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ON THE TATE MODULES OF ABSOLUTELY SIMPLE ABELIAN VARIETIES OF TYPE II

BY

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Abstract. The Tate modules attached to absolutely simple abelian varieties of type II are studied. We construct some explicit Galois submodules, and show that at least one of them is of the right half dimension and carrying with it non-degenerate equivariate alternating form. This method can be used in the study of ℓ-adic representations attached to such abelian varieties.

1. Introduction. Let K be an algebraic number field and let $\overline{K}$ be an algebraic closure of K. Let $G_K = \mathrm{Gal}(\overline{K}/K)$. For an abelian variety A defined over K, we denote by $\mathrm{End}^o(A)$ the endomorphism algebra $\mathrm{End}_{\overline{K}}(A) \otimes_Z Q$ of A. For each prime number ℓ, let T_ℓ be the Tate module of A and let $V_\ell = T_\ell \otimes_{Z_\ell} Q_\ell$. The Galois group G_K acts continuously on T_ℓ. Fixed a K-polarization on A once of all. Let ψ be the associated Riemann form on V_ℓ and $'$ be the induced Rosati involution on $\mathrm{End}^o(A)$. One has $\psi(fv, w) = \psi(v, f'w)$ for all f in $\mathrm{End}^o(A)$ and v, w in V_ℓ.

In this paper, we are interested in the following so called type II absolutely simple abelian varieties: let A be an abelian variety defined over K such that $\dim A = 2gd$ and $\mathrm{End}^o(A) = \mathrm{End}_K(A) \otimes_Z Q = D$ is an indefinite quaternion algebra whose center is a totally real field E of degree g. For such abelian varieties, it is well-known that V_ℓ is a free $E_\ell = E \otimes_Q Q_\ell$-module of rank $4d$ (cf. §2). The action of G_K on V_ℓ is E_ℓ-linear. Moreover,

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$E_\ell = \prod_{\lambda|\ell} E_\lambda$, where λ runs over all primes of E dividing ℓ. Then G_K acts E_λ-linearly on $V_\lambda = V_\ell \otimes_{E_\ell} E_\lambda$. One has the so called λ-adic representation $\rho_\lambda : G_K \longrightarrow \mathrm{Aut}_{E_\lambda}(V_\lambda)$ attached to A (cf. [6]). The Riemann form ψ induces a non-degenerate alternating E_λ-bilinear form ψ_λ on V_λ (cf. §2). It is known that Im ρ_λ lies in $\mathrm{GSp}(V_\lambda, \psi_\lambda)$ the group of symplectic similitudes with respect to ψ_λ. Let G_{V_λ} be the algebraic envelope of $\mathrm{Im}\rho_\lambda$ in GL_{V_λ} and let S_{V_λ} be the connected component of the identity of $G_{V_\lambda} \cap SL_{V_\lambda}$. Then $S_{V_\lambda} \subseteq \mathrm{Sp}(V_\lambda, \psi_\lambda)$.

The main purpose of this paper is to prove the following result:

Theorem A. *Suppose λ is a prime of E where D splits. Then there exists a G_K-submodule W_λ of V_λ such that the following hold:*

(i) *dim $W_\lambda = 2d$.*

(ii) *The Galois module V_λ is isomorphic, over E_λ, to the direct sum of two copies of W_λ.*

(iii) *$\psi_\lambda|_{W_\lambda \times W_\lambda}$ is a non-degenerate S_{V_λ}-equivariant alternating form.*

The main indication of this result is that the study of λ-adic representaions attached to absolutely simple abelian varieties of type II can be treated as the case where dim $A = dg$ and $\mathrm{End}^o(A) = \mathrm{End}_K(A) \otimes_Z Q = E$ as above. In particular, some of the results of J-P. Serre in [7] can be extended to some absolutely simple abelian varieties of type II.

Finally, I would like to express my deep gratitude to Kenneth Ribet for his assistance.

