BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 18, Number 1, March 1990

REARRANGEMENT INEQUALITIES FOR EIGENVALUES OF DIFFERENTIAL EIGENVALUE PROBLEMS

BY

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Abstract. Among equimeasurable rearrangements of the coefficient function of a differential eigenvalue problem, we show that the one in increasing order minimizes the associated least positive eigenvalue and the one in decreasing order maximizes it.

This note is concerned with upper and lower bounds for the least positive eigenvalue of the following eigenvalue problem

$$x^{(2n)} + (-1)^{n+1}\lambda q(t)x = 0, \tag{1}$$

$$x^{(k)}(0) = 0 = (x^{(n+k)})(1),\ k = 0, 1, \ldots, n-1, \tag{2}$$

where $q(t)$ is positive and continuous in $[0,1]$. Among equimeasurable rearrangements of $q(t)$. We shall show that the one in increasing order minimizes the least positive eigenvalue and the one in decreasing order maximizes it.

Two real continuous functions $f(t)$ and $g(t)$ defined in [0,1] are said to be similarly ordered if for each pair of real numbers s and t in $[0,1]$, we have $[f(s)-f(t)][g(s)-g(t)] \geq 0$; $f(t)$ and $g(t)$ are said to be equimeasurable if the measure of $\{t \in [0,1] : f(t) \geq c\}$ is equal to that of $\{t \in [0,1] : g(t) \geq c\}$ for

Received by the editors Sept. 10, 1988.

AMS subject classifications: 34B25, 35P15

Key words: equimeasurable, rearrangement, eigenvalue

each real number c. Let f, f_+, and f_- be equimeasurable, and in addition let $f_+(t)$ and the function $g(t) = t$ be similarly ordered, and $f_-(t)$ and the function $h(t) = 1 - t$ be similarly ordered. The uniquely defined and continuous functions $f_+(t)$ and $f_-(t)$ are called the rearrangement of $f(t)$ in increasing, respectively decreasing order.

Lemma 1. ([3, Theorem 378]) *Let f and g be real nonnegative and continuous functions defined on* $[0,1]$, *then*

$$\int_0^1 f_-g_+ = \int_0^1 f_+g_- \leq \int_0^1 fg \leq \int_0^1 f_+g_+ = \int_0^1 f_-g_-.$$

The following is a slightly modified version of a result of Vollman [5, Theorem 5.1], the proof of which can be obtained by modifying that of Vollman.

Lemma 2. ([5, Theorem 5.1]) *Let $H(s,t)$ be a continuous and nonnegative function defined on* $[0,1] \times [0,1]$ *which is an increasing function when considered as a function of either the variable s, or t. Let $f(t)$, $g(t)$, and $u(t)$ be nonnegative continuous functions defined on [0,1], then*

$$\int_0^1 \int_0^1 h(s,t)f(s)g(s)u(t)\,dudt \leq \int_0^1 \int_0^1 h(s,t)f_+(s)g_+(s)u_+(t)\,dsdt.$$

Let the function $H(s,t)$ be defined by

$$H(s,t) = \begin{cases} \frac{1}{(n-1)!}\int_0^s (s-r)^{n-1}\,dr, & 0 \leq s \leq t \leq 1, \\ \frac{1}{(n-1)!}\int_0^t (t-r)^{n-1}\,dr, & 0 \leq t \leq s \leq 1. \end{cases}$$

If $g(t)$ is any function continuous in the interval $[0,1]$, then it is easily verified that the unique solution of the differential system

$$(-1)^n x^{(2n)} = g$$
$$x^{(k)}(0) = 0 = (x^{(n)})^{(k)}(1), \qquad k = 0, 1, \ldots, n-1$$

is

$$x(t) = \int_0^1 H(s,t)g(s)\,ds.$$

In fact, $H(s,t)$ is the Green's function of the system [2, Lemma 3.1]. Consequently, system (1) – (2) can be transformed into an equivalent integral equation of the form

$$\lambda T x = x, \tag{3}$$

where

$$(Tx)(t) = \int_0^1 H(s,t)q(s)x(s)\,ds$$

is defined in the Banach space

$$B = \{u \in C^{2n-1}[0,1] : u^{(i)}(0) = 0,\ 0 \le i \le n\}$$

equipped with the usual sup norm. By means of the theory of u_0 – *positive* operators, it has been shown by Keener and Travis [4, Lemma 2.4] that the operators T has exactly one (normalized) eigenvector $x(t)$ in the cone

$$\begin{aligned} P =&\{u \in B : u^{(i)}(x) \ge 0, 1 \le i \le n; \\ &(-1)^j u^{(n+j)}(x) \ge 0, x \le 0 \le 1, 0 \le j \le n\}, \end{aligned}$$

and that the corresponding eigenvalue is positive and larger than the absolute value of any other eigenvalue. As a consequence, we have

Lemma 3. *The eigenvalue problem* (1)-(2) *has exactly one (normalized) eigenfunction in P, and the corresponding eigenvalue is positive and smaller than the magnitude of any other eigenvalue.*

We shall denote the least positive eigenvalue stated in the above Lemma by $\lambda(q)$. There is a well known minimum principle for $\lambda(q)$ [1, pp.239-241]:

$$\lambda(q) = \min \frac{\int_0^1 (x^{(n)})^2\,dt}{\int_0^1 q(t)x^2(t)\,dt}, \tag{4}$$

where the minimum is taken over the set S of nontrivial comparison functions $x(t)$ such that (i) $x^{(k)}(t)$ and $(x^{(n+k)})(t)$ exist for $0 \le k \le n$ and $0 \le t \le 1$; and (ii) the boundary conditions in (2) are satisfied. Furthermore, no function other than the corresponding eigenfunction of (1)-(2) yields the minimum. In view of Lemma 3, we may restrict these comparison functions to nonnegative ones.

