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ON THE CONVERGENCE OF WIENER-ITO DECOMPOSITION

BY

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Abstract. It is shown that the Wiener-Ito decomposition of a generalized Brownian functionals on an abstract Wiener space (H, B) with Wiener measure p_t $(t>0)$ may be formally expressed as

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_B D^n p_1 f(0)(x+iy)^n p_1(dy)$$

which makes sense for many classes of functions. Employ the above expression, we are able to discuss various convergences such as uniform convergence on bounded sets, convergence in $L^p(p_1)$ and in generalized sense for the Wiener-Ito decomposition of various classes of functions.

1. **Introduction.** Let $\mathcal{C}[0, 1]$ be the classical Wiener space and w the Wiener measure. The well-known Wiener-Ito decomposition theorem [4] asserts that any w-square integrable function on $\mathcal{C}[0, 1]$ can be decomposed into the orthogonal direct sum of multiple Wiener integrals. On an arbitrary abstract Wiener space (H, B) [1], where B is a real separable Banach space with norm $\|\cdot\|$ and H a dense Hibertian subspace of H with norm $|\cdot|=\sqrt{\langle\cdot,\cdot\rangle}$ such that $\|\cdot\|$ is measurable on H, the Wiener-Ito decomposition theorem remains true with w being replaced by the abstract measure p_1 of variance parameter $t=1$ and the multiple Wiener integrals of order n by the homogeneous chaos of order n. More precisely, let $\{e_n\}$ be a CONS in H consisting of elements in B^* which is identified as a subspace of H in such way that $\langle x, y\rangle = (x, y)$ for $x\in H$, $y\in B^*$, where $(\cdot,\cdot)$ denotes the $B-B^*$ pairing, the homogeneous chaos $\mathcal{H}_n$ of order n is defined by the completion in $L^2(p_1)$ of the span of Hermite polynomials of

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$\{(x, e_k) : k = 1, 2, 3, \cdots\}$ of degree n. Then the Wiener-Ito theorem on B asserts that any p_1-square integrable function f can be written as an orthogonal direct sum of functions $f_n \in \mathscr{H}_n$. The theorem plays a fundamental role in white noise calculus initiated by Hida [3].

In this paper, we first show that f_n has the representation

$$f_n(x) = \frac{1}{n!} \int_B D^n p_1 f(o)(x + iy)^n p_1(dy) \tag{1}$$

for almost all $x \in B$ with respect to p_1, where $p_1 f(x) = \int_B f(x + y) p_1(dx)$ and $D^n g(x)$ denotes the n-th H-derivative of a function g at y (see [2, 5]). Note that $D^n p_1 f(o)$ is only a n-linear Hilbert-Schmidt type operator on $H \times \cdots \times H$ (n copies) [7], it is generally not defined on $B \times \cdots \times B$ (n copies). However the expression on the right hand side of (1) which represents the Gauss transform of the formal expression $D^n p_1 f(o) x^n$ still exists according to [7]. By means of the representation (1), we are able to discuss various convergences such as uniform convergence on bounded sets, convergence in L^p and in generalized sense for the Wiener-Ito decumpositions of various class of functions.

2. **A remark on homogenous Chaos.** In the sequel, let (H, B) be a fixed abstract Wiener space and p_t the abstract Wiener measure with variance parameter $t > 0$.

For a function f on B, define the Gauss transform σf of f by

$$\sigma f(x) = \int_B f(x + iy) p_1(dy) \qquad \text{(if it exists)}.$$

Let $\mathscr{L}^n(B)$ denote the set of all bounded n-linear operator on $B^n := B \times \cdots \times B \to \boldsymbol{R}^1$. For notational convenience, we shall identify $T(x_1, \cdots, x_n)$ with $Tx_1 \cdots x_n$ for $x_1, \cdots, x_n \in B$, and when $x_1 = x_2 = \cdots = x_n$, we simply write Tx^n instead of $T(x, \cdots, x)$. Denote by $\mathscr{L}_2^n(H)$ the Hilbert space of Hilbert-Schmidt operator from H^n into $\boldsymbol{R}^1$ with inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{HS}$ defined by

$$\langle\!\langle T, S \rangle\!\rangle_{HS} := \sum_{i_1 \cdots i_n = 1}^{\infty} (Te_{i_1} \cdots e_{i_n})(Se_{i_1} \cdots e_{i_n}),$$

where $\{e_j\}$ is a CONS in H, and the corresponding norm by $\|\cdot\|_{HS}$. It is well-known that $\mathscr{L}^n(B)$ can be identified as a dense subspace of $\mathscr{L}_2^n(H)$ by restriction (see [5; p. 103]).

