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UNIQUENESS PROPERTY OF LARGE DEFORMATION OF A HEAVY CANTILEVER

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Abstract. In this paper we present a mathematical model to describe the deformations of a cantilever by its own weight. Our primary concern is the uniqueness property of the associated two-point boundary value problem (2.2). We find an optimal condition on the parameter L such that for any α, $-\pi \leq \alpha \leq \pi$, the solution of (2.2) is unique.

1. Introduction. In this paper we are concerned with the question of uniqueness of the deformations of a cantilever by its own weight. We assume that a cantilever of nonuniform density is held fixed at an angle α at one end, say, the origin, and is free at the other end. Let L be the total length of the cantilever, s be the arc length from the origin and $\bar{\theta} = \bar{\theta}(s)$ be the local angle of inclination (see Figure 1). From the derivation in [5], it follows that

$$(1.1) \quad \begin{aligned} &\frac{dm}{ds} - \overline{W}(s)\sin\bar{\theta}(s) = 0, \qquad 0 \leq s \leq L, \\ &\bar{\theta}(0) = \alpha, \quad \frac{d\bar{\theta}}{ds}(L) = 0, \qquad -\pi \leq \alpha \leq \pi, \end{aligned}$$

where $m = m(s)$ is the local moment,

$$(1.2) \quad \overline{W}(s) = \int_s^L \rho(\bar{s})\, d\bar{s}$$

and $\rho = \rho(s)$ is the density function satisfying

$$(1.3) \quad \text{and } \rho(s) \geq \rho_0 > 0, \qquad \text{for all } s \geq 0.$$

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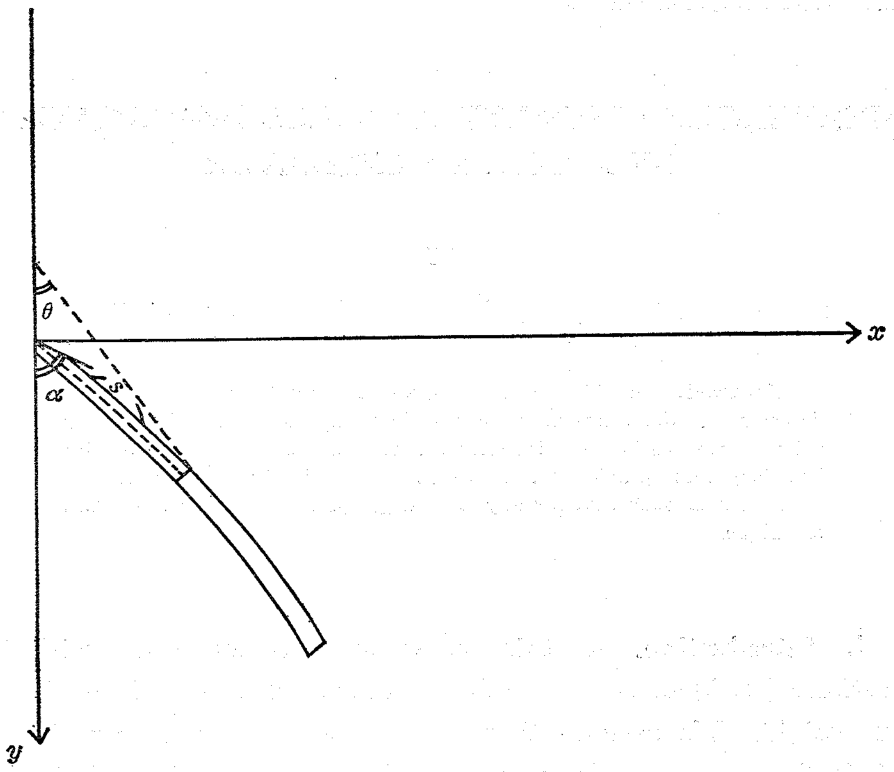

FIG. 1

As in [1], we assume the local moment $m(s)$ is a function of the curvature $d\bar{\theta}/ds$, i.e.,

$$m(s) = M(\psi), \qquad \psi = \frac{d\bar{\theta}}{ds}, \tag{1.4}$$

where the function $M \in C^2(\boldsymbol{R})$ satisfies

$$M(0) = 0 \quad \text{and} \quad M'(\psi) > 0 \qquad \text{for all} \quad \psi \in \boldsymbol{R}. \tag{1.5}$$

