BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 19, Number 1, March 1991

ON REDHEFFER'S TYPE THEOREMS IN DIFFERENTIAL INEQUALITIES ON MANIFOLDS

BY

HSING-HSIA CHEN (陳興夏) AND YI-JUNG HSU (許義容)

Abstract. In the present paper we show that Redheffer's type theorems for C^2 entire solutions of differential inequalities on the Euclidean space can be generalized to complete Riemannian manifolds of nonnegative Ricci curvature.

In the paper [5] Redheffer gave a systematic method for studying C^2 entire solutions of differential inequalities on the Euclidean space $\mathbb{R}^n$. We shall be concerned here with the interesting theorems of Redheffer and try to establish the same type theorems in the case of complete Riemannian manifolds of nonnegative Ricci curvature. We shall see that the proof of Redheffer will essentially work for us except some technical modification. For simplicity, we just only interpret the technique in two special cases, Theorem 1 and Theorem 2. Finally we state the sharper forms, Theorem 3 and Theorem 4, since the proofs build on the same idea but more complicated.

Now we consider the two dimensional case:

Theorem 1. *Let M be a complete Riemannian surface with nonnegative Gaussian curvature. Let u be a real valued C^2 function defined on M with the supremum m. Suppose that there exists a positive constant ϵ such that*

$$\Delta u \geq -\frac{1}{\epsilon}|\nabla u|^2$$

in the set $\{x \in M : m - \epsilon < u(x) < m,\ 0 < |\nabla u|(x) < \epsilon\}$. Then u is

Received by the editors January 5, 1990 and in revised form August 3,1990.

constant.

Proof. Suppose u is not constant, then there are positive numbers R_1 and δ with $\sup_{\overline{B}_1} u = m - \delta$, where $\overline{B}_1$ is the closed geodesic ball with center at x_0 and radius R_1. Denote the function $v : M - \overline{B}_2 \to \mathbb{R}$ by $v(r) = \ln\ln(h(r))$, where R_2 is some constant with $eR_2 > R_1 > R_2 > 0$, $\overline{B}_2$ is the closed geodesic ball with center at x_0 and radius R_2, r is the distance function from x_0 and $h(r) = \dfrac{(e-1)r + R_1 - eR_2}{R_1 - R_2}$. Then at the point where r is smooth, we have

$$|\nabla v|^2 = \left[\frac{e-1}{(R_1 - R_2)\ln(h(r))h(r)}\right]^2$$

and

$$\begin{aligned}
\Delta v &= \frac{(e-1)\Delta r}{(R_1 - R_2)\ln(h(r))h(r)} - |\nabla v|^2 - \frac{(e-1)^2}{(R_1-R_2)^2\ln(h(r))h(r)^2}\\
&\le \frac{(e-1)}{r(R_1 - R_2)\ln(h(r))h(r)} - |\nabla v|^2 - \frac{(e-1)^2}{(R_1-R_2)^2\ln(h(r))h(r)^2} \qquad (1.1)\\
&\le -|\nabla v|^2 + \frac{(e-1)(R_1 - eR_2)}{r(R_1-R_2)^2\ln(h(r))h(r)^2}\\
&< -|\nabla v|^2,
\end{aligned}$$

here we have used the Laplacian Comparison Theorem [7]. Since u has the supremum m, there is a point $\overline{x}$ in $m - \overline{B}_1$ with $u(\overline{x}) > \max(m - \epsilon, m - \delta)$. Choose a small positive constant $\overline{\epsilon}$ so that

$$u(\overline{x}) - \overline{\epsilon} v(r(\overline{x})) \ge \max(m-\epsilon, m-\delta),\ \overline{\epsilon}\frac{e-1}{e(R_1 - R_2)} < \epsilon \text{ and } \overline{\epsilon} < \epsilon. \qquad (1.2)$$

Since $v(r) \to \infty$ as $r \to \infty$ and u is bounded from above, $u(x) - \overline{\epsilon} v(r(x)) \to -\infty$ as $r(x) \to \infty$, there is a R_3 with $R_3 > R_1$ such that $u(x) - \overline{\epsilon} v(r(x)) < m - \epsilon$ for all x in $M - B_3$, where B_3 is the open geodesic ball with center at x_0 and radius R_3. Hence the maximum of $u - \overline{\epsilon} v(r)$ over $M - \overline{B}_1$ is attained at some point $\hat{x}$ in $B_3 - \overline{B}_1$.