For $T \in \mathcal{L}_2^n(B)$, let $f_T(x) = Tx^n$ and define $\lambda_1^n T(x) = \sigma f_T(x)$. Then λ_1^n is a densely defined map from $\mathcal{L}_2^n(H)$ into $L^2(p_1)$. It follows from [7; Theorem 2.4] that λ_1^n is continuous and thus λ_1^n can be extended by continuity to $\mathcal{L}_2^n(H)$. More precisely, if $T \in \mathcal{L}_2^n(H)$, there exists a sequence $\{T_m\}$ of operators in $\mathcal{L}^n(B)$ such that their restrictions $\{\tilde{T}_m\}$ to H^n converges to T in $\mathcal{L}_2^n(H)$ as $m \to \infty$. Then we define

$$(2) \qquad \lambda_1^n T(x) = L^2(p_1) - \lim_{m\to\infty} \sigma f_{T_m}(x)\,.$$

$\lambda_1^n T$ may be regarded as the Gauss transform of the formal expression "Tx^n". Note that Tx^n generally has no meaning, but its Gauss transform $\lambda_1^n T$ always makes sense and may be formally written by

$$\lambda_1^n T(x) = \int_B T(x+iy)^n \, p_1(dy)\,.$$

LEMMA 2.1. *$\mathcal{H}_n = \{\lambda_1^n T : T \in \mathcal{S}\mathcal{L}_2^n(H)\}$, where $\mathcal{S}\mathcal{L}_2^n(H)$ denotes the subspace of all symmetric operators in $\mathcal{L}_2^n(H)$.*

Proof. Let $\mathcal{D}^n(B)$ denote the collection of all operators T of the form

$$(3) \qquad Tx_1\cdots x_n = \sum_{i_1,\cdots,i_n=1}^{M} \lambda_{i_1\cdots i_n}\Big(\prod_{j=1}^{n}(x_j, e_{i_j})\Big),$$

where M is some integer, $\{\lambda_{i_1\cdots i_n}\} \subset \boldsymbol{R}^1$ and $\{e_k : k = 1,\cdots, M\}$ is an ON set in H consisting of dements in B^*. Denote by $\mathcal{S}\mathcal{D}^n(B)$ the subset of all symmetric member in $\mathcal{D}^n(B)$ and by $\mathcal{S}\mathcal{D}^n(H)$ the collection of the restrictions to H^n of operators in $\mathcal{S}\mathcal{D}^n(B)$. Clearly $\mathcal{S}\mathcal{D}^n(H)$ is dense in $\mathcal{S}\mathcal{L}_2^n(H)$.

Note that if $S \in \mathcal{S}\mathcal{D}^n(B)$, then $\sigma f_S = \lambda_1 S \in \mathcal{H}_n$. Now, for any $T \in \mathcal{L}_2^n(H)$, choose a sequence $\{T_m\} \subset \mathcal{S}\mathcal{D}^n(B)$ such that their restrictions $\tilde{T}_m$ to H^n converges to T in $\mathcal{S}\mathcal{L}_2^n(H)$. It follows from Corollary 2.6 in [7] that $\lambda_1^n T = L^2(p_1) - \lim_{m\to\infty} \sigma f_{T_m}(x) \in \mathcal{H}_n$.

Since $\|\lambda_1^n T\|_{L^2(p_1)} = \sqrt{n!}\, \|T\|_{HS}$ (by [7; Proposition 2.8]), $\{\lambda_1^n T : T \in \mathcal{S}\mathcal{L}_2^n(H)\}$ is closed. Thus we have $\mathcal{H}_n = \{\lambda_1^n T : T \in \mathcal{S}\mathcal{L}_2^n(H)\}$.