We note that in the case when the density is uniform and Euler-Bernoullis' law [5] holds, it follows that $\rho(s) \equiv \rho$, $M(\psi) = EI\psi$ where EI is the flexural rigidity. Then, from the equations (1.1), (1.4) we deduce

$$\begin{aligned} &EI\frac{d^2\bar{\theta}}{ds^2} = \rho(L-s)\sin\bar{\theta}(s), \\ &\bar{\theta}(0) = \alpha, \qquad \frac{d\bar{\theta}}{ds}(L) = 0. \end{aligned} \tag{1.6}$$

The special case (1.6) has been extensively studied in [2]. Our main concern in this paper is the following: Find the optimal conditions on the parameter L such that for any α, $-\pi \leq \alpha \leq \pi$, the problem (1.1) & (1.4) has a unique solution. Our main contribution is Theorem 1 in section 2. The result, in particular, solves the conjecture on the uniqueness property stated in [2] for the special case (1.6).

2. **Main Results.** Before we state and prove our main results, we shall reformulated our problem. First of all, we rewrite (1.1) (1.4) as

$$
\begin{aligned}
&M'(\bar{\theta}'(s))\,\bar{\theta}''(s) - \overline{W}(s)\sin\bar{\theta}(s) = 0, \\
&\bar{\theta}(0) = \alpha, \qquad \bar{\theta}'(L) = 0
\end{aligned}
\tag{2.1}
$$

Let $\theta(s) = \bar{\theta}(L-s)$ and $W(s) = \overline{W}(L-s)$. Then (2.1) becomes

$$
\begin{aligned}
&M'(-\theta'(s))\,\theta''(s) - W(s)\sin\theta(s) = 0, \qquad 0 \leq s \leq L, \\
&\theta'(0) = 0, \qquad \theta(L) = \alpha, \qquad -\pi \leq \alpha \leq \pi\,.
\end{aligned}
\tag{2.2}
$$

We note that from (1.2), (1.3) the function $W(s)$ satisfies

$$
\begin{aligned}
&W(s) = \int_{L-s}^{L} \rho(\bar{s})\,d\bar{s} \geq \rho_0\, s \qquad \text{for all} \quad s > 0, \quad \text{and} \\
&W(0) = 0, \quad W'(s) > 0 \quad \text{for} \quad s > 0\,.
\end{aligned}
\tag{2.3}
$$

If $0 < \alpha \leq \pi$ then we set $v(s) = \theta(s) - \pi$ and (2.2) takes the form

$$
\begin{aligned}
&M'(-v'(s))\,v''(s) + W(s)\sin v(s) = 0, \qquad 0 \leq s \leq L, \\
&v'(0) = 0, \quad v(L) = \beta = \alpha - \pi, \quad -\pi < \beta \leq 0\,.
\end{aligned}
\tag{2.4}
$$

If $-\pi \leq \alpha < 0$ then we set $v(s) = \theta(s) + \pi$ and (2.2) is reduced to

$$
\begin{aligned}
&M'(-v'(s))\,v''(s) + W(s)\sin v(s) = 0, \qquad 0 \leq s \leq L, \\
&v'(0) = 0, \qquad v(L) = \beta = \alpha + \pi, \quad 0 \leq \beta < \pi\,.
\end{aligned}
\tag{2.5}
$$

The following lemma asserts the uniqueness property when the initial local angle is zero.

