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# NOTE ON THEOREMS OF HERSTEIN

BY

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**Abstract.** Let $R$ be an associative ring. We prove that if for some fixed positive integer $m$ the mapping $x \to x^m$ in $R$ is onto and $(xy)^m = (yx)^m$ for all $x, y$ in $R$, then $R$ is commutative. Using this result, we can easily show that in Herstein's Theorem 3 of [4], the multiplicative homomorphism can be eliminated. We also prove that if $R$ has identity and the mapping $x \to x^3$ in $R$ is a multiplicative epimorphism then $R$ is commutative.

1. **Introduction.** Let $G$ be a semigroup. Following [3], $G$ is called left (right) cancellative if for all $x, y, z$ in $G$, $xy = xz$ ($yx = zx$) implies $y = z$. We say that $G$ is cancellative if it is both left and right cancellative. We shall denote the center of $G$ by $Z(G)$. Throughout this note, $R$ will denote an associative ring. We shall denote the commutator $xy - yx$ in $R$ by $[x, y]$, the center of $R$ by $Z(R)$, the Jacobson radical of $R$ by $J(R)$, and the set of all positive integers by $N$. For $m, n$ in $N$, we denote the greatest common divisor of $m$ and $n$ by $(m, n)$. If $R$ has identity, then we shall denote the group of units of $R$ by $R^*$.

The following known result is useful.

LEMMA A ([5], p. 221). *If $x, y$ are elements in a ring $R$ such that $[x, y]$ commutes with $x$, then $[x^m, y] = mx^{m-1}[x, y]$ holds for all $m$ in $N$.*

In [4], Herstein proved

THEOREM A ([4], Theorem 2). *Let $R$ be a ring in which for some fixed integer $m > 1$, $(x + y)^m = x^m + y^m$ for all $x, y$ in $R$.*

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*Then every commutator in R is nilpotent, and the nilpotent elements of R form an ideal.*

THEOREM B ([4], Theorem 3). *If $R$ is a ring in which the mapping $x \to x^m$ for a fixed integer $m > 1$ is a homomorphism onto, then $R$ is commutative.*

Recently, the author proves that in Theorem B the multiplicative endomorphism of $R$ is redundant. This is the following

THEOREM C ([11], Theorem 2). *If $R$ is a ring in which the mapping $x \to x^m$ for a fixed integer $m > 1$ is an additive epimorphism, then $R$ is commutative.*

We can use ([11], Theorem 1) and ([9], Theorem 10) to extend the author's previous result ([8], Theorem 1) to arbitrary rings. This is done in [10]. In Theorem B, the multiplicative endomorphism of $R$ seems to be weaker than the additive one. In this note, we prove that in Theorem B if $m=3$ and $R$ has identity, then $R$ is commutative when the additive endomorphism of $R$ is eliminated.

2. **Results.** We begin with

THEOREM 1. *If $R$ is a ring in which the mapping $x \to x^m$ for some fixed positive integer $m$ is onto and $(xy)^m = (yx)^m$ for all $x$, $y$ in $R$, then $R$ is commutative.*

**Proof.** We note first that any nilpotent $a$ in $R$ must be central, because $(1+a)\,x^{m^2}(1+a)^{-1} = \{(1+a)\,xx^{m-1}(1+a)^{-1}\}^m = x^{m^2}$ and so $[a, x^{m^2}] = 0$ for all $x$ in $R$. But, the mapping $x \to x^m$ is surjective, hence $[a, x] = 0$ for all $x$ in $R$.

By the result of [2], the commutator ideal of $R$ is nil. So for all $x$, $y$ in $R$, $[x, y]$ is a nilpotent element. Thus by the results above, $[x, y] \in Z(R)$ for all $x$, $y$ in $R$. Hence, we get $0 = (xy)^m - (yx)^m = mx^{m-1}\,y^{m-1}\,[x, y]$ and so by Lemma A twice $[x^m, y^m] = m^2\,x^{m-1}\,y^{m-1}\,[x, y] = 0$ for all $x$, $y$ in $R$. Since the mapping $x \to x^m$ in $R$ is onto, the commutativity of $R$ follows. This completes the proof of Theorem 1.

We see that the composite of epimorphisms is also an epimorphism. Thus using Theorem A and Theorem 1, we can easily prove Theorem C.