We shall need another minimum principle for $\lambda(q)$.

Lemma 4. ([4, Theorem 3.1]) *The smallest eigenvalue* $\lambda(q)$ *of* (1)-(2) *can be characterized by the extremal principle*

$$\lambda^{-1}(q) = \max_{u\in P, u\neq 0} \frac{\int_0^1\int_0^1 H(s,t)q(s)u(s)u(t)\,dsdt}{\int_0^1 u^2(s)\,ds}. \tag{5}$$

The unique function, except for a constant multiple, which maximizes (5) *is the eigenfunction corresponding to the eigenvalue* $\lambda(q)$.

We are now ready to state and prove the following rearrangement theorem for the least positive eigenvalue of (1)-(2).

Theorem. *Let* $\lambda(q)$ *denote the least positive eigenvalue of the boundary problem* (1)-(2). *We have* $\lambda(q_+) \le \lambda(q) \le \lambda(q_-)$, *where* q_+ *and* q_- *are respectively the rearrangements of* q *in increasing and decreasing order.*

Proof. Let $w(t)$ be an eigenfunction of (1)-(2) corresponding to $\lambda(p_-)$. By Lemma 3, we may assume that $w(t) \ge 0$ and $w'(t) \ge 0$ for $0 \le t \le 1$. Since $< qw, w > \ge < q_-w, w >$ by Lemma 1, we have, in view of (4),

$$\begin{aligned}\lambda(q_-) &= \frac{\int_0^1 (w^{(n)}(t))^2\,dt}{\int_0^1 q_-(t)w^2(t)\,dt} \\ &\ge \frac{\int_0^1 (w^{(n)}(t))^2\,dt}{\int_0^1 q(t)w^2(t)\,dt} \\ &\ge \min_{x\in S} \frac{\int_0^1 (x^{(n)}(t))^2\,dt}{\int_0^1 q(t)x^2(t)\,dt} \\ &= \lambda(q).\end{aligned}$$

Next, let $z(t)$ be an eigenfunction of (1)-(2) corresponding to $\lambda(q)$. By Lemma 3, we may assume that $z(t) \ge 0$. It is easily seen that the function $H(s,t)$ defined right after Lemma 2 satisfies the assumptions of Lemma 2. As a consequence, we have

$$\int_0^1\int_0^1 H(s,t)^2 q(s)z(s)z(t)\,dsdt \le \int_0^1\int_0^1 H(s,t)^2 q_+(s)z_+(s)z_+(t)\,dsdt.$$

In view of Lemma 4,

$$\begin{aligned}\lambda^{-1}(q) &= \frac{\int_0^1\int_0^1 H(s,t)q(s)z(s)z(t)\,dsdt}{\int_0^1 z^2(s)\,ds}\\ &\le \frac{\int_0^1\int_0^1 H(s,t)q_+(s)z_+(s)z_+(t)\,dsdt}{\int_0^1 z_+^2(s)\,ds}\\ &\le \max_{u\in P\ u\neq 0}\frac{\int_0^1\int_0^1 H(s,t)q_+(s)u(s)u(t)\,dsdt}{\int_0^1 u^2(s)\,ds}\\ &= \lambda^{-1}(q_+).\end{aligned}$$

This completes the proof.

Remark. In [4], Keener and Travis studied the more general eigenvalue problem

$$\begin{aligned}&x^{(m)}(t) - (-1)^{m-k}\lambda\sum_{i=0}^{m-1} p_i(t)x^{(i)}(t) = 0,\\ &\qquad\qquad y^{(i)}(0) = 0, \qquad 0\le i\le k-1\\ &\qquad\qquad y^{(i)}(1) = 0, \qquad k\le i\le m-k-1,\end{aligned}$$

where the continuous coefficients p_i are assumed to satisfy certain positivity conditions. This problem has a least positive eigenvalue $\lambda(p_0,p_1,\dots,p_{m-1})$ and a corresponding extremal principle exists for its characterization [4, Theorem 3.1]. In view of these and our technique, we may prove without difficulty that the following inequality holds

$$\lambda(p_0^+,p_1^+,\dots,p_{m-1}^+) \le \lambda(p_0,p_1,\dots,p_{m-1}).$$

The same ideas can be used to deal with the eigenvalue problem

$$\begin{aligned}&(p(t)x^{(n)})^{(n)} + (-1)^{n+1}\lambda q(t)x = 0,\\ &\qquad\qquad x^{(k)}(0) = 0 = (px^{(n)})^{(k)}(1), \qquad 0\le k\le n-1,\end{aligned}$$

where p is positive and $n-times$ continuously differentiable on [0,1]. With the background material in [2,4], we may prove that its least positive eigenvalue $\lambda(q)$ again satisfies $\lambda(q_+)\le\lambda(q)\le\lambda(q_-)$.

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