LEMMA 2.1. *If $\alpha = 0$, then $\theta(s) \equiv 0$ is the unique solution of (2.2) for any $L > 0$.*

Proof. Multiplying (2.2) by $\theta'(s)$ and integrating the resulting equation from 0 to L, we obtain

$$(2.6)\quad \int_0^L M'(-\theta'(s))\,\theta''(s)\,\theta'(s)\,ds = \int_0^L W(s)\sin\theta(s)\,\theta'(s)\,ds\,.$$

We claim that $\theta'(L)=0$. If $\theta'(L)\neq 0$ then the left-hand side of (2.6) is

$$\text{LHS} = \int_0^{\theta'(L)} M'(-\psi)\,\psi\,d\psi > 0$$

by (1.5). However from (2.3) and $\theta(L)=0$ we see that the right-hand side of (2.6) may be computed as follows:

$$\begin{aligned}\text{RHS} &= (-\cos\theta(s))\,W(s)\big|_0^L + \int_0^L \cos\theta(s)\cdot W'(s)\,ds\\ &= -W(L) + \int_0^L \cos\theta(s)\cdot W'(s)\,ds\\ &\leq -W(L) + \int_0^L W'(s)\,ds = 0\,.\end{aligned}$$

Thus we obtain a contradiction and hence $\theta'(L)=0$. Since $\theta(L)=0$, $\theta'(L)=0$, the conclusion $\theta(s)\equiv 0$ follows directly from the uniqueness of solutions of ordinary differential equations.

Before dealing with the case $-\pi<\alpha<\pi$, $\alpha\neq 0$, we shall consider the following initial value problem

$$(2.7)\quad \begin{aligned}&M'(-v'(s))\,v''(s) + W(s)\,v(s)\sin v = 0,\\ &v'(0)=0,\\ &v(0)=a\,.\end{aligned}$$

In (2.7) the function $W(s)$ is assumed to be well-defined on $s\geq 0$ and to satisfy

$$(2.8)\quad \begin{aligned}&W(0)=0,\quad W'(s)>0 \quad \text{for } s>0,\\ &W(s)\geq \rho_0 s \quad \text{for some } \rho_0>0\,.\end{aligned}$$

We denote the solution of (2.7) by $v(s, a)$. From the uniqueness of solutions of ordinary differential equations, it follows that

$$(2.9)\quad v(s, a+2\pi) = 2\pi + v(s, a),$$

$$(2.10)\quad v(s, 0)\equiv 0,\quad v(s,\pi)\equiv \pi,\quad v(s,-\pi)\equiv -\pi\,.$$

From (2.9), (2.10), we shall consider $v(s, a)$ only for $-\pi<a<\pi$. Next we introduce the following notations

$$\Delta(s, a) = \frac{dv}{da}(s, a),$$

$$\phi(s) = \Delta(s, 0).$$

Differentiating (2.7) with respect to a yields

$$\text{(2.11)}\quad \begin{aligned} &M'(-v'(s, a))\,\Delta''(s, a) - M''(-v'(s, a))\,v''(s, a)\,\Delta'(s, a) \\ &\qquad + W(s)(\cos v(s, a))\,\Delta(s, a) = 0, \\ &\Delta(0, a) = 1, \\ &\Delta'(0, a) = 0. \end{aligned}$$

Setting $a = 0$ in (2.11) yields

$$\text{(2.12)}\quad \begin{aligned} &M'(0)\,\phi''(s) + W(s)\,\phi(s) = 0, \\ &\phi(0) = 1, \qquad \phi'(0) = 0. \end{aligned}$$

Since $M'(0) > 0$, from (2.8) it is easy to verify the solution $\phi(s)$ of (2.12) is oscillatory over $[0, \infty]$. Let λ_1 be the first zero of $\phi(s)$.

LEMMA 2.2. *Suppose that*

$$\text{(2.13)}\quad \psi M(\psi) \geq M'(0)\,\psi^2 \quad \textit{for all} \quad \psi \in \boldsymbol{R}.$$

Then for any $a \neq 0$, $-\pi < a < \pi$, *the first zero of* $v(s, a)$ *must be greater than* λ_1.