THEOREM 2. *Let $R$ be a ring with identity* 1 *such that for some fixed integers $m>1$ and $n>1$, $(x+y)^m=x^m+y^m$ and $(x+y)^n=x^n+y^n$ for all $x$, $y$ in $R$. If $(m, n)=1$ or 2, then $R$ is commutative.*

**Proof.** Assume $(m, n)=1$. Let $x, y \in R$. Then $(x+y)^j=x^j+y^j$ implies $[x, y^j]=[x^j, y]$, where $j=m$ and $n$. By Theorem 6 of [1], $R$ is commutative.

Assume $(m, n)=2$. Then either $m\equiv 2 \pmod 4$ or $n\equiv 2 \pmod 4$. By Theorem 1 of [11], $R$ is commutative. This completes the proof of Theorem 2.

In Theorem 2, the restriction on $(m, n)$ is essential as Examples 1 and 2 of [8] show.

THEOREM 3. *Let $R$ be a ring with identity* 1 *in which the mapping $x\to x^3$ is a multiplicative epimorphism. Then $R$ is commutative.*

To prove Theorem 3, we need the following lemmas.

LEMMA 1. *If $G$ is a cancellative semigroup such that $(xy)^3=x^3y^3$ and $x^2\in Z(G)$ for all $x$, $y$ in $G$, then $G$ is abelian.*

**Proof.** By Theorem 14 of [9], $G$ is abelian.

LEMMA 2. *Let $G$ be a cancellative semigroup in which the mapping $x\to x^3$ is an epimorphism. Then $G$ is abelian.*

**Proof.** For all $x$, $y$ in $G$, we have $x^3y^3=(xy)^3=x(yx)^2y$ and $y^3x^3=(yx)^2yx$, and so $x^2y^3=(yx)^2y=y^3x^2$ by cancellation. Since the mapping $x\to x^3$ in $G$ is onto, $x^2\in Z(G)$ for all $x$ in $G$. By Lemma 1, $G$ is abelian.

In the sequel, we assume that all the hypotheses as in Theorem 3 hold.

LEMMA 3. $J(R)^2 \subseteq Z(R)$.

**Proof.** By Lemma 2, $R^*$ is abelian. Let $a, b \in J(R)$. Since $R^*$ is abelian, we get $(1+a)(1+b) = (1+b)(1+a)$ and so $ab = ba$. Thus, $J(R)$ is commutative. So, we have $(ab)x = a(bx) = (bx)a = b(xa) = (xa)b$ for all $x$ in $R$. Hence, $J(R)^2 \subseteq Z(R)$.

LEMMA 4. *For $y, z \in R$, $yz = 0$ implies $zy = 0$.*

**Proof.** Assume $yz = 0$. Since the mapping $x \to x^3$ in $R$ is onto, there exist $a, b \in R$ such that $a^3 = y$ and $b^3 = z$. Then, we get $0 = yz = a^3 b^3 = (ab)^3$ and so $\{(ab)x\}^3 = (ab)^3 x^3 = 0$ for all $x$ in $R$. Hence, $ab \in J(R)$ and $(ab)a \in J(R)$. By Lemma 3, we have $zy = b^3 a^3 = (ba)^3 = b(ab)(aba) = (ab)(aba)b = (ab)^3 = 0$.

**Proof of Theorem 3.** Using Lemmas 2 and 4, the rest of the proof of Theorem 3 is due to Kobayashi [7]. This completes the proof of Theorem 3.

## 3. Remarks.

REMARK 1. In Theorem B, we do not know whether the additive endomorphism can be eliminated.

REMARK 2. It is interesting to find an elementary proof of Theorem 3.

REMARK 3. In Lemma 2, the exponent 3 is essential. We modify Example 3 of [6] to show this fact. Let $m - 1 = n \geq 3$ be fixed. Let $p$ be any arbitrary but fixed prime such that (case 1) $p$ divides $n$ if $n$ is odd, and (case 2) $p$ divides $n/2$ if $n$ is even. This is possible since $n \geq 3$. Let $G$ be a subgroup of the group of units of $3 \times 3$ matrices over $GF(p)$ defined by

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in GF(p) \right\}.$$

It is readily verified that $(xy)^m = x^m y^m$ and $x^m = x$ for all $x, y$ in $G$. However, $G$ is not abelian.

REMARK 4. In Lemma 2, the epimorphism can be replaced by monomorphism because of $x^{m-1}y^m = y^m x^{m-1}$ implies $(x^{m-1}y)^m = x^{(m-1)m}y^m = y^m x^{(m-1)m} = (yx^{m-1})^m$.

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