2. Preliminaries. With the same notations as in §1. For each prime number ℓ, one defines the ℓ-adic Tate modules T_ℓ and V_ℓ attached to A by

$$T_\ell = \varprojlim_n A[\ell^n], \quad V_\ell = T_\ell \otimes_{Z_\ell} Q_\ell,$$

where $A[\ell^n]$ is the kernel of $\ell^n : A(\overline{K}) \longrightarrow A(\overline{K})$. Then T_ℓ is a free Z_ℓ-module of rank 2dim$A$. The Galois group G_K acts continuously on T_ℓ, so one gets a homomorphism

$$\rho_\ell : G_K \longrightarrow \mathrm{Aut}_{Z_\ell}(T_\ell) \subseteq \mathrm{Aut}_{Q_\ell}(V_\ell).$$

ρ_ℓ is called the ℓ-adic representation attached to A.

For the remain of this article, we shall assume that A is an absolutely simple abelian variety of type II as in §1. Namely, dim $A = 2gd$ and $\mathrm{End}^\circ(A) = \mathrm{End}_K(A) \otimes_Z Q = D$ is an indefinite quaternion algebra whose center is a totally real field E of degree g. The action G_K on V_ℓ is E_ℓ-linear. So the image of ρ_ℓ lies in $\mathrm{Aut}_{E_\ell}(V_\ell)$. Further, one has the decomposition $E_\ell = \prod_{\lambda|\ell} E_\lambda$, where λ runs over all primes of E dividing ℓ. For each $\lambda|\ell$, let $V_\lambda = V_\lambda \otimes_{E_\ell} E_\ell$. Then G_K acts E_λ-linearly on V_λ, and one gets a continuous homomorphism $\rho_\lambda : G_\lambda \longrightarrow \mathrm{Aut}_{E_\lambda}(V_\lambda)$. ρ_λ is called the λ-adic representation attached to A.

The followng result is well-known.

Proposition (2.1). *V_ℓ is a free E_ℓ-module of rank $4d$. Accordingly, each V_λ is an E_λ-vector space of dimension $4d$.*

Proof. cf. [6], Theorem (2.1.1).

Let $V_\ell(\mu)$ be the Tate module attached to the group of ℓ-powers of roots of unity in $\overline{K}$, and let $\chi_\ell : G_K \longrightarrow Z_\ell^*$ be the cyclotomic character. For a fixed K-polarization on A, the associated Riemann form on V_ℓ is a non-degenerate alternating form $\psi : V_\ell \times V_\ell \longrightarrow V_\ell(\mu)$. The Weil pairing is well-known to be G_K-equivariant, one has

$$\psi(\sigma v, \sigma w) = \chi_\ell(\sigma) \cdot \psi(v, w) \quad \text{for all} \quad \sigma \in G_K \quad \text{and} \quad v, w \text{ in } V_\ell.$$

Thus the image of ρ_ℓ lies in $\mathrm{GSp}(V_\ell, \psi)$ the group of symplectic similitudes with respect to ψ. Let G_{V_ℓ} be the algebraic envelope of $\mathrm{Im}\rho_\ell$ in GL_{V_ℓ}. Then $G_{V_\ell} \subseteq \mathrm{GSp}(V_\ell, \psi)$.

In the type of abelian variety we considered, the Rosati involution fixes the center E of D. Hence one has

$$\psi(ev, w) = \psi(v, ew) \quad \text{for all} \quad e \text{ in } E \quad \text{and} \quad v, w \text{ in } V_\ell.$$

By Q_ℓ-linearity, one has

$$\psi(e_\ell v, w) = \psi(v, e_\ell w) \quad \text{for all} \quad e_\ell \text{ in } E_\ell \quad \text{and} \quad v, w \text{ in } V_\ell.$$

The following lemma is well-known.

Lemma (2.2.) *There exists a unique non-degenerate E_ℓ-bilinear alternating form $\Phi : V_\ell \times V_\ell \longrightarrow E_\ell$ such that*

$$\mathrm{Tr}_{E_\ell/Q_\ell}(\Phi(v,w)) = \psi(v,w) \quad \textit{for all } v, w \textit{ in } V_\ell.$$

Proof. cf.[3], Sublemma 4.7.

