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APPROXIMATIONS TO OPTIMAL STOPPING RULES FOR NORMAL RANDOM VARIABLES

BY

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Abstract. For $X_1, X_2, \ldots$ i.i.d. with finite mean and $Y_n = \max\{X_1, \ldots, X_n\} - cn$, c positive, a number of authors have considered the problem of determining an optimal stopping rule for the reward sequence Y_n. In general, it requires complete knowledge of the distribution of the X_i. Martinsek (1984) examined the problem of approximating the optimal expected reward when the X_i are exponentially distributed with unknown mean. This paper deals with the case where X_i are $\mathrm{N}(\theta, 1)$ distributed with θ unknown. Stopping rules designed to approximate the optimal rule (which can be used only when θ is known) are proposed. Under certain conditions the difference between the expected reward using the proposed stopping rules and the optimal expected reward vanishes as c approaches zero.

1. Introduction. In the theory of optimal stopping procedures there arises an important class of problems of the following nature:

Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with $E|X_1| < \infty$. We observe the X_i's sequentially and are allowed to stop observing at any stage. If we stop with n-th observation, we receive a reward Y_n, where Y_n is some measurable function of $X_1, X_2, \ldots, X_n$. The problem is to find a stopping rule which maximizes the expected reward. Such a rule, if one exists, will be called optimal stopping rule.

We shall be considering a particular class of reward functions of the form:

$$Y_n = \max\{X_1, X_2, \ldots, X_n\} - cn, \quad c > 0. \tag{1.1}$$

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This problem and variations on it have been treated extensively by MacQueen and Miller (1960), Derman and Sacks (1960), Sakaguchi (1961), Chow and Robbins (1961, 1963), Yahav (1966), Cohn (1967) and DeGroot (1968).

The optimal stopping rule for this problem, i.e., the rule which maximizes $E(Y_T)$ over all stopping rules T with $E(Y_T^-) < \infty$, is

$$T_c^* = \inf\{n \geq 1 : X_n \geq r_c\}, \tag{1.2}$$

where

$$E(X_1 - r_c)^+ = c \tag{1.3}$$

and

$$E(Y_{T_c^*}) = E(X_{T_c^*}) - cE(T_c^*) = r_c. \tag{1.4}$$

(for a proof of this result, see Chow, Robbins and Siegmund 1971, pp.56-58). However, in order to use the stopping rule T_c^* it is necessary to know r_c, which in turn requires knowledge of the distribution of the X_i. If only partial information about the distribution is available, it may not be possible to compute r_c, and in such cases it would be desirable to approximate the optimal rule T_c^* and the optimal reward $E(Y_{T_c^*})$ as well.

This type of approximation problem has been considered previously by Bramblett (1965) and Martinsek (1984). Bramblett (1965) showed that for certain cases involving unknown location parameters, the ratio of the expected reward using an approximating stopping rule to the optimal expected reward approaches one as c goes to zero. In other words, he showed that certain approximating stopping rules are asymptotically optimal in the sense of Kiefer and Sacks (1963) and Bickel and Yahav (1967, 1968). But he was unable to get results about the vanishing of the difference in expected rewards as c approaches zero.

Martinsek (1984) considered the case where X_i has an exponential distribution with unknown mean, and proved that the difference between the optimal expected reward and the expected reward using an approximating stopping rule vanishes as c goes to zero. But as he mentioned in the last section, results about the performance of the approximating stopping rule depend on the properties of the underlying distribution. Therefore, his method cannot be applied to any case other than exponential.

The purpose of this paper is to consider the special case of the normal distribution with unknown mean, and to prove a result which suggests that a certain approximation to the optimal rule performs well asymptotically.