Proof. The existence of the first zero of $v(s, a)$ is obvious from equation (2.7) and conditions (2.8) on $W(s)$. We first consider the case $0 < a < \pi$ and we shall prove the lemma by contradiction. Suppose there exists a, $0 < a < \pi$, such that first zero L^* of $v(s, a)$ satisfies $L^* \leq \lambda_1$. We may rewrite the equation in (2.7) in the following form

$$\text{(2.14)}\quad -\frac{d}{ds}(M(-v'(s)) + W(s)\sin v(s) = 0.$$

Multiplying (2.14) by ϕ and multiplying (2.12) by v, subtracting the resulting expressions from each other and integrating the final equation from 0 to L^*, we obtain

$$\text{(2.15)}\quad \begin{aligned} &-\int_0^{L^*} \phi(s)(M(-v'(s)))'\,ds - M'(0)\int_0^{L^*} v(s)\,\phi''(s)\,ds \\ &\qquad = \int_0^{L^*} W(s)\,\phi(s)\,v(s)\left(1 - \frac{\sin v(s)}{v(s)}\right)ds. \end{aligned}$$

Since $\phi(s)>0$, $v(s)>0$ for $0\le s<L^*$, it follows that the right-hand side of (2.15) is positive. In order to get a contradiction, it suffices to show that the left-hand side of (2.15) is nonpositive. In fact, the left-hand side of (2.15) is

$$\begin{aligned} \text{LHS} = & -\phi(L^*)M(-v'(L^*))+\phi(0)M(0) \\ & +\int_0^{L^*} M(-v'(s))\,\phi'(s)\,ds \\ & -M'(0)[v(L^*)\,\phi'(L^*)-v(0)\,\phi'(0)] \\ & +M'(0)\int_0^{L^*} v'(s)\,\phi'(s)\,ds \\ = & -\phi(L^*)M(-v'(L^*)) \\ & +\int_0^{L^*}\phi'(s)[M(-v'(s))-M'(0)(-v'(s))]\,ds\,. \end{aligned} \tag{2.16}$$

Since $v'(s)<0$ and $\phi'(s)<0$ on $[0, L^*]$ from the assumption (2.13), it follows that LHS ≤ 0. Thus we complete the proof for the case $0<a<\pi$. The case $-\pi<a<0$ can be argued in a similar way. In that case, $v(s)<0$, $v'(s)>0$, $\phi(s)>0$, $\phi'(s)<0$ for $0<s<L^*$. From assumption (2.13), it can be shown that from (2.16) the left-hand side of (2.15) is nonnegative while the left-hand side of (2.15) is nonnegative while the right-hand side of (2.15) is negative.

In the following, we shall state and prove our main result.

THEOREM 1. *Let*

$$\psi M''(\psi)\ge 0 \qquad \textit{for all} \quad \psi\in \mathbf{R}. \tag{2.17}$$

Then for any α, $-\pi\le\alpha\le\pi$, *the problem* (2.2) *has a unique solution provided that* $L\le\lambda_1$.

Proof. From Lemma 2.1, it suffices to consider two cases, namely, $0<\alpha\le\pi$ and $-\pi\le\alpha<0$. We first consider the case, $0<\alpha\le\pi$. Let $v(s)=\theta(s)-\pi$. Then the problem (2.2) takes the form of (2.4),

$$\begin{aligned} & M'(-v'(s))\,v''(s)+W(s)\sin v(s)=0, \\ & v'(0)=0, \quad v(L)=\beta, \quad -\pi<\beta\le 0\,. \end{aligned} \tag{2.4}$$

It is easy to verify that the function $M(\psi)$ with property

(2.17) must also satisfy (2.13). Hence by Lemma 2.2, the first zero of $v(s, \alpha)$, $0<\alpha<\pi$, must be greater than λ_1. Since $L \leq \lambda_1$, the initial value α^* which gives rise to $v(L, \alpha^*)=\beta$, $-\pi<\beta<0$, must satisfy $-\pi<\alpha^*<\beta$. (Note that the case $\beta=0$ is already taken care of by Lemma 2.2.) Thus let $y_1(\alpha, \beta)$ be the first zero of $v(s, \alpha)-\beta=0$, $-\pi<\alpha<\beta$. Obviously from (2.10) and the continuous dependence on initial data, we have $\lim_{\alpha\to\beta^-} y_1(\alpha, \beta)=0$ if $\beta\neq 0$, and $\lim_{\alpha\to(-\pi)^+} y_1(\alpha, \beta)=+\infty$. To complete the proof, it suffices to show that

$$\frac{d}{d\alpha}(y_1(\alpha, \beta))<0 \qquad \text{for} \quad -\pi<\alpha<\beta<0. \tag{2.18}$$