PROPOSITION 2.2. *Define the operator* Q_n *on* $L^2(p_1)$ *by* $Q_n f = (1/n!)\,\lambda_1^n[D^n p_1 f(o)]$. *Then* Q_n *is an orthogonal projection from* $L^2(p_1)$ *onto* $\mathscr{H}_n$.

Proof. First we note that $D^n p_1 f(o) \in \mathscr{S}\mathscr{L}_2^n(H)$ for $f \in L^2(p_1)$ (by [7; Proposition 3.2] or [6; Theorem 3.3]). Thus $Q_n f = (1/n!)\,\lambda_1^n[D^n p_1 f(o)] \in \mathscr{H}_n$ by Lemma 2.1.

Next, for real-valued functions f, $g \in L^2(p_1)$,

$$\begin{aligned}
&\int_B Q_n f(x)\, g(x)\, p_1(dx)\\
&\qquad = \frac{1}{n!}\int_B \lambda_1^n\,[D^n p_1 f(o)](x)\, g(x)\, p_1(dx)\\
(4)\qquad &\qquad = \frac{1}{n!}\langle\!\langle D^n p_1 f(o),\, D^n p_1 g(o)\rangle\!\rangle_{HS}\\
&\qquad\qquad\qquad\qquad \text{(by [7; Proposition 3.2])}\\
&\qquad = \int_B f(x)\, Q_n g(x)\, p_1(dx).
\end{aligned}$$

This proves that Q_n is self-adjoint. Moreover, for $T \in \mathscr{L}_2^n(H)$, we have

$$\begin{aligned}
&\int_B Q_n(\lambda_1^n T)(x)\, g(x)\, p_1(dx)\\
&\qquad = \int_B \lambda_1^n T(x)\, Q_n g(x)\, p_1(dx)\\
&\qquad = \frac{1}{n!}\langle\!\langle T,\, D^n p_1 g(o)\rangle\!\rangle_{HS}\\
&\qquad = \int_B \lambda_1^n T(x)\, g(x)\, p_1(dx),
\end{aligned}$$

for all $g \in L^2(p_1)$. Therefore $Q_n(\lambda_1^n T) = \lambda_1^n T$ which, in turn, implies that Q_n maps $L^2(p_1)$ onto $\mathscr{H}_n$.

It remains to show that $Q_n^2 = Q_n$. Note that, for $h_1 \cdots h_n \in H$,

$$\begin{aligned}
&D_1^n p_1(Q_n f)(o)\, h_1 \cdots h_n\\
&\qquad = \int_B Q_n f(x)\, \lambda_1^n\Big[\bigotimes_{k=1}^n h_k\Big](x)\, p_1(dx)\\
&\qquad\qquad\qquad\qquad \text{(by [7; Proposition 3.1])}\\
&\qquad = \frac{1}{n!}\int_B \lambda_1^n\,[D^n p_1 f(o)](x)\, \lambda_1^n\Big[\bigotimes_{k=1}^n h_k\Big](x)\, p_1(dx)\\
&\qquad = D^n p_1 f(o)\, h_1 \cdots h_n \qquad \text{(by [7; Proposition 2.8])},
\end{aligned}$$

where $\otimes_{k=1}^{n} h_k$ denotes the symmetric tensor product of $h_1 \cdots h_k$. Thus $D^n p_1(Q_n f)(o) = D^n p_1 f(o)$.