Recall that $E_\ell = \prod_{\lambda|\ell} E_\ell$ and $V_\lambda = V_\lambda \otimes_{E_\ell} E_\lambda$. Let $\pi_\lambda : E_\ell \longrightarrow E_\lambda$ be the natural projection. Then there exists a unique non-degenerate E_λ-bilinear alternating form ψ_λ on V_λ such that

$$\psi_\lambda(v \otimes_{E_\ell} a, w \otimes_{E_\ell} b) = ab\pi_\lambda(\Phi(v,w)) \text{ for all } a, b \text{ in } E_\lambda \text{ and } v,\ w \text{ in } V_\ell.$$

(cf. [2], ch. 9, §1, $n°4$).

Let $\chi_{\ell/\lambda} : G_K \longrightarrow E_\lambda^*$ be the composition map $G_K \xrightarrow{\chi_\ell} Z_\ell^* \subseteq E_\lambda^*$. Then we have the following.

Lemma (2.3). (i) $\psi_\lambda(\sigma v_\lambda, \sigma w_\lambda) = \chi_{\ell/\lambda}(\sigma)\psi(v_\lambda, w_\lambda)$ *for all* σ *in* G_K *and* v_λ, w_λ *in* V_λ.

(ii) $\psi_\lambda(fv_\lambda, w_\lambda) = \psi_\lambda(v_\lambda, f'w_\lambda)$ *for all* f *in* D *and* v_λ, w_λ *in* V_λ.

Proof. For (i), it suffices to show that $\Phi(\sigma v, \sigma w) = \chi_\ell(\sigma)\Phi(v,w)$ for all σ in G_K and v, w in V_ℓ. For a fixed $\sigma \in G_K$, one considers the map Φ' defined by

$$\Phi'(v,w) = \chi_\ell(\sigma^{-1})\Phi(\sigma v, \sigma w).$$

It is easy to see that Φ' is an E_ℓ-bilinear form: $V_\ell \times V_\ell \longrightarrow E_\ell$. Further,

$$\mathrm{Tr}_{E_\ell/Q_\ell}(\chi_\ell(\sigma^{-1})\Phi(\sigma v, \sigma w)) = \chi_\ell(\sigma^{-1})\mathrm{Tr}_{E_\ell/Q_\ell}(\Phi(\sigma v, \sigma w)) = \psi(v,w).$$

By the uniqueness of Φ, one concludes that $\Phi(\sigma v, \sigma w) = \chi_\ell(\sigma)\Phi(v,w)$.

For (ii), it suffices to show that $\Phi(fv, w) = \Phi(v, f'w)$ for all f in D and v, w in V_ℓ. For a fixed f in D, one considers the maps Φ', Φ'' defined by

$$\Phi'(v,w) = \Phi(fv, w) \quad \text{and} \quad \Phi''(v,w) = \Phi(v, f'w).$$

It is easy to see that Φ', Φ'' are E_ℓ-bilinear forms. Further,

$$\mathrm{Tr}_{E_\ell/Q_\ell}(\Phi'(v,w)) = \psi(fv,w) = \psi(v,f'w) = \mathrm{Tr}_{E_\ell/Q_\ell}(\Phi''(v,w)).$$

By the uniqueness of Φ, one concludes that

$$\Phi(fv,w) = \Phi(v,f'w).$$

This completes the proof.

Let G_{V_λ} be the algebraic envelope of $\mathrm{Im}\rho_\lambda$ in GL_{V_λ}. Then one has $G_{V_\lambda} \subseteq \mathrm{GSp}\,(V_\lambda, \psi_\lambda)$. Let S_{V_λ} be the connected component of the identity of $G_{V_\lambda} \cap SL_{V_\lambda}$. Then $S_{V_\lambda} \subseteq \mathrm{Sp}(V_\lambda, \psi_\lambda)$.

We now recall some consequences of G. Faltings' well-known Theorems ([4], §5, Satz 3, Satz 4).

Proposition (2.4). (i) *G_K acts on V_λ semisimply. Consequently, G_{V_λ} is a reductive algebraic group over E_λ.*

(ii) *$End_{G_K}(V_\lambda) = D_\lambda = D \otimes_E E_\lambda$.*

(iii) *If G_{V_ℓ} is connected, then $G_{V_\lambda} = S_{V_\lambda} \cdot G_m$.*

Proof. (i) follows immediately from Faltings' Satz 3.