If r is smooth at $\hat{x}$, then at $\hat{x}$

$$|\nabla u| = \overline{\epsilon}\,|\nabla v|, \qquad (1.3)$$

$$\Delta u \leq \bar{\epsilon} \Delta v. \tag{1.4}$$

Since at $\hat{x}$, $|\nabla u| = \bar{\epsilon}|\nabla v| > 0$, $m > u \geq m - \epsilon + \bar{\epsilon} v > m - \epsilon$. Furthermore, by (1.2) and (1.3), $|\nabla u| = \bar{\epsilon}|\nabla v| < \bar{\epsilon} v'(R_1) = \bar{\epsilon}\dfrac{e-1}{e(R_1 - R_2)} < \epsilon$. We conclude that $m - \epsilon < u(\hat{x}) < m$ and $0 < |\nabla u|(\hat{x}) < \epsilon$. But by (1.1), (1.2), (1.3) and (1.4), at $\hat{x}$

$$\Delta u \leq \bar{\epsilon} \Delta v < -\bar{\epsilon}|\nabla v|^2 = -(\bar{\epsilon})^{-1}|\nabla u|^2 < -\frac{1}{\epsilon}|\nabla u|^2,$$

which is a contradiction.

If r is not smooth at $\hat{x}$, then we modify the proof by a technique given by S. Y. Cheng [2] as follows. Let $\sigma : [0, r(\hat{x})] \to M$ be the minimizing geodesic joining x_0 and $\hat{x}$. Choose a small positive number τ such that $\sigma(\tau)$ is not conjugate to $\hat{x}$. Then there exists a geodesic cone C with vertex at $\sigma(\tau)$ and contains a neighborhood of $\hat{x}$ such that r_τ, the distance function from $\sigma(\tau)$, is smooth inside C. Let $\bar{r} = r_\tau + \tau$. Then by using triangle inequality the function $u - \bar{\epsilon} v(\bar{r})$ also attains a local maximum at $\hat{x}$. Since $\bar{r}$ is smooth at $\hat{x}$, for τ small enough, $|\nabla v(\bar{r})|^2 = \left[\dfrac{e-1}{(R_1 - R_2)\ln(h(r))h(r)}\right]^2$ and

$$\begin{aligned}
\Delta v(\bar{r}) &= v''(\bar{r}) + v'(\bar{r})\Delta r_\tau \\
&\leq \frac{e-1}{r_\tau(R_1 - R_2)\ln(h(r))h(r)} - |\nabla v|^2 - \frac{(e-1)^2}{(R_1 - R_2)^2 \ln(h(r))h(r)^2} \\
&= -|\nabla v|^2(\bar{r}) + \frac{(e-1)(\tau(e-1) + (R_1 - eR_2))}{r_\tau(R_1 - R_2)^2 \ln(h(r))h(r)^2} \\
&< -|\nabla v|^2(\bar{r}),
\end{aligned}$$

still hold at $\hat{x}$. The same argument as above gives a contradiction. This completes the proof.

Remarks: (a) The assumption of nonnegative curvature is necessary, as is seen from the function $u(x, y) = x^2 + y^2$ on the standard Poincaré disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ of Gaussian curvature -1.

(b) The result is false in higher dimension. There are counterexamples in the Euclidean space of dimension greater than two ([5]).

(c) The exponent of $|\nabla u|$ is optimum since we can not replace by any exponent α with $\alpha < 2$ ([5]).

As an immediate application, we have

Corollary. *Let M be a complete Riemannian surface with nonnegative Gaussian curvature. Let u be a real valued C^2 function defined on M such that $\Delta u \geq f(u)$ on M. If there is a positive continuous function g defined on some interval $[a, \infty)$and a positive constant c with*

$$\int_b^\infty \frac{d\tau}{(\int_a^\tau (c+g(t))dt)^{\frac{1}{2}}} < \infty,$$

for some constant b, $b > a$ and

$$\inf_{x\geq L} \frac{f(x)}{g(x)} \geq \left[\int_L^\infty \frac{d\tau}{(2\int_a^\tau (c+g(t))dt)^{\frac{1}{2}}} \right]^2,$$

for some $L > b$, then u is bounded from above.