Assume throughout the rest of this paper that the distribution of X_i is $\mathrm{N}(\theta, 1)$. An easy computation shows that

$$E(X_1 - r)^+ = \varphi(r-\theta) - (r-\theta)[1-\Phi(r-\theta)], \tag{1.5}$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy.$$

Therefore, the optimal stopping rule is

$$T_c^* = \inf\{n \geq 1 : X_n \geq r_c\}, \tag{1.6}$$

where

$$\varphi(r_c-\theta) - (r_c-\theta)[1-\Phi(r_c-\theta)] = c. \tag{1.7}$$

Unfortunately, it is not possible to give closed-form expressions for r_c as Martinsek (1984) did for the exponential case. The methods of Martinsek (1984) cannot be used to approximate the optimal stopping rule in such situation. We will do it implicitly, using the following idea. For each n, we can estimate r_c by $\hat{r}_{c,n}$, where $\hat{r}_{c,n}$ satisfies

$$\varphi(\hat{r}_{c,n} - \overline{X}_n) - (\hat{r}_{c,n} - \overline{X}_n)[1-\Phi(\hat{r}_{c,n} - \overline{X}_n)] = c, \tag{1.8}$$

and approximate the optimal rule T_c^* by

$$\hat{T}_c = \inf\{n \geq n_c : X_n \geq \hat{r}_{c,n}\}, \tag{1.9}$$

where n_c is a positive integer depending on c. We hope that $E(Y_{\hat{T}_c})$ is close to $E(Y_{T_c^*})$. In the next section it is proved that if $n_c = \delta c^{-\alpha}$ for some $\delta > 0$ and $0 < \alpha < 1$, then

$$E(Y_{T_c^*}) - E(Y_{\hat{T}_c}) \to 0, \qquad \text{as} \quad c \to 0. \tag{1.10}$$

2. Some properties of r_c and T_c^*. In this section we prove some inequalities for r_c and T_c^* which will be needed later.

Lemma 2.1. *Let r_c be as defined in* (1.7). *Then for* $0 < c < 1/\sqrt{2\pi}$,

$$c^{-1}e^{-(r_c-\theta)^2/2} \geq \sqrt{2\pi}\ [1+(r_c-\theta)^2].$$

Proof. For $x > 0$, we have

$$\frac{x}{1+x^2}e^{-x^2/2} \leq \int_x^\infty e^{-y^2/2}dy \leq \frac{1}{x}e^{-x^2/2}. \tag{2.1}$$

(cf. Freedman (1983), (4) Lemma, p. 11–12). By (1.5) and (2.1), if $r > \theta$.

$$\begin{aligned} E(X_1-r)^+ &= \frac{1}{\sqrt{2\pi}}e^{-(r-\theta)^2/2} - (r-\theta)\int_{r-\theta}^\infty \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\ &\leq \frac{1}{\sqrt{2\pi}}e^{-(r-\theta)^2/2} - \frac{1}{\sqrt{2\pi}}\frac{(r-\theta)^2}{1+(r-\theta)^2}e^{-(r-\theta)^2/2} \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{1+(r-\theta)^2}e^{-(r-\theta)^2/2}. \end{aligned} \tag{2.2}$$

For $0 < c < 1/\sqrt{2\pi}$, it is easy to see that $r_c > \theta$. Let r_c' be the number satisfying

$$\frac{1}{\sqrt{2\pi}}\frac{1}{1+(r_c'-\theta)^2}e^{-(r_c'-\theta)^2/2} = c.$$

By (2.2), we have $r_c \leq r_c'$. Therefore

$$\begin{aligned} c^{-1}e^{-(r_c-\theta)^2/2} &\geq c^{-1}e^{-(r_c'-\theta)^2/2} \\ &= \sqrt{2\pi}\ [1+(r_c'-\theta)^2] \\ &\geq \sqrt{2\pi}\ [1+(r_c-\theta)^2], \end{aligned}$$

completing the proof.

Lemma 2.2. *For* $0 < c < 1/\sqrt{2\pi}$,

$$\frac{P\{X_1 \geq r_c\}}{c} \geq r_c - \theta.$$

Proof. By (1.7),

$$\frac{1}{\sqrt{2\pi}}c^{-1}e^{-(r_c-\theta)^2/2} - (r_c-\theta)\frac{P\{X_1 \geq r_c\}}{c} = 1.$$

From Lemma 2.1,

$$[1+(r_c-\theta)^2]-(r_c-\theta)\frac{P\{X_1\geq r_c\}}{c}\leq 1.$$

Lemma 2.2 follows immediately.