By Faltings' Satz 4, $\mathrm{End}_{G_K}(V_\ell) = D_\ell = D \otimes_Q Q_\ell$. Since E is the center of D, one has $\mathrm{End}_{G_K,E_\ell}(V_\ell) = D_\ell$. Assertion (ii) follows from the decompostions $\mathrm{End}_{G_K,E_\ell}(V_\ell) = \prod_{\lambda|\ell} \mathrm{End}_{G_K,E_\lambda}(V_\lambda)$ and $D_\ell = (D \otimes_E E) \otimes_Q Q_\ell = \prod_{\lambda|\ell} D_\lambda$.

To prove (iii), let $\mathcal{G}_\ell$ be the Lie algebra of the ℓ-adic Lie group $\mathrm{Im}\rho_\ell$. By Faltings' Satz 3, $\mathcal{G}_\ell$ is a reductive Lie algebra over Q_ℓ. So, $\mathcal{G}_\ell = [\mathcal{G}_\ell, \mathcal{G}_\ell] \oplus \mathcal{C}_\ell$, where $[\mathcal{G}_\ell, \mathcal{G}_\ell]$ is the derived subalgebra of $\mathcal{G}_\ell$ and $\mathcal{C}_\ell$ is the center of $\mathcal{G}_\ell$. Let det: $\mathrm{Im}\rho_\ell \longrightarrow Z_\ell^*$ be the determinant. The Lie algebra $\mathcal{G}_\ell^\circ$ of Ker(det) is an ideal of $\mathcal{G}_\ell$ containing $[\mathcal{G}_\ell, \mathcal{G}_\ell]$ and of codimension 1 in $\mathcal{G}_\ell$. $\mathcal{G}_\ell^\circ$ is again reductive. Let $\mathcal{G}_\ell^\circ = [\mathcal{G}_\ell^\circ, \mathcal{G}_\ell^\circ] \oplus \mathcal{C}_0$, where $\mathcal{C}_0$ is the center of $\mathcal{G}_\ell^\circ$.

It is clear that $[\mathcal{G}_\ell, \mathcal{G}_\ell] = [\mathcal{G}_\ell^\circ, \mathcal{G}_\ell^\circ]$ and $\mathcal{C}_0$ is a subspace of $\mathcal{C}_\ell$ of codimension 1. By the theorem of Bogomolov ([1], Corollary 1), one has $\mathcal{C}_\ell = \mathcal{C}_0 + Q_\ell\cdot\mathrm{id}$, where $Q_\ell\cdot$id denotes the scalar multiples of the identity linear transformation in $\mathrm{End}(V_\ell)$.

Since $\mathcal{C}_0$ is contained in the symplectic Lie algebra $\mathrm{sp}(V_\ell, \psi)$, one has $\psi(cv, w) + \psi(v, cw) = 0$ for all v,w in V_ℓ, c in $\mathcal{C}_0$.

Note that by Faltings' Satz 4, $\mathcal{C}_0$ is contained in E_ℓ. Hence one has $c + c' = 0$. But $'$ acts as identity on E_ℓ, one concludes that $\mathcal{C}_0 = 0$. Thus $\mathcal{G}_\ell = [\mathcal{G}_\ell, \mathcal{G}_\ell] \oplus Q_\ell$·id.

If G_{V_ℓ} is connected, then its commutator subgroup $[G_{V_\ell}, G_{V_\ell}]$ is closed and connected. In particular, $S_{V_\ell} = [G_{V_\ell}, G_{V_\ell}]$. By the above equality, one concludes that $G_{V_\ell} = S_{V_\ell} \cdot G_m$. Consequently, $G_{V_\lambda} = S_{V_\lambda} \cdot G_m$.

This completes the proof.

Remark. 1^o It is known that G_{V_ℓ} is connected if one replaces the base field K by a finite extension of K (cf. [7]).

2^o By Proposition (2.4), one has $\mathrm{End}_{S_{V_\lambda}}(V_\lambda) = D_\lambda$.

3. Proof of the main theorem. In this section, we are assuming that K is large enough that G_{V_ℓ} is connected. Let A be an absolutely simple abelian variety of type II defined over K as in §1, 2.

