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RANKS OF CHORDAL GRAPHS

BY

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Abstract. A tree is an n-tree if a base path can be chosen such that every vertex is at distance $< n$ to this path. The extent of a tree is the least n such that it is an n-tree. A chordal graph is the intersection graph of some subtrees of a representing tree. The rank of a chordal graph is defined to be the least extent of such representing trees. In this paper, we provide an algorithm for determining the extent and the intersection of all base paths of a tree. We show that a chordal graph of rank >1 has three simplicial vertices. We establish a rank reduction theorem for chordal graphs. Then we use it to determine the rank of a tree regarded as a chordal graph and to investigate a certain kind of betweeness property within a chordal graph.

1. **Introduction.** All graphs in this paper will be finite and have no loops or multiple edges. We use $G=(V(G), E(G))$ to denote a graph, where $V(G)$ and $E(G)$ are its vertex and edge sets, respectively. If $X \subseteq V(G)$, we use $G \backslash X$ to denote the graph obtained from G by deleting vertices in X and all edges incident upon them. The cardinality of a set X is written as $|X|$. A set of pairwise adjacent vertices is said to be a *clique*. A clique is a maximal clique if it is not properly included in another clique. The distance $d(x, y)$ between two vertices is the length of a shortest path connecting x to y and $d(x, y)$ is defined to be ∞ if x and y are in different components. The distance between a vertex x and a set $X \subseteq V(G)$ is defined to be $\min\{d(x, y) \mid y \in X\}$. A *chord* of a cycle C in G is an edge joining two nonconsecutive vertices of C, i.e., an edge of G which is not in C but joins two vertices of C. A graph G is said to be *chordal* if every cycle of length ≥ 4 has at least one chord. Equivalently, G does not contain an induced subgraph which is a cycle of length ≥ 4. In the

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literature, chordal graphs have also been called *triangulated*, *rigid-circuit*, *monotone transitive*, and *perfect elimination* graphs.

The importance of chordal graphs was recognized when they were shown to be one of the first classes of graphs that are *perfect*. Thus they ushered in the study of the theory of perfect graphs. The usefulness of chordal graphs has been strengthened when efficient algorithms were successfully designed for some basic problems which were shown to be *NP*-complete for general graphs. For a survey of chordal graphs, see Golumbic [4].

There is a class of chordal graphs, the so-called *interval graphs*, which are among the most useful mathematical structures for modeling real world problems. A graph G is an interval graph if its vertices can be put into one-to-one correspondence with a set S of intervals of the real line such that two vertices are adjacent in G if and only if their corresponding intervals have nonempty intersection. We call S an interval representation for G. Equivalently, S can be regarded as a set of subpaths of a given path.

Chordal graphs can be characterized as the intersection graphs of subtrees of trees. In particular, the special class of interval graphs are the intersection graphs of subpaths of paths, which are the least complex trees. Our research presented in this paper started with an effort to make this sense of complexity more precise. We will introduce a notion of extent for trees at the first stage. Then we define the rank of a chordal graph to be the least extent of a tree representation for that graph. In view of this concept of rank, we expect to generalize results concerning interval graphs to similar ones for chordal graphs of higher ranks.

When we remove all end vertices, i. e. vertices of degree 1, of a tree, the extent is naturally reduced by 1. Trees are chordal graphs by the trivial reason that they have no cycles. The role of an end vertex of a tree is played by a *simplicial* vertex in a general chordal graph. We will establish a rank reduction theorem by the removal of simplicial vertices.

Two applications of this reduction will be given. One determines the rank of a tree and the other generalizes a betweeness property in interval graphs to chordal graphs of higher ranks.

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2. **Extent and Base Line.** A tree T is said to be an *n-tree* if there is a path P in T such that $d(x, P) < n$ for all vertices x. In this terminology, a 1-tree is an ordinary path, a 2-tree is what usually called a *caterpiller*, and an n-tree is automatically an $(n+1)$-tree.

The *extent* $e(T)$ of a tree T is defined to be the smallest n such that T is an n-tree. Any path P in T, satisfying $d(x, P) < n$ for all vertices x, is called a *base path* of T. A base path can be exhibited if we iteratively test whether the remaining tree is a path and remove all end vertices from the tree. The number of iterations turns out to be the extent of the tree. The complexity of this algorithm is linear. The base path so produced is called the *base line* which plays a special role among all base paths.

THEOREM 1. *The base line is the intersection of all base paths.*

Proof. Let B be the base line of a tree T. Let P be any arbitrary base path. Suppose that two ends of B are the vertices x and y (which could be identical).

If B is not a subpath of P, then at most one of x and y belongs to P. There are distinct paths P_x and P_y going out of x and y, respectively, such that both have length $e(T)-1$. Suppose that x does not belong to P. After the removal of x, the tree T separates into components. Now P is included in one of the components. Thus we can either extend P_x or extend P_y to have a path reaching P and having length $> e(T)-1$. This contradicts the fact that P is a base path. Therefore B is a subpath of P.