Proof. Denote the function w defined on $[b, \infty)$ by

$$w(x) = \int_b^x \frac{d\tau}{(2\int_a^\tau (c+g(t))dt)^{\frac{1}{2}}}.$$

It is easy to see that $w'' + c(w')^3 + g(w')^3 = 0$ and $0 < w'(x) < 1$ for x large enough, $x \geq x_0$. In particular, we have $w'' + c(w')^2 + g(w')^3 \geq 0$ on $[x_0, \infty)$. Let

$$\epsilon = \int_{\tilde{L}}^\infty \frac{d\tau}{(2\int_a^\tau (c+g(t))dt)^{\frac{1}{2}}},$$

where $\tilde{L}$ is a large constant so that $\tilde{L} > L$, $c\,\epsilon < 1$ and $w^{-1}(\sup w - \epsilon) \geq x_0$. For the remainder of the proof, we smoothly extend w to the whole real line so that $w' > 0$.

Now suppose that u is not bounded from above. Let $v = w(u)$. Then we have

$$\left(\frac{w''}{(w')^3} + \frac{c}{w'}\right)|\nabla v|^2 + f(u) \geq -g(u)|\nabla v|^2 + f(u) > 0$$

and

$$\frac{w''}{(w')^2}|\nabla v|^2 + w'f(u) > -c|\nabla v|^2,$$

for all points in M at which simultaneously $\sup w - \epsilon < v < \sup w$ and $0 < |\nabla v| < \epsilon$. Which implies at such points

$$\begin{aligned}\Delta v &= w''|\nabla u|^2 + w'\Delta u \\ &\geq \frac{w''}{(w')^2}|\nabla v|^2 + w'f(u) \\ &> -c|\nabla v|^2 \\ &\geq -\frac{1}{\epsilon}|\nabla v|^2.\end{aligned}$$

Consequently, v is constant and u is constant, a contradiction.

Remarks. (a) Some problems which are related to the previous result have been studied in S. Y. Cheng and S. Y. Yau [3] (Theorem 8) and R. Redheffer [5]. (b) When the function f depends also on u and $|\nabla u|$, using the same method, we can find a sufficient condition for u to be bounded from above on M (for the detail, see H. H. Chen [1]).

Combining Theorem 1 with an interior gradient estimate of S. T. Yau [7], we also have the following result (Li and Tam [4]).

Corollary. *Let M be a complete Riemannian surface with nonnegative Gaussian curvature. Let E be the space of all harmonic functions defined on M with the growth condition*

$$\limsup_{r(x)\to\infty} \frac{|u(x)|}{r(x)} < \infty,$$

where r is the distance function from a fixed point x_0 at M. Then (a) $\dim E \leq 3$, (b) $\dim E = 3$ *if and only if M is the standard two dimensional Euclidean space,* (c) $\dim E = 2$ *if and only if M is the standard flat cylinder $S^1 \times \mathbb{R}^1$ and* (d) $\dim E = 1$ *if and only if M is the Möbius band, or M is closed, or the curvature of M is positive somewhere.*

Proof. Suppose that $u \in E$, then the Bochner-Lichnerowicz formula reduces to

$$\frac{1}{2}\Delta|\nabla u|^2 = \sum u_{ij}^2 + K|\nabla u|^2 \geq 0,$$

where K is the Gaussian curvature of M. The interior gradient estimate [7] shows that $|\nabla u|$ is a bounded function on M and hence, by Theorem 1, $|\nabla u|^2$ is constant. The hypothesis of curvature implies that M is flat, provided $\dim E \neq 1$. The proof then follows from the classification theorem of complete Riemannian surfaces of flat Gaussian curvature [6].

In the higher dimensional case, we have

Theorem 2. *Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Let u be a real valued C^2 function defined on M with the supremum m. Suppose that there exists a positive constant ϵ such that*

$$\Delta u \geq \epsilon|\nabla u|^{\frac{n}{n-1}}$$

in the set $\{x \in M : m - \epsilon < u(x) < m,\ 0 < |\nabla u|(x) < \epsilon\}$. Then u is constant.

Proof. We may assume that $\epsilon > n-1$ and $n \geq 3$. In fact, if $\epsilon \leq n-1$, then we can replace the Riemannian metric ds^2 of M by the conformal metric $d\tilde{s}^2 = c^2 ds^2$, the function u by $\tilde{u} = c^{-\alpha}u$ and ϵ by $\tilde{\epsilon} = c^{\beta}\epsilon$, where $\alpha = (n/2) - 1$, $\beta = (2-n)/2(n-1)$ and c is a small positive constant with $\tilde{\epsilon} > n-1$.