Since

$$v(y_1(\alpha, \beta), \alpha)-\beta=0, \qquad \text{for all} \quad \alpha\in(-\pi, \beta), \tag{2.19}$$

differentiating (2.19) with respect to α yields

$$v'(y_1(\alpha, \beta), \alpha)\frac{dy_1}{d\alpha}(\alpha, \beta)+\Delta(y_1(\alpha, \beta), \alpha)=0,$$

or

$$\frac{dy_1}{d\alpha}(\alpha, \beta)=\frac{-\Delta(y_1(\alpha, \beta), \alpha)}{v'(y_1(\alpha, \beta), \alpha)}. \tag{2.20}$$

Since $v'(y_1(\alpha, \beta), \alpha)>0$, from (2.20) it suffices to show that

$$\Delta(y_1(\alpha, \beta), \alpha)>0. \tag{2.21}$$

Now we consider the equation (2.11) for $\Delta(s, \alpha)$ and rewrite it in the following form

$$\begin{aligned} &\frac{d}{ds}(M'(-v'(s, \alpha))\Delta'(s, \alpha))+W(s)(\cos v(s, \alpha))\Delta(s, \alpha) \\ &\qquad =0, \\ &\Delta(0, \alpha)=1, \\ &\Delta'(0, \alpha)=0. \end{aligned} \tag{2.22}$$

From now on for convenience we denote $v(s)\equiv v(s, \alpha)$ and $\Delta(s)\equiv\Delta(s, \alpha)$. We claim that $\Delta(s)>0$ for $0\leq s\leq y_1(\alpha, \beta)$. This implies that, in particular, (2.21) holds. We shall argue by contradiction. Suppose that the claim is not true, then we have two possible cases.

Case 1: *There exists* s^*, $0 < s^* \le y_1(a, \beta)$ *such that* $\Delta(s^*) = 0$, $\Delta'(s^*) < 0$ *and* $\Delta(s) > 0$, $\Delta'(s) < 0$ *for* $0 < s < s^*$.

We first note that from the uniqueness of solutions of ordinary differential equations, it is impossible to have $\Delta(s^*) = 0$, $\Delta'(s^*) = 0$ for equation (2.22). Now we compare the following two identities:

$$-\frac{d}{ds}(M(-v'(s))) + W(s)\sin v(s) = 0, \tag{2.14}$$

and

$$\frac{d}{ds}(M'(-v'(s))\,\Delta'(s)) + W(s)(\cos v(s))\,\Delta(s) = 0. \tag{2.22}$$

Multiplying (2.14) by Δ and multiplying (2.22) by v, subtracting the resulting equations from each other and integrating the final expression from 0 to s^*, we obtain

$$\begin{aligned} &-\int_0^{s^*} \Delta(s)\frac{d}{ds}(M(-v'(s)))\,ds \\ &-\int_0^{s^*} v(s)\frac{d}{ds}(M'(-v'(s))\,\Delta'(s))\,ds \\ &+\int_0^{s^*} W(s)\,\Delta(s)\,v(s)\left[\frac{\sin v(s)}{v(s)} - \cos v(s)\right]ds = 0. \end{aligned} \tag{2.23}$$

Since $\sin v/v \ge \cos v$ for $v \in [-\pi, \pi]$ and $\Delta(s) > 0$, $v(s) < 0$ for $0 \le s < s^*$, it follows that

$$\int_0^{s^*} W(s)\,\Delta(s)\,v(s)\left[\frac{\sin v(s)}{v(s)} - \cos v(s)\right]ds < 0. \tag{2.24}$$

Let

$$I_1 = -\int_0^{s^*} \Delta(s)\frac{d}{ds}(M(-v'(s)))\,ds,$$

and

$$I_2 = -\int_0^{s^*} v(s)\frac{d}{ds}(M'(-v'(s))\,\Delta'(s))\,ds.$$

In order to get a contradiction, it suffices to show that $I_1 + I_2 \le 0$. Since $M(0) = 0$, $\Delta'(0) = 0$, $\Delta(s^*) = 0$, it follows from integrating by parts that

$$I_1 = \int_0^{s^*} \Delta'(s)\,M(-v'(s))\,ds,$$

and

$$I_2 = -v(s^*)\,M'(-v'(s^*))\,\Delta'(s^*) + \int_0^{s^*} M'(-v'(s))\,v'(s)\,\Delta'(s)\,ds\,.$$