With respect to the base line B of the tree T, a unique *level number* can be assigned to each vertex x. We say that x is at level n if $d(x, P) = n-1$. The highest level number is equal to the extent of the tree. If x is at level $n > 1$, then there is a unique vertex y at level $n-1$ which is adjacent to x. This vertex y is said to be the *predecessor* of x.

3. **Rank reduction.** Let F be a family of nonempty sets. We allow members of F to be identical. The *intersection graph* of F is obtained by representing each set in F by a vertex and connecting two vertices by an edge if and only if their corresponding sets have nonempty intersection. A graph G is a chordal graph if and only if G is the intersection graph of a family of distinct subtrees of a tree. (Buneman [1], Gavril [3], and Walter [7].) It is straightforward to verify that this characterization still holds when we allow repeated occurrences of subtrees. Now we say that a chordal graph G is an *n-bush* if it is the intersection graph of some subtrees of an n-tree. Thus an interval graph is a 1-bush. An n-bush is automatically an $(n+1)$-bush. The *rank* of a chordal graph G, denoted by $rk(G)$, is defined to be the smallest n such that G is an n-bush.

Using this rank notion, results for general chordal graphs could be refined. The following theorem is an improvement of the well-known fact that a chordal graph has two nonajacent simplicial vertices if it is not a clique. (Dirac [2]) A vertex is called a *simplicial vertex* if all vertices adjacent to it form a clique.

THEOREM 2. *If G is a chordal graph with $rk(G)>1$, then G has at least three mutually nonadjacent simplicial vertices.*

Proof. Among all representations for G as the intersection graph of subtrees of a tree, we choose a tree T with the smallest number of vertices and $e(T)=rk(G)$. Let B be the base line of T. Since $e(T)>1$, T has at least three end vertices x_1, x_2, and x_3. By the minimality condition on T, each vertex x_i must occur in some subtree representing a vertex of G, otherwise it can be deleted. If x_i itself is not a representing subtree, then x_i belonging to the intersection of two subtrees will imply the predecessor of x_i belonging to the intersection. It follows that x_i can be deleted without affecting the intersection graph. Let each x_i represent a vertex v_i of G. Evidently, v_1, v_2, and v_3 are simplicial and mutually nonadjacent.

To obtain the rank of a chordal graph, we shall apply a procedure similar to the one for determining the extent of a tree based on the following reduction.

THEOREM 3. *Let G' be obtained from the chordal graph G by deleting all simplicial vertices of G. If $rk(G)>1$, then $rk(G') = rk(G)-1$.*

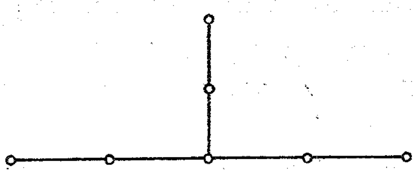
Proof. Among all representations for G as the intersection graph of subtrees of an $(n+1)$-tree, we may choose an $(n+1)$-tree T with subtrees $T_1, T_2, \cdots, T_m$ such that $|V(T)|$ is minimum. Let $x_1, x_2, \cdots, x_k$ be all end vertices of T. Using the minimality condition on T, we can reason as we did in the proof of Theorem 2 to show that each of $x_1, x_2, \cdots, x_k$ is a representing subtree. Since a subtree of T can represent different vertices of G. Each x_i is a subtree representing a clique of simplicial vertices. After trimming $x_1, x_2, \cdots, x_k$ off the tree T, we obtain a tree T'. A representation for G' can be drawn on T'. Thus $rk(G') \leq rk(G)-1$.

Let $K_1, K_2, \cdots, K_k$ be the partition of all simplicial vertices of G into maximal cliques. Let G' be the intersection graph of some subtrees of an n-tree T. Since each K_i consists of simplicial vertices, we can use the same subtree to represent every vertex in K_i. Therefore, without loss of generality, we may assume that each K_i is a single simplicial vertex v_i. Now suppose that vertices adjacent to v_1 in G are represented by subtrees $T_1, T_2, \cdots, T_m$. Any pair of these subtrees has nonempty intersection since v_1 is a simplicial vertex. It is well-known that a family of subtrees of a tree satisfies the Helly property, i. e. the intersection of the whole family is nonempty if the intersection of any two members is nonempty. (See Golumbic [4 p. 92, Proposition 4.7].) Now there is vertex z belonging to all of $T_1, T_2, \cdots, T_m$. We create a new vertex x_1 and make z the predecessor of x_1. Let $\{x_1\}$ be the new subtree representing v_1. Modify each T_i so that $x_1 z$ becomes an edge of T_i. In this manner, we can successively attach new vertices $x_1, x_2, \cdots, x_k$ to T to obtain a new tree T' together with modified subtrees to represent G. The extent of T' is at most $n+1$, i.e. $rk(G) \leq rk(G')+1$.