Consequently

$$Q_n^2 f = \lambda_1^n [D^n p_1(Q_n f)(o)] = \lambda_1^n [D^n p_1 f(o)] = Q_n f .$$

COROLLARY 2.3. (Wiener-Ito decomposition theorem). *For $f \in L^2(p_1)$, f can be uniquely represented by*

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_1^n [D^n p_1 f(o)] .$$

3. **Convergence of Wiener-Ito decomposition.** Let $\mathcal{E}_a$ denote the class of analytic functions f on the complexification B_c of B such that there exists constants c, c' such that $|f(z)| \leq c \exp(c'\|z\|)$ for $z \in B_c$, where $\|z\| = \sqrt{\|x\|^2 + \|y\|^2}$ if $z = x + iy$ $(x, y \in B)$. Denote by $\mathcal{E}_a(B)$ the class of function f defined on B such that $f(z)$ $(z \in B_c)$ is defined and $f \in \mathcal{E}_a$. For any integer $m = 1, 2, 3, \cdots$, define on $\mathcal{E}_a(B)$ a norm $\|f\|_m = \sup_{z \in B_c} |f(z)| \exp(-m\|z\|)$, and let $\mathcal{E}_a^m(B) = \{f \in \mathcal{E}_a(B) : \|f\|_m < \infty\}$. $\{(\mathcal{E}_a^m(B), \|\cdot\|_m)\}$ is an increasing sequence of Banach spaces and $\bigcup_m \mathcal{E}_a^m(B) = \mathcal{E}_a(B)$. Let $\mathcal{E}$ denote the space $\mathcal{E}_a(B)$ which is topologized by the inductive limit topology induced by $\mathcal{E}_a^m(B)$. Then $\mathcal{E}$ is a bornological topological space. For more details about $\mathcal{E}$, we refer the reader to [8].

LEMMA 3.1. [8]. *For $f \in \mathcal{E}$, the sequence $\sum_{n=0}^{m} (1/n!) \lambda_1^n [D^n p_1 f(o)]$ converges to f in $\mathcal{E}$ as $m \to \infty$.*

PROPOSITION 3.2. *Let $f \in \mathcal{E}$. Then we have*

(a) *The series $\sum_{n=0}^{\infty} (1/n!) \lambda_1^n [D^n p_1 f(o)](x)$ converges to $f(x)$ uniformly on bounded set.*

(b) *The series $\sum_{n=0}^{\infty} (1/n!) \lambda_1^n [D^n p_1 f(o)]$ converges to f in $L^\alpha(p_1)$ for $\alpha \geq 1$.*

Proof. (a) Observe that, for any constant $c > 0$ and for any positive integer m, we have

$$\sup_{\|x\| \leq c} |f(x)| \leq \|f\|_m \cdot e^{mc} \tag{5}$$

for all $f \in \mathcal{E}_m^a(B)$. Then (a) follows from Lemma 3.1 and the inequality (5).

(b) follows from Lemma 3.1 and the following inequality:

$$\int_B |f(x)|^\alpha \, p_1(dx) \leq \left(\int_B e^{m\alpha\|x\|} \, p_1(dx)\right)\|f\|_m^\alpha \quad \text{for all} \quad f \in \mathcal{E}_\alpha^m(B).$$

In the remaining of the paper, we discuss the convergence of the Wiener-Ito decomposition for a generalized Brownian functional in the sense of [8]. By a generalized Brownian functional on B or, simply, a generalized function, we mean a member of the dual $\mathcal{E}^*$ of $\mathcal{E}$. A sequence $\{F_n\}$ in $\mathcal{E}^*$ is said to be convergent to $F \in \mathcal{E}^*$ if, for $\varphi \in \mathcal{E}$, $\langle\langle F_n, \varphi\rangle\rangle \to \langle\langle F, \varphi\rangle\rangle$, where $\langle\langle\cdot, \cdot\rangle\rangle$ denotes the $\mathcal{E}^* - \mathcal{E}$ pairing.

LEMMA 3.3. *$Q_n : \mathcal{E} \to \mathcal{E}$ is continuous.*

Proof. For any $f \in \mathcal{E}_\alpha^m(B)$, there holds the inequality

$$\|Q_n f\|_1 \leq K\|f\|_m \tag{6}$$

for $f \in \mathcal{E}_\alpha^m(B)$ (by [8; Proposition 3.1 and 3.2]), where $K = e^{mn}\left(\int_B e^{m\|x\|} \, p_1(dx)\right)\left(\int_B e^{\|x\|} \, p_1(dx)\right)$.

The Lemma follows immediately from the inequality (6).