Suppose u is not constant. Then there are positive numbers R_1, δ and a point x_1 in $M - \overline{B}_1$ with $m > u(x_1) > \max(m-\epsilon, m-\delta)$, $\sup_{\overline{B}_1} u = m - \delta$ and $(2(n-1)\delta)^{\frac{n-1}{n-2}} < \epsilon$, where $\overline{B}_1$ is the closed geodesic ball with center at x_0 and radius R_1. For each constant a with $a > R_1$, denote the function $v_a : (R_1, a) \to \mathbb{R}$ by

$$v_a(t) = \int_{R_1}^{t} g(\tau)d\tau,$$

where $g(\tau) = (\tau \ln \frac{a}{\tau})^{1-n}$. Choose a large enough with $g(R_1) = (R_1 \ln \frac{a}{R_1})^{1-n} < \epsilon$ and

$$u(x_1) - v_a(r(x_1)) > \max(m-\epsilon, m-\delta),$$

where r is the distance function from x_0. Since $v_a(t) \to \infty$ as $t \to a^-$, there is a constant R_2 with $a > R_2 > R_1$ and $u(x) - v_a(r(x)) < m - \epsilon$ on $B_a - \overline{B}_2$ where $B_a(\overline{B}_2$ resp.) is the open (closed resp.) geodesic ball with center at x_0 and radius a (R_2 resp.). Thus the function $u - v_a(r)$ attains a local maximum at some point x_a in $B_2 - \overline{B}_1$.

If r is smooth at x_a, then at x_a

$$|\nabla u| = |\nabla v_a| \tag{2.1}$$

and

$$\Delta u \le \Delta v_a. \tag{2.2}$$

Since $|\nabla u| = |\nabla v_a| = (r_a \ln \frac{a}{r_a})^{1-n} > 0$ and $u - v_a(r) > m - \epsilon$ at x_a, $m > u(x_a) > m - \epsilon$. To prove $0 < |\nabla u|(x_a) < \epsilon$, it suffices to show that $0 < |\nabla v_a(r_a)| < \epsilon$. In fact, since $m - v_a(r_a) > u(x_a) - v_a(r_a) > m - \delta$, $\delta > v_a(r_a)$, there is a constant $\hat{r}$ with $a > \hat{r} > r_a$ and $v_a(\hat{r}) = \delta$. If $g(\hat{r}) > g(R_1)$, then $g(\hat{r}) > \min g = g(\frac{a}{e})$ and hence $\hat{r} > \frac{a}{e}$. It is easy to see that g is convex. Thus the area of the region bounded by the tangent line of g at $(\hat{r}, g(\hat{r}))$, the t axis and the line $t = \hat{r}$ is less than $v_a(\hat{r})$, that is

$$\frac{g(\hat{r})^2}{2g'(\hat{r})} < \delta. \tag{2.3}$$

On the other hand, $g'(t) = (1-n)(t \ln \frac{a}{t})^{-n}(\ln \frac{a}{t} - 1)$ and hence

$$g'(\hat{r}) < (n-1) g(\hat{r})^{\frac{n}{n-1}}. \tag{2.4}$$

From (2.3) and (2.4) we have

$$g(\hat{r}) < (2(n-1)\delta)^{\frac{n-1}{n-2}} < \epsilon.$$

Thus $g(r_a) \le \max(g(R_1), g(\hat{r})) < \epsilon$, the claim is true. Combining (2.1) and (2.2) together, we have at x_a

$$\begin{aligned}\epsilon|\nabla u|^{\frac{n}{n-1}} &\le \Delta u \le \Delta v_a \\ &\le (n-1)\Big(r_a \ln\frac{a}{r_a}\Big)^{-n}\Big(1-\ln\frac{a}{r_a}\Big) + \frac{n-1}{r_a}\Big(r_a \ln\frac{a}{r_a}\Big)^{1-n} \\ &\le (n-1)|\nabla v_a|^{\frac{n}{n-1}} \\ &= (n-1)|\nabla u|^{\frac{n}{n-1}} \\ &< \epsilon|\nabla u|^{\frac{n}{n-1}},\end{aligned}$$

a contradiction. Here we have used the Laplacian Comparison Theorem [7]. If r is not smooth at x_a, modifying the proof as the proof of Theorem 1, we have

$$\begin{aligned}\epsilon|\nabla u|^{\frac{n}{n-1}} &\le \Delta u \le \Delta v_a(\bar{r}) \\ &\le (n-1)|\nabla v_a(\bar{r})|^{\frac{n}{n-1}} + \frac{(n-1)\tau}{r_a-\tau} r_a^{-n}\Big(\ln\frac{a}{r_a}\Big)^{1-n} \\ &= (n-1)|\nabla u|^{\frac{n}{n-1}} + \frac{(n-1)\tau}{r_a-\tau} r_a^{-n}\Big(\ln\frac{a}{r_a}\Big)^{1-n} \\ &< \epsilon|\nabla u|^{\frac{n}{n-1}},\end{aligned}$$

for some small τ. So we also get a contradiction. This completes the proof of Theorem 2.