4. **Applications.** In this section, we will derive some consequences of Theorem 3. The first one concerns the rank of a tree which is obviously a chordal graph.

LEMMA 1. *Let T be a tree. Then $rk(T)=1$ if and only if T is a 2-tree.*

Proof. Necessity. Since $rk(T)=1$, the tree T is an interval graph. By the forbidden subgraphs characterization for interval graphs by Lekkerkerker and Boland [7], T does not contain the graph in the following figure as an induced subgraph.



Therefore the extent $e(T)$ cannot be greater than 2.

Sufficiency. Suppose $e(T) \leq 2$. Let the base line of T consist of the path $x_1 x_2 \cdots x_m$. Each x_i is adjacent to a stable set of vertices $\{y_1^i, y_2^i, \cdots, y_{k_i}^i\}$ which could be empty. In the intersection graph representation for T, we use the path $x_1^i z_1^i z_2^i \cdots z_{k_i}^i x_2^i$ to represent the vertex x_i and each z_j^i to represent y_j^i. Then we identify x_2^{i-1} with x_1^i, for $i=2,3,\cdots,m$, to obtain the entire path. It follows $rk(T)=1$.

THEOREM 4. *Let T be a tree such that $e(T)>1$. Then $rk(T)=e(T)-1$.*

Proof. Use induction on $e(T)$. The conclusion holds for $e(T)=2$ by Lemma 1. Now suppose $e(T)>2$. By Lemma 1, $rk(T)>1$. Simplicial vertices of T are precisely end vertices of T. If T' is obtained from T by deleting all end vertices, then by the algorithm in Sections 2 $e(T')=e(T)-1>1$. By the induction hypothesis, $rk(T') = e(T') - 1$. Theorem 3 implies $rk(T') = rk(T) - 1$. Therefore we have $rk(T) = e(T) - 1$.

Our next application concerns a kind of closeness concept within a chordal graph. For three sets of vertices Q, R, and S, we say that Q is *n-between* R and S if, for any $y \in R$ and $z \in S$ and any path P joining y and z, we have $d(x, P) \leq n$ for any $x \in Q$. Obviously, being n-between implies being $(n+1)$-between.

Halin [5] has established the characterization that a graph G is an interval graph if and only if, for any three maximal cliques of G, one is separating the other two. We can generalize the necessary condition to higher ranks.

THEOREM 5. *Let G be a chordal graph such that $rk(G) = n \geq 1$. Then, for any three cliques Q_1, Q_2 and Q_3, one is n-between the other two.*

Proof. If Q_1 and Q_2 are in different components, then Q_3 is trivially n-between them since no counterexample path from Q_1 to Q_2 could be constructed. So we may assume that G is connected. We prove the theorem by induction on $rk(G) = n$.

Suppose $n = 1$. Extend each Q_i to a maximal clique Q_i'. By Halin's theorem, we may assume that Q_1' is separating Q_2' from Q_3'. Thus any path P joining a vertex in Q_2 to a vertex in Q_3 will pass through Q_1'. So every vertex in Q_1 is at most at unit distance to P.

Now assume that $rk(G) = n + 1$ and the theorem holds for lesser ranks. All simplicial vertices of G are partitioned into maximal cliques $K_1, K_2, \cdots, K_k$ such that no edge joins vertices in different cliques. By Theorem 3, the graph $G \backslash (K_1 \cup K_2 \cup \cdots \cup K_k)$ is of rank n. Each Q_i can intersect at most one K_j. Suppose $v \in Q_i \cap K_j$. Then any vertex $x \in K_j$ and any vertex $y \in Q_i$ are adjacent to v. It follows that x and y are adjacent since v is simplicial. If we use $N(S)$ to denote the set $\{x \in V(G) \backslash S \mid x \text{ is adjacent to all vertices in } S\}$, then we have obtained $Q_i \backslash K_j \subseteq N(K_j)$. By the connectivity of G, we know $N(K_j) \neq \emptyset$. Obviously, $N(K_j)$ is a clique. Now we define Q_i' as follows.

$$Q'_i = \begin{cases} Q_i & \text{if none of } K_j \text{ intersects } Q_i, \\ N(K_j) & \text{if only } K_j \text{ intersects } Q_i. \end{cases}$$

Then Q'_1, Q'_2, and Q'_3 are nonempty cliques in $G \backslash (K_1 \cup K_2 \cup \cdots \cup K_k)$. By the induction hypothesis, we may assume that Q'_1 is n-between Q'_2 and Q'_3. Let P be a path joining a vertex in Q_2 to a vertex in Q_3. A subpath of P joins a vertex in Q'_2 to a vertex in Q'_3. Every vertex in Q'_1 is at distance $\leq n$ to P. So every vertex in Q_1 is at distance $\leq n+1$ to P, i.e., Q_1 is $(n+1)$-between Q_2 and Q_3.

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