Lemma 2.3. *For any $\beta>0$, $r_c\leq o(c^{-\beta})$ as $c\to 0$.*

Proof. By (2.2), if $r>\theta$,

$$\begin{aligned} E(X_1-r)^+ &\leq \frac{1}{\sqrt{2\pi}}\frac{1}{1+(r-\theta)^2}e^{-(r-\theta)^2/2} \\ &\leq \frac{1}{\sqrt{2\pi}}e^{-(r-\theta)^2/2}. \end{aligned} \tag{2.3}$$

Let r_c'' be the number greater than θ satisfying

$$\frac{1}{\sqrt{2\pi}}e^{-(r_c''-\theta)^2/2}=c.$$

By (2.3), we have

$$r_c\leq r_c''=\theta+\big(2\log\frac{1}{c\sqrt{2\pi}}\big)^{1/2}=o\big(c^{-\beta}\big)\qquad\text{as}\quad c\to 0,\qquad\text{for}\quad\beta>0.$$

Lemma 2.4. *Let T_c^* be as defined in (1.6) and (1.7). Then $E[(T_c^*)^p]=o(c^{-p})$ as $c\to 0$, for all $p>0$.*

Proof.

$$\begin{aligned} E(cT^*) &= c\cdot\sum_{k=1}^{\infty}kP\{T_c^*=k\} \\ &= c\cdot\sum_{k=1}^{\infty}kP\{X_1<r_c,\dots,X_{k-1}<r_c,X_k\geq r_c\} \\ &= c\cdot\sum_{k=1}^{\infty}k\big(P\{X_1<r_c\}\big)^{k-1}P\{X_1\geq r_c\} \\ &= \frac{c}{P\{X_1\geq r_c\}}. \end{aligned}$$

By Lemma 2.2,

$$E(cT_c^*)\leq\frac{1}{r_c-\theta}\to 0\qquad\text{as}\quad c\to 0. \tag{2.4}$$

Since T_c^* is geometrically distributed, for all $p > 0$ there exists $M_p > 0$ such that

$$\begin{aligned} E\left[(cT_c^*)^p\right] &= c^p E\left[(T_c^*)^p\right] \\ &\leq c^p M_p \cdot \left[E(T_c^*)\right]^p = M_p \cdot \left[E(cT_c^*)\right]^p \end{aligned}$$

Lemma 2.4 follows from (2.4).

3. Performance of $\hat{T}_c$. Unlike T_c^*, the stopping rule $\hat{T}_c$ defined by (1.9) is not a geometric random variable. The key to studying the behavior of $\hat{T}_c$ is to approximate $\hat{T}_c$ by appropriate geometrically distributed random variables $T_{c,\beta}^+$ and $T_{c,\beta}^-$ which are defined as follows.

$$T_{c,\beta}^+ = \inf\left\{n \geq 1 : X_n \geq r_c + c^\beta\right\} \tag{3.1}$$

and

$$T_{c,\beta}^- = \inf\left\{n \geq 1 : X_n \geq r_c - c^\beta\right\}, \tag{3.2}$$

where $\beta > 0$.

Lemma 3.1. *For any $\beta > 0$, both $ET_{c,\beta}^+/ET_c^*$ and $ET_{c,\beta}^-/ET_c^*$ go to one as $c \to 0$.*

Proof.

$$\frac{ET_c^+}{ET_c^*} = \frac{P\{X_1 \geq r_c\}}{P\{X_1 \geq r_c + c^\beta\}}.$$

By (2.1)

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \frac{r_c - \theta}{1 + (r_c - \theta)^2} e^{-(r_c-\theta)^2/2} &\leq P\{X_1 \geq r_c\} \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{1}{r_c - \theta} e^{-(r_c-\theta)^2/2} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \frac{r_c - \theta + c^\beta}{1 + (r_c - \theta + c^\beta)^2} e^{-(r_c-\theta+c^\beta)^2/2} &\leq P\{X_1 \geq r_c + c^\beta\} \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{1}{r_c - \theta + c^\beta} e^{-(r_c-\theta+c^\beta)^2/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{(r_c - \theta)^2 + c^\beta(r_c - \theta)}{1 + (r_c - \theta)^2} e^{(r_c-\theta)c^\beta + c^\beta/2} \\ &\leq P\{X_1 \geq r_c\}/P\{X_1 \geq r_c + c^\beta\} \\ &\leq \frac{1 + (r_c - \theta)^2 + 2c^\beta(r_c - \theta) + c^{2\beta}}{(r_c - \theta)^2 + c^\beta(r_c - \theta)} e^{(r_c-\theta)c^\beta + c^\beta/2}. \end{aligned} \tag{3.3}$$

By Lemma 2.3, it is not difficult to see that both sides in (3.3) go to one as $c \to 0$. For $ET^{-}_{c,\beta}/ET^{*}_{c}$, the argument is quite similar.

