|\varphi T^k f\|^2\\ &+\Big(\frac{4+2nk}{S_0}\Big)(\|\bar{\partial}\varphi T^k u_\delta\|^2+\|\bar{\partial}^*\varphi T^k u_\delta\|^2)\\ &+2(G_1+G_2+G_3)\|\varphi T^k u_\delta\|^2\\ &+\delta\sum_{j=1}^{n}\Big(\Big(\frac{4+k}{S_0}\Big)\Big(\Big\|\frac{\partial}{\partial z_j}\varphi T^k u_\delta\Big\|^2\\ &+\Big\|\frac{\partial}{\partial \bar{z}_j}\varphi T^k u_\delta\Big\|^2\Big)+2G_4\|\varphi T^k u_\delta\|^2\Big). \end{aligned} \tag{3.25}$$

Now choose $S_0=\max(8+4nk,20C(1+2\gamma))$, where C is given in (3.12), we get

$$\begin{aligned} III_k &\le C(k,S_0,S)\|f\|_k^2+2S_0\|\varphi T^k f\|^2\\ &+\Big(\frac{2}{S_0}+4(G_1+G_2+G_3)+4n\delta G_4\Big)\|\varphi T^k u_\delta\|^2. \end{aligned} \tag{3.26}$$

It follows that

$$(3.27)\qquad II_k = \sum_{j=1}^{n} ||\overline{L}_j\varphi T^k u_\delta||^2 + \sum_{j=1}^{n-1} ||L_j\varphi T^{k-1}u_\delta||^2 + ||\varphi T^k u_\delta||^2 + III_k \leq C(k,S)||f||_k^2 + C_0||\varphi T^k u_\delta||^2,$$

where $C_0 = C(\frac{2}{S_0} + 4(G_1+G_2+G_3) + 4n\delta G_4)$. Since S_0 has been fixed, by shrinking the set V_1 and letting $\delta > 0$ be sufficiently small and letting $S > 0$ be sufficiently large, we have $C_0 \leq \frac{1}{2}$. It follows that

$$(3.28)\qquad II_k \leq C(k)||f||_k^2.$$

This completes the estimate for II_k. In particular it shows that

$$(3.29)\qquad ||\varphi T^k u_\delta||^2 \leq C(k)||f||_k^2,$$

where the constant $C(k)$ is independent of $\delta > 0$.

Next we have to estimate the mixed tangential derivatives of u_δ, i.e., $\varphi Op(s,k)u_\delta$ for all $s \leq k$, and normal derivatives of u_δ. The estimates for these derivatives are standard, e.g., see [6]. Hence the proof for the estimate (3.8) is now complete.

Once we have the a priori estimate (3.8), it is standard to see that the $\overline{\partial}$-Neumann problem is globally regular for all order by applying the regularization theorem developed by Kohn-Nirenberg [15]. This also completes the proof of Theorem 1.

References

1. H.P. Boas, *Smalls sets of inifinite type are benign for the $\overline{\partial}$-Neumann problem*, Proc. Amer. Math. Soc., Vol. **103**, No. **2**(1988), 569-578.

2. H.P. Boas, and E.J. Straube, *Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in C^n admitting a defining function that is plurisubharmonic on the boundary*, Math. Z., **206**(1991), 81-88.

3. D. Catlin, *Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains*, Ann. Math., **126**(1987), 131-191.

4. D. Catlin, *Global regularity of the $\overline{\partial}$-Neumann problem*, Proc. Sympos. Pure Math., **41**(1984), Amer. Math. Soc., Providence, R.I.

5. S.C. Chen, *Global analytic hypoellipticity of the $\overline{\partial}$-Neumann problem on circular domains*, Invent. Math., **92**(1988), 173-185.

6. S.C. Chen, *Global regularity of the $\overline{\partial}$-Neumann problem on circular domains*, Math. Ann., **285**(1989), 1-12.

7. S.C. Chen, *Regularity of the $\overline{\partial}$-Neumann problem*, Proc. Amer. Math. Soc., Vol. **111**, No. **3**(1991), 779-785.

8. S.C. Chen, *Global regularity of the $\overline{\partial}$-Neumann problem in dimension two*, Proceedings of the AMS Summer Research Institute on Several Complex Variables and Complex Geometry, July 1989, (in press).

9. J. D'Angelo, *Real hypersurfaces, order of contact, and applications*, Ann. of Math., **115**(1982), 615-637.

10. M. Derridj, and D.S. Tartakoff, *On the global real analyticity of solutions to the $\overline{\partial}$-Neumann problem, commun. Partial Differ. Equations I*, (1976), 401-435.

11. G.B. Folland, and J.J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. of Math. Studies, **75**(1972), Princeton, Princeton University Press.

12. J.J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds, I, II*, Ann. of Math., **78, 79**(1963,64), 112-148, 450-472.

13. J.J. Kohn, *Boundary behavior of $\overline{\partial}$ on weakly pseudo-convex manifolds of dimension two*, J. Differ. Geom., **6**(1972), 523-542.

14. J.J. Kohn, *Subellipticity of the $\overline{\partial}$-Neumann problem on pseudo-convex domains: sufficient conditions*, Acta Math., **142**(1979), 79-122.

15. J.J. Kohn, and L. Nirenberg, *Non-coercive boundary value problems*, Comm. Pure Appl. Math., **18**(1965), 443-492.

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