Let $\{\sigma_1, \ldots, \sigma_g\}$ be the set of all distinct embeddings of E into R. Since D is an indefinite quaternion algebra over E, for each $\sigma_i : E \longrightarrow R$, one has $D_i = D \otimes_{E \nearrow \sigma_i} R \simeq M_2(R)$. Let $x^* = \mathrm{Tr}_{D/E}(x) - x$ be the canonical involution of D over E (cf. [5], pp. 195). By the Skolem-Noether theorem, there is an a in $D - \{0\}$ such that $x' = ax^*a^{-1}$ for all x in D.

On the other hand, let us denote $D \otimes_Q R$ by D_R. The Rosati involution extends R-linearly to an involution (again, denote it by $'$) on D_R as follows:

We have $D_R \simeq D_1 \times \cdots \times D_g$ and each $D_i \simeq M_2(R)$. For each i, let a_i be the image of a in $D_i \simeq M_2(R)$, so that on this factor, the R-linear extension of the Rosati involution takes the form $x_i' = a_i(\mathrm{Tr}_{D_i/R}(x_i) - x_i)a_i^{-1}$ for all $x_i \in D_i$. Then the R-linear extension of Rosati involution on D_R is given by $(x_1, \ldots, x_g) \longrightarrow (x_1', \ldots, x_g')$.

Further, it is well-known that the Rosati involution is positive definite. By applying the theorem of Skolem-Noether, one can choose an isomorphism $D_R \xrightarrow{\sim} M_2(R) \times \cdots \times M_2(R)$ (g factors) such that the involution $'$ passed to the right side is given by

$$(x_1,\dots,x_g)\longrightarrow(x_1^t,\dots,x_g^t)\quad(\text{ cf. [5], §21, Theorem 2.})$$

Lemma (3.1). *The set $\{x\in D|x'=x\}$ is an E-vector subspace of D of dimension 3.*

Proof. It is suffices to show that for each $i,\widetilde{S}_i=\{x_i\in D_i|x_i'=x_i\}$ is an R-vector subspace of D_i of dimension 3. But this follows easily from the fact that $S=\{X_i\in M_2(R)|X_i^t=X_i\}$ is an R-vector subspace of $M_2(R)$ of dimension 3.

Remark. In general, over an arbitrary field F of characteristic 0, any involution $'$ on $M_2(F)$ is of the form $X'=B^{-1}X^tB$, where B is a symmetric or skew-symmetric invertible matrix in $M_2(F)$. In particular, up to conjugation, over the field of complex number C, there are two types of involutions on $M_2(C)$. Namely, $X'=J^{-1}X^tJ$ with $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ or $X'=X^t$. The former one is called the canonical involution on $M_2(C)$.

For each i, let σ_i be the embedding $E\overset{\sigma_i}{\hookrightarrow}R\overset{\text{id}}{\hookrightarrow}C$. By the preceding lemma, it is easy to see that there is an isomorphism $D\otimes_{E\nearrow\sigma_i}C\simeq M_2(C)$ such that the involution extended to the right side by C-linearity is given by $X\longrightarrow X^t$.

Let $D_\lambda=D\otimes_E E_\lambda$. Then G_K acts D_λ-linearly on V_λ. As we have seen in §2, $G_{V_\lambda}=S_{V_\lambda}\cdot G_m$, where S_{V_λ} is a connected semisimple algebraic group over E_λ which is contained in the commutant of D_λ in the symplectic group $\mathrm{Sp}(V_\lambda,\Psi_\lambda)$. By Proposition (2.4), V_λ is a faithful semisimple representation of S_{V_λ} such that $\mathrm{End}_{S_{V_\lambda}}(V_\lambda)=D_\lambda$.

For almost all primes λ in $E,D_\lambda\simeq M_2(E_\lambda)$ the 2×2 matrix algebra over E_λ. Let λ be such a prime for the moment. We denote again by $'$ the E_λ-linear extension of the Rosati involution to D_λ. Fix an isomorphism of D_λ onto $M_2(E_\lambda)$. Through this isomorphism, $M_2(E_\lambda)$ acts on V_λ and $M_2(E_\lambda)$ commutes with the action of the Galois group G_K. On the other hand, the involution $'$ on D_λ passes to $M_2(E_\lambda)$; we denote it again by $'$.