Lemma 3.2. *Define* $\hat{T}_c$ *by* (1.8) *and* (1.9) *with* $n_c = \delta c^{-\alpha}$ *for some* $\delta > 0$ *and* $0 < \alpha < 1$. *Then for* $0 < \beta < \alpha/2$, *as* $c \to 0$,

$$E(\hat{T}_c) \le o(1) + n_c - 1 + E(T^{+}_{c,\beta}). \tag{3.4}$$

Proof. Define

$$L_{c,\beta} = \sup\{n \ge 1 : |\overline{X}_n - \theta| \ge c^{\beta}\}, \qquad 0 < \beta < \alpha/2.$$

By Theorem 7 of Chow and Lai (1975),

$$\{(c^{2\beta}L_{c,\beta})^p : c < 1/\sqrt{2\pi}\} \text{ is uniformly integrable for all } p > 0. \tag{3.5}$$

In particular, for all $p > 0$,

$$E(L^{p}_{c,\beta}) = O(c^{-2\beta p}) \qquad \text{as} \quad c \to 0.$$

Since $T^{+}_{c,\beta}$ is geometric distributed, Lemma 2.4 and Lemma 3.1 imply that

$$\{(cT^{+}_{c,\beta})^p : c < 1/\sqrt{2\pi}\} \text{ is uniformly integrable for all } p > 0. \tag{3.7}$$

For K sufficiently large, $Kc^{-1} > 2n_c$ for all $c < 1/\sqrt{2\pi}$. We have, treating $Kc^{-1}/2$ as an integer,

$$\begin{aligned} &P\{c\hat{T}_c > K\} \\ &= P\{L_{c,\beta} > Kc^{-1}/2, c\hat{T}_c > K\} + P\{L_{c,\beta} \le Kc^{-1}/2, c\hat{T}_c > K\} \\ &\le P\{c^{2\beta}L_{c,\beta} > K/2\} + P\{L_{c,\beta} \le Kc^{-1}/2, \hat{T}_c > Kc^{-1}\}. \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\{L_{c,\beta} \le Kc^{-1}/2, \hat{T}_c > Kc^{-1}\} \\ &\subset \{\theta - c^{\beta} \le \overline{X}_n \le \theta + c^{\beta}, X_n < \hat{r}_{c,n} = \overline{X}_n + (r_c - \theta) \\ &\qquad\qquad \text{for all } Kc^{-1}/2 < n \le Kc^{-1}\} \\ &\subset \{X_n < (r_c + c^{\beta}) \text{ for all } Kc^{-1}/2 < n \le Kc^{-1}\} \\ &\subset \{\tilde{T}^{+}_{c,\beta} \ge Kc^{-1}/2\}, \end{aligned}$$

where

$$\tilde{T}^+_{c,\beta} = \inf\{m \geq 1 : X_{m+K/2c} \geq r_c + c^\beta\}.$$

Since $P\{\tilde{T}^+_{c,\beta} \geq Kc^{-1}/2\} = P\{T^+_{c,\beta} \geq Kc^{-1}/2\} = P\{cT^+_{c,\beta} \geq K/2\}$, by (3.8)

$$P\{c\hat{T}_c > K\} \leq P\{c^{2\beta} L_{c,\beta} > K/2\} + P\{cT^+_{c,\beta} > K/2\}.$$

Hence by (3.5) and (3.7),

(3.9) $\{(c\hat{T}_c)^p : c < 1/\sqrt{2\pi}\}$ is uniformly integrable for all $p > 0$.