Lemma 3.3 yields the following

DEFINITION 3.4. For $F \in \mathcal{E}^*$, define $Q_n F$ by

$$\langle\langle Q_n F, \varphi\rangle\rangle : \langle\langle F, Q_n \varphi\rangle\rangle.$$

The Lemma 3.3 ensures that $Q_n F \in \mathcal{E}^*$. Q_n ejoys the following properties

PROPOSITION 3.5.

(a) *$Q_n : \mathcal{E}^* \to \mathcal{E}^*$ is continuous.*

(b) *$Q_n^2 F = Q_n F$ for all $F \in \mathcal{E}^*$.*

(c) *If $m \neq n$, then $\langle\langle Q_n F, \varphi\rangle\rangle = 0$ for all $\varphi \in \mathcal{H}_m$. Equivalently, $Q_n Q_m = Q_m Q_n = 0$.*

(d) *If $Q_n F = 0$ for all n, then $F = 0$.*

Let $\mathcal{H}_n^*$ denote the class of generalized functions $F \in \mathcal{E}^*$ satisfying $Q_n F = F$. Then we have

THEOREM 3.6. *For any $F \in \mathcal{E}^*$, we have*

$$F = \sum_{n=0}^{\infty} Q_n F, \tag{7}$$

where the sum is convergent in $\mathcal{E}^*$. Furthermore, if there exist a sequence $\{F_n\} \subset \mathcal{E}^*$ such that $F_n \in \mathcal{H}_n^*$ and $F = \sum_{n=0}^{\infty} F_n$, then we must have $F_n = Q_n F$.

Proof. For any $\varphi \in \mathcal{E}$, we have

$$\begin{aligned}\lim_{k\to\infty} \left\langle\!\left\langle \sum_{n=0}^{k} Q_n F, \varphi \right\rangle\!\right\rangle &= \lim_{k\to\infty} \left\langle\!\left\langle F, \sum_{n=0}^{k} Q_n \varphi \right\rangle\!\right\rangle \\ &= \langle\!\langle F, \varphi \rangle\!\rangle \qquad \text{(by Lemma 3.1).}\end{aligned}$$

This proves (7).

The second assertion follows from the equality (7) and Proposition 3.5.

REMARK 3.7. The equality (7) will be called the generalized Wiener-Ito decomposition of the generalized function F. It follows from the above theorem that we may decompose $\mathcal{E}^*$ into the direct sum

$$\mathcal{E}^* = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n^*.$$

COROLLARY 3.8 *If $f \in L^{\alpha}(p_1)$ for some $\alpha > 1$, then $f \in \mathcal{E}^*$, $Q_n f = (1/n!)\, \lambda_1^n [D^n p_1 f(o)]$ and $f = \sum_{n=0}^{\infty}(1/n!)\, \lambda_1^n [D^n p_1 f(o)]$ (in $\mathcal{E}^*$).*

Proof. The fact that $f \in \mathcal{E}^*$ has been proved in [8]. To prove the Corollary, we only have to verify that $Q_n f = (1/n!)$ $\lambda_1^n [D^n p_1 f(o)]$. In fact, for any $\varphi \in \mathcal{E}$, we have

$$\begin{aligned}\langle\!\langle Q_n f, \varphi \rangle\!\rangle &= \langle\!\langle f, Q_n \varphi \rangle\!\rangle \\ &= \int_B f(x) \left\{ \frac{1}{n!} \lambda_1^n [D^n p_1 \varphi(o)](x) \right\} p_1(dx) \\ &= \frac{1}{n!} \langle\!\langle D^n p_1 f(o), D^n p_1 \varphi(o) \rangle\!\rangle_{HS}\end{aligned}$$

$$= \int_B \left\{ \frac{1}{n!} \lambda_1^n \left[D^n p_1 f(o) \right](x) \right\} \varphi(x) p_1(dx)$$

(by [7; Proposition 3.2])

$$= \left\langle\!\left\langle \frac{1}{n!} \lambda_1^n \left[D^n p_1 f(o) \right],\ \varphi \right\rangle\!\right\rangle.$$

Consequently, $Q_n f = (1/n!)\, \lambda_1^n \left[D^n p_1 f(o) \right]$.

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