In $M_2(E_\lambda)$, let $t=\begin{pmatrix}1&0\\0&-1\end{pmatrix},u=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. We have $t^2=u^2=1$ and

$tu = -ut = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Put $e = \frac{1}{2}(1+t) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), X = eV_\lambda$, and $Y = (1-e)V_\lambda$. Then, one has $V_\lambda = X \oplus Y$ as G_K-modules. In fact, X is the eigenspace for the eigenvalue 1 of t, and Y is the eigenspace for eigenvalue -1 of t. Further, u is an isomorphism of X onto Y and $u^{-1} = u$. One concludes that

$$\dim X = \dim Y = \frac{1}{2}\dim V_\lambda = 2d \text{ over } E_\lambda.$$

Similarly, put $f = \frac{1}{2}(1+u) = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right), X' = fV_\lambda$, and $Y' = (1-f)V_\lambda$. Then, one has $V_\lambda = X' \oplus Y'$ as G_K-modules and $t = t^{-1}$ is an isomorphism of X' onto Y'. So dim X' = dim $Y' = 2d$ over E_λ.

Lemma (3.2). *The following statements are true:* (i) X *and* Y (*resp.* X' *and* Y') *are isomorphic as* G_K*-modules, and hence as* S_{V_λ}*-modules.*

(ii) *The representation* V_λ *of* S_{V_λ} *is isomorphic, over* E_λ, *to two copies of the representation* X *or* Y (*resp.* X' *or* Y') *of* S_{V_λ}.

(iii) *The representation* X *or* Y (*resp.* X' *or* Y') *of* S_{V_λ} *is faithful and absolutely irreducible.*

Proof. Since the action of G_K on V_λ commutes with $M_2(E_\lambda), X$ and Y (resp. X' and Y') are G_K-modules; and hence are S_{V_λ}-modules. On the other hand, since u, t commute with G_K, hence (i) and (ii) follows easily.

Further, by $\text{End}_{S_{V_\lambda}}(V_\lambda) = D_\lambda \simeq M_2(E_\lambda)$ and (ii), one has $\text{End}_{S_{V_\lambda}}(X) = E_\lambda$. Moreover, V_λ is faithful and semisimple. Therefore X is a faithful, absolutely irreducible representation (over E_λ) of S_{V_λ}.

This completes the proof of this lemma.

Recall that the representation V_λ of S_{V_λ} admits the non-degenerate alternating bilinear form Ψ_λ. On the other hand, X, Y (resp. X', Y') are irreducible S_{V_λ}-modules. Hence any S_{V_λ}-equivariant bilinear form on X, Y (resp. X', Y') is either 0 or non-degenerate. Thus, either X is a maximal isotropic subspace of V_λ with respect to Ψ_λ, or $\Psi_\lambda|_{X \times X}$ is a non-degenerate S_{V_λ}-equivariant alternating form on X. Similiarly, this property holds for the S_{V_λ}-modules Y, X' and Y'.

Lemma (3.3). *If* X *and* Y *are maximal isotropic subspaces of* V_λ *with respect to* Ψ_λ, *then* X' *and* Y' *are orthogonal and hence* $\Psi_\lambda|_{X' \times X'}$ *is a non-degenerate alternating form.*

Proof. Recall that we have

$$\Psi_\lambda(Mv, w) = \Psi_\lambda(v, M'w) \text{ for all } v, w \text{ in } V_\lambda; M \text{ in } M_2(E_\lambda).$$

Suppose that X and Y are isotropic. We divide the remaining proof into four steps as follows:

Step 1. $e' = 1 - e$ and $(1 - e)' = e$

Since X (resp. Y) is isotropic, so

$$\Psi_\lambda(ev, ew) = 0 \text{ (resp. } \Psi_\lambda((1-e)v, (1-e)w) = 0) \text{ for all } v, w \text{ in } V_\lambda.$$

By the non-degeneracy of Ψ_λ, one concludes that $e'e = 0$ and $(1-e)'(1-e) = 0$. Thus $e' = 1 - e$ and $(1 - e)' = e$.