Then

$$I_1 + I_2 = -v(s^*)\,M'(-v'(s^*))\,\Delta'(s^*) + \int_0^{s^*} \Delta'(s)[M'(-v'(s))\,v'(s) + M(-v'(s))]\,ds\,. \tag{2.25}$$

Since $\Delta'(s^*) < 0$, $v(s^*) \leq 0$, $M'(-v'(s^*)) > 0$, the first term on the right-hand side of (2.25) is nonpositive. To complete our proof for Case 1, we now only have to show that

$$M'(-v'(s))\,v'(s) + M(-v'(s)) \geq 0 \qquad \text{for} \quad 0 \leq s \leq s^*\,. \tag{2.26}$$

Since $\psi = -v'(s) \leq 0$ for $0 \leq s \leq s^*$, this follows from

$$M(\psi) \geq M'(\psi)\,\psi \qquad \text{for} \quad \psi \leq 0, \tag{2.27}$$

which, in turn, follows from our hypothesis on M, namely, $M(0) = 0$ and $M''(\psi) \leq 0$ for $\psi \leq 0$. Thus we have arrived at a contradiction in Case 1.

Case 2: *There exist* γ *and* s^* *such that* $\gamma < s^* \leq y_1(a, \beta)$, *such that* $\Delta(\gamma) > 0$, $\Delta'(\gamma) = 0$, $\Delta(s^*) = 0$, $\Delta'(s^*) < 0$ *and* $\Delta(s) > 0$, $\Delta'(s) < 0$ *for* $\gamma < s < s^*$. We proceed an in Case 1 except replacing the lower limit 0 of integrals in (2.23) by γ. We then have

$$\tilde{I}_1 + \tilde{I}_2 + \int_\gamma^{s^*} W(s)\,\Delta(s)\,v(s)\left[\frac{\sin v(s)}{v(s)} - \cos v(s)\right] ds = 0, \tag{2.28}$$

where

$$\tilde{I}_1 = -\int_\gamma^{s^*} \Delta(s)\,\frac{d}{ds}\,(M(-v'(s)))\,ds,$$

$$\tilde{I}_2 = -\int_\gamma^{s^*} v(s)\,\frac{d}{ds}\,(M'(-v'(s))\,\Delta'(s))\,ds\,.$$

The third term in (2.28) is again negative, and to reach a contradiction, it suffices to show that $\tilde{I}_1 + \tilde{I}_2 \leq 0$. In fact, following similar arguments in Case 1, we deduce that

$$\begin{aligned}\tilde{I}_1 + \tilde{I}_2 &= \Delta(\gamma) M(-v'(\gamma)) - v(s^*) M'(-v'(s^*)) \Delta'(s^*) \\ &\quad + \int_\gamma^{s^*} \Delta'(s)[M'(-v'(s)) v'(s) + M(-v'(s))] \, ds \\ &\leq 0.\end{aligned}$$

This finishes the proof for Case 2.

Next consider the case $-\pi \leq \alpha < 0$. The proof is similar to that of the case $0 < \alpha \leq \pi$. We let $v(s) = \theta(s) + \pi$. Then problem (2.2) takes the form of (2.5)

$$\begin{aligned}&M'(-v'(s)) v''(s) + W(s) \sin v(s) = 0, \\ &v'(0) = 0, \quad v(L) = \beta, \quad 0 \leq \beta < \pi.\end{aligned} \tag{2.5}$$

As before the case $\beta = 0$ is taken care of by Lemma 2.2. So we assume $0 < \beta < \pi$. Since $\lim_{a \to \beta^-} y_1(a, \beta) = 0$ (if $\beta > 0$) and $\lim_{a \to \pi^-} y_1(a, \beta) = +\infty$, it suffices to show that

$$\frac{d}{da}(y_1(a, \beta)) > 0 \quad \text{for} \quad 0 \leq \beta < a < \pi.$$

By (2.20) and the fact that $v(s) > 0$, $v'(s) < 0$ for $0 < s \leq y_1(a, \beta)$, we only have to prove that

$$\Delta(y_1(a, \beta), a) > 0 \qquad \text{for} \quad a \in (\beta, \pi).$$

As in the case $0 < \alpha \leq \pi$, we claim that $\Delta(s, a) > 0$ for $0 \leq s \leq y_1(a, \beta)$. The proof of this assertion is similar to that of the case $0 < \alpha \leq \pi$ (with some obvious modification) and is therefore omitted.