Then from (3.6) with $p > (\alpha/2 - \beta)^{-1}$ and (3.9) with $p = 2$,

$$\begin{aligned} E(\hat{T}_c) &= E(\hat{T}_c \cdot I_{\{L_{c,\beta} \geq n_c\}}) + E(\hat{T}_c \cdot I_{\{L_{c,\beta} < n_c\}}) \\ &\leq [E(\hat{T}_c^2)]^{1/2} P^{1/2}\{L_{c,\beta} \geq n_c\} + E\big(\inf\{n \geq n_c : X_n \geq r_c + c^\beta\}\big) \\ &\leq [E(\hat{T}_c^2)]^{1/2} n_c^{-p/2} \cdot E^{1/2}(L^p_{c,\beta}) + (n_c - 1) + E(T^+_{c,\beta}) \\ &= o(1) + (n_c - 1) + E(T^+_{c,\beta}). \end{aligned}$$

Lemma 3.3. *Define* $\hat{T}_c$ *by* (1.8) *and* (1.9) *with* $n_c = \delta c^{-\alpha}$ *for some* $\delta > 0$ *and* $0 < \alpha < 1$. *Then for* $0 < \beta < \alpha/2$, *as* $c \to 0$,

$$E(\hat{T}_c) \geq (n_c - 1) + E(T^-_{c,\beta}) - o(c^q) \quad \textit{for} \quad q > 0.$$

Proof.

$$\begin{aligned} E(\hat{T}_c) &\geq E(\hat{T}_c \cdot I_{\{L_{c,\beta} < n_c\}}) \\ &\geq E[(\inf\{n \geq n_c : X_n \geq r_c - c^\beta\})(I_{\{L_{c,\beta} < n_c\}})] \\ &= E(\inf\{n \geq n_c : X_n \geq r_c - c^\beta\}) - E[\inf\{n \geq n_c : X_n \geq r_c - c^\beta\})(I_{\{L_{c,\beta} \geq n_c\}})] \\ &\geq (n_c - 1) + E(T^-_{c,\beta}) - E^{1/2}(\inf\{n \geq n_c : X_n \geq r_c - c^\beta\})^2 \cdot P^{1/2}\{L_{c,\beta} \geq n_c\} \\ &= (n_c - 1) + E(T^-_{c,\beta}) - E^{1/2}[(n_c - 1) + T^-_{c,\beta}]^2 \cdot P^{1/2}\{L_{c,\beta} \geq n_c\}. \end{aligned}$$

By Lemma 2.4 and Lemma 3.1,

$\{(cT^-_{c,\beta})^p : c < 1/\sqrt{2\pi}\}$ is uniformly integrable for all $p > 0$.

In particular, for all $p > 0$,

$$E(T^-_{c,\beta})^p \leq O(c^{-p}).$$

Therefore,

$$\begin{aligned}
&E^{1/2}[(n_c-1)+T^{-}_{c,\beta}]^2 \cdot P^{1/2}\{L_{c,\beta} \ge n_c\} \\
&\le [(n_c-1)^2+2(n_c-1)E(T^{-}_{c,\beta})+E(T^{-}_{c,\beta})^2]^{1/2} \cdot [n_c^{-p}E(L^p_{c,\beta})]^{1/2} \\
&= [(O(c^{-2\alpha})+O(c^{-\alpha}\cdot c^{-1})+O(c^{-2})]^{1/2} \cdot O(c^{((\alpha/2)-\beta)p}) \\
&= O(c^{((\alpha/2)-\beta)p-1}) = o(c^q).
\end{aligned}$$

By (3.10), the proof is completed.

Lemma 3.4. *If $n_c = \delta c^{-\alpha}$ for some $\delta > 0$ and $0 < \alpha < 1$, then for every $\beta \in (0, \alpha/2)$,*

$$\sum_{j=n_c}^{\infty} E\big(|X_j| I_{\{|\bar{X}_j-\theta|>c^\beta\}}\big) \to 0,$$

as $c \to 0$.

Proof. Choosing p in (3.5) large enough so that $\beta < \alpha(p-2)/2p$, we have

$$\begin{aligned}
\sum_{j=n_c}^{\infty} E\big(|X_j| I_{\{|X_j-\theta|>c^\beta\}}\big) &\le \sum_{j=c_n}^{\infty} E^{1/2}(X_j^2)P^{1/2}\{L_{c,\beta} \ge j\} \\
&= O(1)\sum_{j=n_c}^{\infty} j^{-p/2}E^{1/2}(L^p_{c,\beta}) \\
&\le O(1)\sum_{j=n_c}^{\infty} j^{-p/2}\cdot c^{-\beta p} \\
&= O\Big(c^{-\beta p}\int_{n_c}^{\infty} x^{-p/2}dx\Big) \\
&= O(c^{\alpha(p/2-1)-\beta p}) = o(1).
\end{aligned}$$

Theorem. *Define $\hat{T}_c$ by (1.8) and (1.9) with $n_c = \delta c^{-\alpha}$ for some $\delta > 0$ and $0 < \alpha < 1$. Then as $c \to 0$,*

$$E(Y_{T^*_c}) - E(Y_{\hat{T}_c}) \to 0.$$

That is, the expected loss due to not knowing θ and using the approximating rule $\hat{T}_c$ vanishes as $c \to 0$.