Step 2. $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)' = \left(\begin{smallmatrix}0&0\\c&0\end{smallmatrix}\right)$ for some $c \neq 0, c^2 = 1$. Similarly, $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)' = \left(\begin{smallmatrix}0&b\\0&0\end{smallmatrix}\right)$ for some $b \neq 0, b^2 = 1$.

Let $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)' = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. By $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, we have $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)'\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)' = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)'$. Thus, $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. This implies that $a = b = 0$. On the other hand, by $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, we have $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$. This implies that $b = d = 0$. So $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)' = \left(\begin{smallmatrix}0&0\\c&0\end{smallmatrix}\right)$, for some $c \neq 0$ with $c^2 = 1$.

Step 3. $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)' = \left(\begin{smallmatrix}d&b\\c&a\end{smallmatrix}\right)$ for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in M_2(E_\lambda)$.

By $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, we have

$$\begin{pmatrix}0&0\\c&0\end{pmatrix}\begin{pmatrix}0&b\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&bc\end{pmatrix} = \begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Hence $bc = 1$ and $b^2 = c^2 = 1$. Suppose $b = c = -1$, then $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)' = \left(\begin{smallmatrix}d&-b\\-c&a\end{smallmatrix}\right)$. Thus, over C, $'$ is the canonical involution $J^{-1}M^tJ$ which contradicts Lemma (3.1). Hence $b = c = 1$ and $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)' = \left(\begin{smallmatrix}d&b\\c&a\end{smallmatrix}\right)$.

Step 4. X' and Y' are orthogonal with respect to Ψ_λ.

By Step 3, one knows that $u' = u$. On the other hand, $ux = x$ for all $x \in X'$, and $uy = -y$ for alll $y \in Y'$. One concludes that $\Psi_\lambda(x, y) = \Psi_\lambda(ux, y) = \Psi_\lambda(x, uy) = \Psi_\lambda(x, -y)$.

Thus, one has $\Psi_\lambda(x,y)=0$ for all $x\in X', y\in Y'$. This completes the proof of this Lemma.

To sum up, we have the following main result:

Theorem A. *Suppose λ is a prime of E where D splits. Then there exists a G_K-submodule W_λ of V_λ such that the following hold:*

(i) $\dim W_\lambda = 2d$.

(i) *The Galois modules V_λ is isomorphic, over E_λ, to the dirct sum of two copies of W_λ.*

(iii) $\Psi_\lambda|_{W_\lambda\times W_\lambda}$ *is a non-degenerate S_{V_λ}-equivariant alternating form.*

Proof. By Lemma (3.2) an (3.3), one of the G_K-modules X, Y, X' and Y' satisfies (i), (ii) and (iii); which can be taken as W_λ.

Remark. Let C be the completion of an algebraic closure $\overline{Q}_\ell$ of Q_ℓ containing E_λ. For primes λ in E such that D_λ is a division algebra over E_λ, by extension of scalars to C and the same argument as Theorem A, one has an analogue result for the representation $V_\lambda\otimes_{E_\lambda} C$ of $S_{V_\lambda/C}$.

4. Concluding remark. If A is an abelian variety defined over K such that $\dim A = d$, where $d=2$ or odd; and $\mathrm{End}_K(A)=\mathrm{End}_{\overline{K}}(A)=Z$, then J-P. Serre[7] proved that the Lie algebra $\mathcal{G}_\ell$ of $\mathrm{Im}\rho_\ell$ is isomorphic to $\mathrm{sp}(2d,Q_\ell)\oplus Q_\ell\cdot\mathrm{id}$.

By Theorem A, one sees that Serre's method also works for the following absolutely simple abelian varieties of type II: $\dim A=2d$, where d is 2 or odd; and $\mathrm{End}^\circ(A)=D$ is an indefinite quaternion algebra over Q. More precisely, for primes ℓ where D splits, the Lie algebra $\mathcal{G}_\ell$ attached to the above type II absolutely simple abelian varieties is isomorphic to $\mathrm{sp}(2d,Q_\ell)\oplus Q_\ell\cdot\mathrm{id}$. We shall give the detail in a separate paper.

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