REMARK 1. In the case of liner elasticity, $M(\psi) = EI\psi$, $EI > 0$, a simple proof of Theorem 1 is possible. Since $M'(\psi) \equiv EI$, (2.4) becomes

$$\begin{aligned}&EI v''(s) + W(s) \sin v(s) = 0, \\ &v'(0) = 0, \quad v(L) = \beta, \quad \beta \in (-\pi, 0].\end{aligned} \tag{2.29}$$

The case $\beta = 0$ follows from Lemma 2.2, thus we assume $\beta < 0$. Suppose $v_1(s)$, $v_2(s)$ are two distinct solutions of (2.29). We may assume that there exists $L^* \leq L$ such that $0 > v_1(s) > v_2(s)$ for $0 < s < L^*$, $v_1(L^*) = v_2(L^*)$ and $v_2'(L^*) > v_1'(L^*)$. Compare the following two identities

$$EI\,v_1''(s)+W(s)\sin v_1(s)=0 \tag{2.30}$$

$$EI\,v_2''(s)+W(s)\sin v_2(s)=0\,. \tag{2.31}$$

Multiplying (2.30) by $v_2(s)$ and multiplying (2.31) by $v_1(s)$, subtracting the resulting equation from each other and integrating the final equation from 0 to L^*, we obtain

$$\begin{aligned} EI\int_0^{L^*}&[v_1''(s)\,v_2(s)-v_2''(s)\,v_1(s)]\,ds \\ &+\int_0^{L^*}W(s)\,v_1(s)\,v_2(s)\left(\frac{\sin v_1(s)}{v_1(s)}-\frac{\sin v_2(s)}{v_2(s)}\right)ds \\ =0\,. \end{aligned} \tag{2.32}$$

Since the function $\sin v/v$ is strictly increasing in $-\pi<v<0$, it follows that

$$\int_0^{L^*}W(s)\,v_1(s)\,v_2(s)\left(\frac{\sin v_1(s)}{v_1(s)}-\frac{\sin v_2(s)}{v_2(s)}\right)ds>0\,.$$

On the other hand, we have

$$\begin{aligned} \int_0^{L^*}&[v_1''(s)\,v_2(s)-v_2''(s)\,v_1(s)]\,ds \\ &=v_1'(L^*)\,v_2(L^*)-v_2'(L^*)\,v_1(L^*) \\ &=v_1(L^*)(v_1'(L^*)-v_2'(L^*))>0\,. \end{aligned}$$

This gives rise to a contradiction in view of (2.32) and the uniqueness of (2.29) is established.

REMARK 2. The conclusion of Theorem 1 also holds when the function $M(\psi)$ is replaced by $M(\psi,s)$ where $M(\psi,s)$ satisfies

(i) $M(0,s)\equiv 0$ for all $s\geq 0$

(ii) $\dfrac{\partial M}{\partial\psi}(\psi,s)>0$ for all $\psi\in\boldsymbol{R}$, $s\geq 0$

(iii) $\dfrac{\partial^2 M}{\partial\psi\partial s}(0,s)\leq 0$, $s\geq 0$

(iv) $\psi\dfrac{\partial^2 M}{\partial\psi^2}(\psi,s)\geq 0$ for all $\psi\in\boldsymbol{R}$, $s\geq 0$.

REMARK 3. Let $y_1(a)$ be the first zero of $v(s,a)$, $a\neq 0$. From [3] or [4], we may show that

$$\lim_{a\to 0}y_1(a)=\lambda_1\,.$$

From this and Lemma 2.2, the condition $L \leq \lambda_1$ is *optimal* for the uniqueness property of problem (2.2) for any α, $-\pi \leq \alpha \leq \pi$.

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