Remarks. (a) The assumption of nonnegative Ricci curvature is necessary since we have a C^2 increasing radial function u defined on the space form $M = \{x \in \mathbb{R}^n : |x| < 1\}$ of constant sectional curvature -4 such that

$$u(x) = \frac{1}{2}\int_0^\rho e^{-(n-1)t}\sqrt{1-t}\, dt, \quad \text{for } \rho \text{ near } 1,$$

where $\rho = |x|$.

(b) The exponent of $|\nabla u|$ is optimum since we can not replace it by any exponent α with $\alpha > \frac{n}{n-1}$ ([5]).

The simplest case of our results can be stated in the following way.

Corollary. *Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Let u be a real valued C^2 function defined* *on M with the supremum m. Suppose that there exists a positive constant ϵ such that*

$$\Delta u \geq \epsilon |\nabla u|$$

in the set $\{x \in M : m - \epsilon < u(x) < m,\ 0 < |\nabla u|(x) < \epsilon\}$. Then u is constant.

We conclude this paper by stating the sharper forms of Theorem 1 and Theorem 2 as follows

Theorem 3. *Let M be a complete Riemannian surface with nonnegative Gaussian curvature. Let h be a positive C^1 function defined on $(0, \infty)$ such that*

$$h(t)[-\ln(t)]^a \text{ decreases and } h(t)[-\ln(t)]^b \text{ increases}, \ 0 < t < 1 - a,$$

for some constants a and b, $a < 1$. Let u be a real valued C^2 function defined on M with the supremum m. Suppose that there exists a positive constant ϵ such that

$$\Delta u \geq -|\nabla u|^2 h(|\nabla u|) \tag{3.1}$$

in the set $\{x \in M : m - \epsilon < u(x) < m,\ 0 < |\nabla u|(x) < \epsilon\}$. If

$$\int_0^1 \frac{dt}{h(t) t \ln(t)} = -\infty, \tag{3.2}$$

then u is constant.

Theorem 4. *Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Let h be a positive, increasing, C^1 function defined on $(0, \infty)$ such that*

$$t^a h(t) \text{ decreases, near } t = 0^+,$$

for some constant a. Let u be a real valued C^2 function defined on M with the supremum m. Suppose that there exists a positive constant ϵ such that

$$\Delta u \geq |\nabla u|^{\frac{n}{n-1}} h(|\nabla u|) \tag{4.1}$$

in the set $\{x \in M : m - \epsilon < u(x) < m,\ 0 < |\nabla u|(x) < \epsilon\}$. *If*

$$\int_0^1 \frac{h(t)}{t} dt = \infty, \tag{4.2}$$

then u *is constant.*

Remarks. (a) The proof of these theorems essentially follows that of [5] except we need make sure that the technical process as in the proof of previous theorems still works.

(b) The interest of these theorems is that, in the case of the Euclidean space, if the integral (3.2) ((4.2) resp.) converges, then there is a nonconstant entire function u satisfying (3.1) ((4.1) resp.) (Theorem V and Theorem VII in [5]).

Acknowledgement. The authors are grateful to the referee for the valuable suggestions.

References

1. H. H. Chen, *On R. Redheffer's method in differential inequalities on complete Riemannian surfaces*, Master Thesis, Chiao Tung Univ., 1989.

2. S. Y. Cheng, *Liouville theorem for harmonic map*, Proceeding of Symposia in Pure Mathematics, **36** (1980), 147-151.

3. S. Y. Cheng and S. Y. Yau, *Differential equations on Riemannian manifolds*, Communications on Pure and Applied Mathematics, **27** (1975), 333-354.

4. P. Li and L. F. Tam, *Linear growth harmonic functions on a complete manifold*, Journal of Differential Geometry, **29** (1989), 421-425.

5. R. Redheffer, *On the inequality* $\Delta u \geq f(u, |\text{grad}\, u|)$, Journal of Mathematical Analysis, (1960), 277-299.

6. J. A. Wolf, *Space of constant curvature*, Mcgraw-Hill, New York, 1967.

7. S. T. Yau, *Harmonic function on complete Riemannian manifolds*, Communication on Pure and Applied Mathematics, **17** (1975), 201-228.

Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, R. O. C.