Proof. Because T_c^* is optimal and $E(Y_{T_c^*}) = r_c$,

$$0 \le E(Y_{T_c^*}) - E(Y_{\hat{T}_c}) \le r_c - E(X_{\hat{T}_c}) + cE\hat{T}_c. \tag{3.11}$$

From Lemma 3.4, for $0 < \beta < \alpha/2$ and $c < 1/\sqrt{2\pi}$, by independence of the X_i,

$$\begin{aligned}
E(X_{\hat{T}_c}) &= \sum_{j=n_c}^{\infty} E(X_j I_{\{\hat{T}_c=j\}}) \\
&\ge \sum_{j=n_c}^{\infty} E(X_j I_{\{\hat{T}_c=j,|\overline{X}_j-\theta|\le c^\beta\}}) + o(1) \\
&\ge \sum_{j=n_c}^{\infty} E(X_j I_{\{\hat{T}_c\ge j,|\overline{X}_j-\theta|\le c^\beta, X_j\ge r_c+c^\beta\}}) + o(1) \\
&= \sum_{j=n_c}^{\infty} E(X_j I_{\{\hat{T}_c\ge j, X_j\ge r_c+c^\beta\}}) - \sum_{j=n_c}^{\infty} E(|X_j| I_{\{|\overline{X}_j-\theta|>c^\beta\}}) + o(1) \\
&= \sum_{j=n_c}^{\infty} P\{\hat{T}_c \ge j\} E(X_j I_{\{X_j\ge r_c+c^\beta\}}) + o(1) \\
&= E(X_1 \cdot I_{\{X_1\ge r_c+c^\beta\}}) \cdot [E(\hat{T}_c) - (n_c-1)] + o(1), \quad \text{as} \quad c \to 0.
\end{aligned}$$

By Lemma 3.3,

$$E(X_{\hat{T}_c}) \ge E(X_1 \cdot I_{\{X_1\ge r_c+c^\beta\}}) \cdot E(T_{c,\beta}^-) + o(1), \quad \text{as} \quad c \to 0.$$

$$\begin{aligned}
&E(X_1 \cdot I_{\{X_1\ge r_c+c^\beta\}}) \cdot E(T_{c,\beta}^-) \\
&= \left[\frac{1}{\sqrt{2\pi}} e^{-(r_c+c^\beta-\theta)^2/2} + \theta P\{X_1 \ge r_c + e^\beta\}\right] \Big/ P\{X_1 \ge r_c - c^\beta\}.
\end{aligned}$$

From (2.1),

$$P\{X_1 \ge r_c - e^\beta\} \le \frac{1}{\sqrt{2\pi}} \frac{1}{(r_c - c^\beta - \theta)} e^{-(r_c-c^\beta-\theta)^2/2}.$$

Therefore, by Lemma 2.3,

$$\begin{aligned}
E(X_{\hat{T}_c}) &\ge (r_c - c^\beta - \theta) e^{[(r_c-c^\beta-\theta)^2-(r_c+c^\beta-\theta)^2]/2} \\
&\qquad + \theta \cdot \frac{P\{X_1 \ge r_c + c^\beta\}}{P\{X_1 \ge r_c - c^\beta\}} + o(1) \\
&= r_c \cdot e^{-2c^\beta(r_c-\theta)} + o(1) \\
&= r_c(1 + o(c^\beta)) + o(1).
\end{aligned} \tag{3.12}$$

By Lemma 2.4, Lemma 3.1 and Lemma 3.2, as $c \to 0$,

$$cE(\hat{T}_c) \to 0.$$

Therefore, by (3.11), (3.12) and Lemma 2.3,

$$E(Y_{T_c^*}) - E(Y_{\hat{T}_c}) \to 0, \quad \text{as} \quad c \to 0.$$

The proof is completed.

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